The Global Glimm Problem

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The Global Glimm Problem

'Problem'

Assume that a task can be performed by finitely many elements. Can we bound the number of elements needed?

- Any finite loopless planar graph G admits a finite colouring. In fact, one has $\chi(G) \leq 4$.
- Every finitely generated subgroup of $(\mathbb{Q}, +)$ is singly generated.
- Finite dimensional von Neumann algebras are singly generated. Is every finitely generated von Neumann algebra singly generated?
- For infinite dimensional, separable, simple, unital, nuclear C^*-algebras, $\dim_{\mathsf{nuc}}(A) < \infty$ if and only if $\dim_{\mathsf{nuc}}(A) \leq 1$. (Castillejos-Evington-Tikuisis-White-Winter)

The Global Glimm Problem

Lemma (Glimm)

Let A be a C^*-algebra and $k \in \mathbb{N}$. If A has an irreducible representation of dimension at least k, then A admits a nonzero *-homomorphism $C_0((0,1], M_k) \to A$.

Proposition

Let A be a C^*-algebra. Then, no hereditary sub-C^*-algebra of A admits a finite-dimensional irreducible representation if and only if for each $a \in A_+$, each natural number $k \geq 2$, and each $\varepsilon > 0$, there exist finitely many *-homomorphisms

$$\varphi_1, \ldots, \varphi_n \colon C_0((0,1], M_k) \to \overline{aAa}$$

such that $(a - \varepsilon)_+$ belongs to the ideal of A generated by the combined images of $\varphi_1, \ldots, \varphi_n$.

The Global Glimm Problem

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$$\varphi_1, \ldots, \varphi_n \colon C_0((0,1], M_k) \to \overline{aAa}$$

such that $(a - \varepsilon)_+$ belongs to the ideal of A generated by the combined images of $\varphi_1, \ldots, \varphi_n$.

A C^*-algebra is *nowhere scattered* if and only if it satisfies the above property.

The Global Glimm Problem

Assuming that A is nowhere scattered, can we always set $n = 1$?

The Global Glimm Property (Kirchberg-Rørdam 2002)

A C^*-algebra A has the *Global Glimm Property* if for every $a \in A_+$, $k \geq 2$, and $\varepsilon > 0$ there exists a *-homomorphism $\varphi \colon C_0((0,1], M_k) \to \overline{aAa}$ such that $(a - \varepsilon)_+$ belongs to the ideal of A generated by the image of φ.

The Global Glimm Problem

The Global Glimm Problem

Does every nowhere scattered C^*-algebra have the Global Glimm Property?

1. Nowhere scatteredness
2. The Global Glimm Property
3. The Global Glimm Problem

Nowhere scatteredness

Definition (cf Kuratowski 1942)

A topological space X is *scattered* if every nonempty closed subset C of X has an isolated point in C.

Definition (Jensen 1977, Ghasemi-Koszmider 2018)

A C^*-algebra is *scattered* if every nonzero quotient A/I contains a nonzero projection p such that $p(A/I)p = \mathbb{C}p$.

Remark

$C_0(X)$ is scattered if and only if X is.

Theorem (Kusuda 2012)

Let A be a C^*-algebra. Then, the following are equivalent:

- A is scattered;
- every sub-C^*-algebra of A is of real rank zero;
- every sub-C^*-algebra of A is locally AF.

Nowhere scatteredness

Definition

A C^*-algebra A is *nowhere scattered* if no quotient A/I contains a nonzero projection p such that $p(A/I)p = \mathbb{C}p$.

Theorem

Let A be a C^*-algebra. Then, the following are equivalent:

- A is nowhere scattered;
- A has no nonzero scattered ideal-quotients;
- A has no nonzero elementary ideal-quotients;
- no hereditary sub-C^*-algebra of A admits a nonzero one-dimensional representation;
- no hereditary sub-C^*-algebra of A admits a finite-dimensional irreducible representation.

Nowhere scatteredness

Examples

- $C_0(X)$ is never nowhere scattered.
- A simple C^*-algebra is nowhere scattered if and only if it is not elementary.
- $\mathcal{Z}$-stable C^*-algebras are nowhere scattered.
- Traceless C^*-algebras are nowhere scattered.
- Purely infinite (and weakly purely infinite) C^*-algebras are nowhere scattered.

Nowhere scatteredness

Proposition

Let A be a nowhere scattered C^*-algebra. Then every quotient and every hereditary sub-C^*-algebra of A is nowhere scattered.

Proposition

Let A be C^*-algebra and let I be an ideal of A. Then, A is nowhere scattered if and only if I and A/I are nowhere scattered.

Proposition

Inductive limits of nowhere scattered C^*-algebras are nowhere scattered.

Nowhere scatteredness

Unitization and multipliers

$\tilde{A}$ is never nowhere scattered. When is $M(A)$ nowhere scattered? What if we assume A to be simple?

- There exists a (simple) nowhere scattered C^*-algebra A such that $M(A)$ admits a nonzero one-dimensional representation.
- If a simple C^*-algebra A has a continuous scale, $M(A)/A$ is purely infinite. Thus, $M(A)$ is nowhere scattered.

