Hereditary C*-Subalgebra Lattices

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Cuntz Subequivalence

- When $A = C_0(X)$, for locally compact Hausdorff X, Cuntz subequivalence $\precsim_{\mathrm{Cu}}$ corresponds to $\subseteq$ on supports, i.e.

$$a \precsim_{\mathrm{Cu}} b \qquad \Leftrightarrow \qquad \mathrm{supp}(a) \subseteq \mathrm{supp}(b),$$

  for all $a, b \in A_+$, where $\mathrm{supp}(a) = \{x \in X : a(x) \neq 0\}$.
- X is 2nd countable $\Rightarrow$ supports are the open subsets $\mathcal{O}(X)$ so

$$\text{Cuntz equivalence classes} \quad \approx \quad \text{open set lattice } \mathcal{O}(X)$$

  with lattice operations $O \wedge N = O \cap N$ and $O \vee N = O \cup N$.
- For noncommutative A, Cuntz equivalence classes may not be a lattice, hence we stabilise to obtain a semigroup structure

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+ / \sim .$$

Hereditary C*-Algebras

- A C*-subalg B of A is hereditary if $0 \leq a \leq b \in B \Rightarrow a \in B$.
- When $A = C_0(X)$, for any locally compact Hausdorff X,

  $$\text{Open Subsets } \mathcal{O}(X) \quad \leftrightarrow \quad \text{Hereditary C*-Subalgebras } \mathcal{H}(A),$$

  where $B \in \mathcal{H}(A)$ corresponds to $\mathrm{supp}(B) = \bigcup_{b \in B} \mathrm{supp}(b)$.
- Also $B \subseteq C \;\Leftrightarrow\; \mathrm{supp}(B) \subseteq \mathrm{supp}(C)$, i.e. $\mathcal{O}(X) \approx \mathcal{H}(A)$.
- But $\mathcal{H}(A)$ is a lattice even for noncommutative C*-algebra A:

  $$B \wedge C = B \cap C \qquad \text{and} \qquad B \vee C = HC^*(B \cup C)$$

- Surprisingly little known about $\mathcal{H}(A)$. Notable exceptions:
  - Akemann's work on open projections in A^{**} (late 60's/70's).
  - Giles and Kummer's work on q-topology (70's).
  - Work of Mulvey and Rosický et al on quantales (80's).
  - Related work on comparison of open projections, e.g. Peligrad-Zsido (2000), Rørdam-Ortega-Thiel (2011).

Unital vs Compact

- Can we detect whether A is unital from the lattice $\mathcal{H}(A)$?

Definition

Any lattice L with maximum 1 is said to be compact if, for any $C \subseteq L$ with $\bigvee C = 1$, we have finite $F \subseteq C$ with $\bigvee F = 1$.

$$\mathcal{O}(X) \text{ is compact} \quad \Leftrightarrow \quad X \text{ is compact.}$$
$$\mathcal{H}(C_0(X)) \text{ is compact} \quad \Leftrightarrow \quad C_0(X) \text{ is unital.}$$

- Does this extend to non-commutative A? Yes.

Proposition (Rosický 1989)

A is unital precisely when $\mathcal{H}(A)$ is compact.

- Proof: Assume A is non-unital.
- Case 1: $A = A_b = \overline{bAb}$ for some $b \in A_+$.
- Take $f_n \in C([0,1])_+$ which is 0 precisely on $[0, 1/n]$.
- $A_{f_n(a)}$ satisfy $\bigvee A_{f_n(a)} = A$ but $\bigvee_{n \leq m} A_{f_n(a)} = A_{f_m(a)} \neq A$.

Unital vs Compact

Proposition (Rosický 1989)

A is unital precisely when $\mathcal{H}(A)$ is compact.

- Case 2: $A \neq A_b (= \overline{bAb})$ for any $b \in A_+$.
- A_b satisfy $\bigvee_{b \in A_+} A_b = A$ but, for any finite $F \subseteq A_+$, $\bigvee_{b \in F} A_b = A_{\sum F} \neq A$. In either case $\mathcal{H}(A)$ is not compact.
- Converse holds by a result of Akemann (1971).
- Alternatively use the order anti-isomorphism from hereditary C*-subalgebras to weak*-closed faces of states $S(A)$:

$$B \mapsto B^0 = \{\phi \in S(A) : \phi[B] = \{0\}\}.$$

- If $\mathcal{C} \subseteq \mathcal{H}(A)$ and $\bigvee \mathcal{C} = A$ then $\bigwedge_{B \in \mathcal{C}} B^0 = \emptyset$.
- If A is unital, $S(A)$ is compact so we have finite $\mathcal{F} \subseteq \mathcal{C}$ with $\bigwedge_{B \in \mathcal{C}} B^0 = \emptyset$ and hence $\bigvee \mathcal{F} = A$, i.e. $\mathcal{H}(A)$ is compact. $\square$

Corners vs Complements

- If $p \in \mathcal{P}(A) = \{p \in A : p = pp^*\}$ then $A_p = pAp$ is a corner.
- Can we detect corners in $\mathcal{H}(A)$ from the lattice structure?