The Global Glimm Problem

The Global Glimm Problem

Does every nowhere scattered C^*-algebra have the Global Glimm Property?

1. Nowhere scatteredness
2. The Global Glimm Property

The Global Glimm Property

The Global Glimm Property (Kirchberg-Rørdam 2002)

A C^*-algebra A has the *Global Glimm Property* if for every $a \in A_+$, $k \geq 2$, and $\varepsilon > 0$ there exists a *-homomorphism $\varphi \colon C_0((0,1], M_k) \to \overline{aAa}$ such that $(a-\varepsilon)_+$ belongs to the ideal of A generated by the image of φ.

Theorem

Let A be a C^*-algebra. Then, the following are equivalent:

1. A has the Global Glimm Property;
2. for every $a \in A_+$ and $\varepsilon > 0$ there exists an element $r \in \overline{aAa}$ with $r^2 = 0$ such that $(a-\varepsilon)_+ \in \overline{\mathrm{span}} ArA$.

'The Global Glimm Problem'

Assume that a C^*-algebra A has no nonzero elementary ideal-quotients. Then, does every hereditary sub-C^*-algebra of A have a nilpotent (close to) full element?

The Global Glimm Property

Examples

- A simple C^*-algebra has the Global Glimm Property if and only if it is not elementary.
- $\mathcal{Z}$-stable C^*-algebras have the Global Glimm Property.

Theorem

The Global Glimm Property passes to hereditary sub-C^*-algebras and quotients.
Further, if I is an ideal in a C^*-algebra A, then A has the Global Glimm Property if and only if I and A/I do.

Proposition

The Global Glimm Property passes to inductive limits.

Proposition

The Global Glimm Property is invariant under Morita equivalence.

The Global Glimm Problem

The Global Glimm Problem

Does every nowhere scattered C^*-algebra have the Global Glimm Property?

1. Nowhere scatteredness
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Purely infinite and weakly purely infinite C^*-algebras

Definition (Kirchberg-Rørdam 2002)

A C^*-algebra A is *purely infinite* if for every $x \in \mathrm{Cu}(A)$ one has $x = 2x$.
A is said to be *weakly purely infinite* if there exists $n \geq 1$ such that $nx = 2nx$ for each $x \in \mathrm{Cu}(A)$.

Question (Kirchberg-Rørdam 2002)

Is every weakly purely infinite C^*-algebra purely infinite?

Proposition (Kirchberg-Rørdam 2002)

A weakly purely infinite C^*-algebra is purely infinite if and only if it has the Global Glimm Property.

Purely infinite and weakly purely infinite C^*-algebras

Remark

- Every weakly purely infinite C^*-algebra is nowhere scattered.
- Every purely infinite C^*-algebra has the Global Glimm Property.

If every nowhere scattered C^*-algebra has the Global Glimm Property, every weakly purely infinite C^*-algebra is purely infinite.

Question

Can one characterize nowhere scatteredness and the Global Glimm Property in terms of the Cuntz semigroup?

Divisibility conditions on the Cuntz semigroup

Definition (Robert-Rørdam 2013)

$\mathrm{Cu}(A)$ is said to be *weakly* $(2,\omega)$*-divisible* if, whenever $x' \ll x$ in $\mathrm{Cu}(A)$, there exist $y_1, \ldots, y_n \in \mathrm{Cu}(A)$ such that

$$x' \leq y_1 + \ldots + y_n \quad \text{and} \quad 2y_j \leq x$$

for each j.

$\mathrm{Cu}(A)$ is said to be $(2,\omega)$*-divisible* if, whenever $x' \ll x$ in $\mathrm{Cu}(A)$, there exist $y \in \mathrm{Cu}(A)$ and $n \in \mathbb{N}$ such that

$$x' \leq ny \quad \text{and} \quad 2y \leq x.$$

Examples

- $\overline{\mathbb{N}}$ is not weakly $(2,\omega)$-divisible. For example, $1 \ll 1$ does not admit the previous division.
- $[0,\infty]$ is $(2,\omega)$-divisible.

Divisibility conditions on the Cuntz semigroup

Theorem

Let A be a C^*-algebra. Then,

- A is nowhere scattered if and only if $\mathrm{Cu}(A)$ is weakly $(2, \omega)$-divisible.
- A has the Global Glimm Property if and only if $\mathrm{Cu}(A)$ is $(2, \omega)$-divisible.

Theorem

Let S be a Cu-semigroup satisfying (O5), (O6) and (O8). Then, S is weakly $(2, \omega)$-divisible if and only if S has no nonzero ideal-quotients isomorphic to $\overline{\mathbb{N}}$.

The Global Glimm Problem (Cu-version)

Is every weakly $(2, \omega)$-divisible Cuntz semigroup $(2, \omega)$-divisible?

Divisibility conditions on the Cuntz semigroup

We identify two conditions in the Cuntz semigroup that capture what a nowhere scattered C^*-algebra needs to have the Global Glimm Property.

Recall that A is nowhere scattered if and only if for every $a \in A_+$ and every $\varepsilon > 0$ there exist *-homomorphisms

$$\varphi_1, \dots, \varphi_n \colon C_0((0,1], M_2) \to \overline{aAa}$$

such that $(a-\varepsilon)_+$ belongs to the ideal of A generated by the combined images of $\varphi_1, \dots, \varphi_n$.

- Nowhere scatteredness
  $\to$ Finitely many *-homomorphisms.
- Nowhere scatteredness + weak version of $\wedge$ in $\mathrm{Cu}(A)$
  $\to$ 2 *-homomorphisms.
- Nowhere scatteredness + weak versions of $\wedge$ and $\vee$ in $\mathrm{Cu}(A)$
  $\to$ 1 *-homomorphism (ie the Global Glimm Property).

Ideal-filteredness

Definition

A Cuntz semigroup $\mathrm{Cu}(A)$ is *ideal-filtered* if, for every $v', v, x, y \in \mathrm{Cu}(A)$ such that

$$v' \ll v \leq \infty x, \infty y,$$

there exists $u \in \mathrm{Cu}(A)$ such that

$$v' \leq \infty u \quad \text{and} \quad u \leq x, y.$$

If $\mathrm{Cu}(A)$ has infima, $\mathrm{Cu}(A)$ is ideal-filtered (setting $u = x \wedge y$).

Lemma

Let A be a C^*-algebra such that $\mathrm{Cu}(A)$ is ideal-filtered. Then, for every $x' \ll x$, there exists $a \in A_+$ such that

$$x' \leq \infty[a] \quad \text{and} \quad [a] \leq x.$$

Ideal-filteredness

Proposition

Every $(2, \omega)$-divisible Cuntz semigroup is ideal-filtered.
That is, every C^*-algebra with the Global Glimm Property has an ideal-filtered Cuntz semigroup.

Proposition

Every C^*-algebra of stable rank one has an ideal-filtered Cuntz semigroup.

Theorem

Every separable C^*-algebra of topological dimension zero has an ideal-filtered Cuntz semigroup.

Example

$\mathrm{Cu}(C(S^2))$ is not ideal-filtered.

Ideal-filteredness

Theorem

Let A be a nowhere scattered C^*-algebra such that $\mathrm{Cu}(A)$ is ideal-filtered. Then for each $a \in A_+$, and each $\varepsilon > 0$, there exist *-homomorphisms

$$\varphi_1, \varphi_2 \colon C_0((0,1], M_2) \to \overline{aAa}$$

such that $(a-\varepsilon)_+$ belongs to the ideal of A generated by the combined images of φ_1 and φ_2.

Property (V)

Definition

A Cuntz semigroup $\mathrm{Cu}(A)$ is said to have *property (V)* if for all elements $c, d_j', d_j, x \in \mathrm{Cu}(A)$ satisfying $d_j' \ll d_j$ for $j = 1, 2$, and

$$d_1, d_2 \ll c, \quad \text{and} \quad c + d_1, c + d_2 \ll x,$$

there exist $y, z \in \mathrm{Cu}(A)$ such that

$$y + z \leq x, \quad \text{and} \quad d_1' + d_2' \leq \infty y, \infty z.$$

If $\mathrm{Cu}(A)$ has suprema, $\mathrm{Cu}(A)$ has property (V) (setting $y = c$ and $z = d_1 \vee d_2$)

Property (V)

Proposition

Every C^*-algebra with the Global Glimm Property has a Cuntz semigroup with property (V).

Proposition

Let A be a residually stably finite C^*-algebra. Then $\mathrm{Cu}(A)$ has property (V).

Theorem

Every real rank zero C^*-algebra has a Cuntz semigroup with property (V).

Theorem

Let A be a separable C^*-algebra with strict comparison of positive elements and with topological dimension zero. Then $\mathrm{Cu}(A)$ has property (V).

The Global Glimm Problem

Theorem

Let A be a C^*-algebra. Then the following are equivalent:

1. A has the Global Glimm Property;
2. $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible;
3. $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible, ideal-filtered and has property (V);
4. A is nowhere scattered and $\mathrm{Cu}(A)$ is ideal-filtered and has property (V).

The Global Glimm Problem

Let A be a nowhere scattered C^*-algebra. Is $\mathrm{Cu}(A)$ ideal-filtered and has property (V)?

Proposition (Antoine-Perera-Robert-Thiel, Elliott-Rørdam)

Let A be a C^*-algebra with stable rank one/real rank zero. Then, A has the Global Glimm Property if and only if A is nowhere scat.

The Global Glimm Problem

Proposition

Let A be a separable C^*-algebra with topological dimension zero. Assume that A is residually stably finite or has strict comparison of positive elements. Then A has the Global Glimm Property if and only if A is nowhere scattered.

Questions

- Does $\mathrm{Cu}(A)$ have Property (V) for every C^*-algebra A?
- Is $\mathrm{Cu}(A)$ ideal-filtered whenever A is nowhere scattered?
- In particular, are these two properties satisfied in the Cuntz semigroup of a weakly purely infinite C^*-algebra?

Thank you!