Definition

In a lattice L with max 1 and min 0, $p, q \in L$ are complements if

$$p \vee q = 1 \qquad \text{and} \qquad p \wedge q = 0.$$

$$\begin{aligned} O, N \in \mathcal{O}(X) \text{ are complements} \quad &\Leftrightarrow \quad O \cup N = X \text{ and } O \cap N = \emptyset. \\ O \in \mathcal{O}(X) \text{ has a complement} \quad &\Leftrightarrow \quad O \text{ is clopen.} \\ B \in \mathcal{H}(C_0(X)) \text{ has a complement} \quad &\Leftrightarrow \quad B \text{ is a corner (if } X \text{ is compact).} \end{aligned}$$

- Does this extend to non-commutative A? Yes.

Theorem (Akemann-B. 2015)

If A is unital, $B \in \mathcal{H}(A)$ is a corner iff it has a complement $C \in \mathcal{H}(A)$. Then $B = A_p$ and $C = A_q$ for $p, q \in \mathcal{P}(A)$ with

$$\|1 - p - q\| < 1.$$

Murray-von Neumann Equivalence vs Perspectivity

Definition (von Neumann 30's)

In a lattice L with max 1 and min 0, we call $p, q \in L$ perspective, written $p \sim_{\mathrm{per}} q$, if they have a common complement.

- If A is unital and A_p and A_q have a common complement A_r then, by the previous result, $\|r^\perp - p\|, \|r^\perp - q\| < 1$ and hence

$$p \sim_{\mathrm{MvN}} r^\perp \sim_{\mathrm{MvN}} q.$$

⇒ Perspectivity implies Murray-von Neuman equivalence.

- Could a weakening of perspectivity imply Cuntz equivalence?
- Possible approach: note we have a way-below analog in $\mathcal{H}(A)$:

$$B \ll C \qquad \Leftrightarrow \qquad \exists c \in C_+ \ \forall b \in B \ (bc = b = cb).$$

- When $B \ll C$, let us call $D \in \mathcal{H}(A)$ a complement of (B, C) if

$$B \wedge D = \emptyset \qquad \text{and} \qquad C \vee D = A.$$

- Call C and E weakly perspective if, for all $B \ll C$ and $D \ll E$, (B, C) and (D, E) have a common complement.
- Conjecture(?) $A_b \sim_{\mathrm{wper}} A_c$ implies $b \sim_{\mathrm{Cu}} c$.

Strong Orthogonality vs the Del Relation

- Can we detect when $B, C \in \mathcal{H}(A)$ are 'far apart'?
- Define orthogonality $\perp$ and strong orthogonality $\triangledown$ by

$$B \perp C \;\Leftrightarrow\; BC = \{0\} \qquad \text{and} \qquad B \triangledown C \;\Leftrightarrow\; BAC = \{0\}.$$

- We immediately see that, for any $B, C \in \mathcal{H}(A)$,

$$B \triangledown C \qquad \Rightarrow \qquad B \perp C \qquad \Rightarrow \qquad B \wedge C = \{0\}.$$

- Converses also hold when A is commutative.
- But $\perp$ is not detectable from the lattice structure in general.
- E.g. $\mathcal{H}(M_2) \cong \mathcal{P}(M_2)$ and any permutation of $\mathcal{P}(M_2)$ preserving rank will be an order isomorphism.
- But $\triangledown$ is a different story...

Strong Orthogonality vs the Del Relation

Theorem (Akemann-B. 2015)

For $B, C \in \mathcal{H}(A)$, the following are equivalent.

- $B \mathbin{\triangledown} C$.
- $B \vee C = B \oplus C$.
- $D = (B \vee D) \wedge (C \vee D)$, for all $D \in \mathcal{H}(A)$.
- $B \wedge D = B \wedge (C \vee D)$, for all $D \in \mathcal{H}(A)$ (the del relation).
- Every primitive/prime ideal in $\mathcal{H}(A)$ contains B or C.
- Every maximal/irreducible in $\mathcal{H}(A)$ contains B or C.

- D is irreducible if $B \wedge C = D$ implies $B = D$ or $C = D$.
- Prime means irreducible in the ideal lattice $\mathcal{I}(A)$.
- Primitive $\Rightarrow$ prime and conversely for separable A.
- Converse can fail for nonseparable A (Weaver 2001).
- Maximal $\Rightarrow$ irreducible (conversely for commutative A).

Question (Akemann 2015)

Is every irreducible in $\mathcal{H}(A)$ maximal?

Ideals vs Distributors

- Can we detect the ideals $\mathcal{I}(A)$ in the lattice $\mathcal{H}(A)$?

Definition

In a lattice L, we call $p \in L$ a distributor if, for all $q, r \in L$,

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r).$$

- L is distributive $\Leftrightarrow$ every $p \in L$ is a distributor.
- $\mathcal{O}(X)$ is always distributive while $\mathcal{H}(A)$ is not.
- $\mathcal{I}(A) \subseteq$ distributors (Borceaux-Rosický-Bossche 1989).

Theorem (Akemann-B. 2015)

The ideals are precisely of the distributors of $\mathcal{H}(A)$.

- (Mulvey 1986) quantale=lattice with a special & operator.
- $\mathcal{H}(A)$ is a quantale where $B\&C = B \cap \overline{\mathrm{span}}(ACA)$ (in terms of the corresponding right ideals, $I\&J = \overline{IJ}$)
- But $\overline{\mathrm{span}}(ACA)$ is the smallest distributive cover of C.

$\Rightarrow$ Order on $\mathcal{H}(A)$ already determines the quantale structure.

Annihilators

- The annihilator of any $B \subseteq A$ is defined by

$$B^\perp = \{a \in A : Ba = Ba^* = \{0\}\}.$$

- We denote the collection of all annihilators by

$$\mathcal{A}(A) = \{B^\perp : B \subseteq A\} = \{B \subseteq A : B = B^{\perp\perp}\} \subseteq \mathcal{H}(A).$$

- $A_p \in \mathcal{A}(A)$, for $p \in \mathcal{P}(A)$. Converse holds if A is a vN algebra.
- Annihilators still plentiful in C*-algebras, unlike projections.
- If $B \subseteq C_0(X)$ then $\mathrm{supp}(B^\perp) = \mathrm{int}(X \setminus \mathrm{supp}(B))$.

$\Rightarrow$ Annihilator supports are regular(=interior of a closed set).

- Conversely, $B \in \mathcal{H}(A)$ and $\mathrm{supp}(B)$ is regular $\Rightarrow B = B^{\perp\perp}$.
- So annihilators $\approx$ regular open subsets.

Annihilators vs Separative Elements

- Can we detect annihilators in the lattice $\mathcal{H}(A)$?

Definition

In a lattice L with minimum 0, we say p separates q from r when

$$0 \neq p \leq r \qquad \text{and} \qquad p \wedge q = 0.$$

If q is separated from every $r \nleq q$ then we call q separative.

- Say $O, R \in \mathcal{O}(X)$, R is regular and $O \nsubseteq R$. Then $O \nsubseteq \overline{R}$ (otherwise $O = O^\circ \subseteq \overline{R}^\circ = R$) so $O \setminus \overline{R}$ separates R from O.
- If $N \in \mathcal{O}(X)$ is not regular then $N \subsetneqq \overline{N}^\circ$. But N is not separated from $\overline{N}^\circ$, by the definition of closure.
- So in $\mathcal{O}(X)$, separative $\Leftrightarrow$ regular. Thus, for $B \in \mathcal{H}(C_0(X))$,

$$B \text{ is separative} \qquad \Leftrightarrow \qquad B \text{ is an annihilator.}$$

- Does this generalise? No (e.g. $A = C([0,1]) \otimes \mathcal{K}$) but

Theorem (Akemann-B. 2015)

Every annihilator is separative in $\mathcal{H}(A)$.

Type Decompositions

- $\mathcal{A}(A)$ is a separative ortholattice with orthocomplement $B^\perp$,

$$B \wedge C = B \cap C \qquad \text{and} \qquad B \vee C = (B \cup C)^{\perp\perp}.$$

- This automatically gives us various type decompositions.
- E.g. let us call an element p of a lattice L,
  - distributive if $p\downarrow = \{q \in L : q \leq p\}$ is a distributive sublattice.
  - semi-distributive if each $q \leq p$ dominates a distributive $d \neq 0$.
  - anti-distributive if no non-zero $q \leq p$ is distributive.

Proposition

If L is a sep. ortholattice, we have unique central complements $p, q \in L$ such that p is semi-distributive and q is anti-distributive.

- If $B \in \mathcal{A}(A)$ then

B is distributive	$\Leftrightarrow$	B is commutative
B is semi-distributive	$\Leftrightarrow$	B is discrete (Peligrad-Zsido 2001)
B is anti-distributive	$\Leftrightarrow$	B is antiliminary

Type Decompositions

Corollary (Akemann-B. 2015) (Ng-Wong 2016)

Any C*-algebra A has unique orthogonal annihilator ideals B and C such that $A = B \vee C$, B is discrete and C is antiliminary.

- If A is a von Neumann or AW^* algebra above then B is the type I part and C is type II+III part of A (and $A = B \oplus C$).
- Replacing 'distributive' with 'modular' we get another decomposition $A = B \vee C$. If A is an AW* algebra then in this case B is the type I+II part while C is the type III part.
- Replacing with 'orthomodular', we get another decomposition $A = B \vee C$. If A is an AW* algebra then in this case $B = A$ and $C = \{0\}$ because $\mathcal{A}(A) \approx \mathcal{P}(A)$ is always orthomodular.
- However, annihilators are not always orthomodular so for general C*-algebras this 'type IV part' may be non-zero.

Question

Do there exist any type IV C*-algebras?