WORKSHOP ‘CUNTZ SEMIGROUPS’ 2022: OPEN PROBLEMS

Abstract. A list of problems proposed during the workshop *Cuntz semigroups*, 19-23 September 2022 at the University of Kiel.

Progress on 2021 problems

(1) N. C. Phillips has provided a lower bound for the radius of comparison of $C(X)$ in terms of the covering dimension of X, namely:

$$\mathrm{rc}(C(X)) \geq \frac{1}{2}(\dim(X) - 7).$$

This is not as good as the inequality supplied by [1, Theorem 1.1] when the covering dimension of X agrees with its rational cohomological dimension, but is valid for arbitrary compact Hausdorff spaces X.

(2) In joint work with W. Winter, S. Geffen has demonstrated a partial converse to [3, Corollary 8.12]: if A is a simple, separable, unital, stably finite, (exact) monotracial C*-algebra such that $\mathrm{Cu}(A)$ is almost unperforated and almost divisible, then A has stable rank one. Removing all assumptions about the trace space is work in progress.

2022 problems

(1) Is $\mathrm{Cu}(A)$ ideal-filtered (in the sense of [5, Definition 4.1]) when A is nowhere scattered (that is, admits no nonzero elementary ideal-quotients)? Does it always have property (V) [5, Definition 5.1]?

(2) Given a unital, simple, separable, exact, nonelementary C*-algebra A and $a \in A_+$, let

$$S_a = \{b \in A_+ \mid [b] = [a]\}.$$

It was shown in [7] that if A is approximately divisible and of real rank zero, then S_a is path connected. In fact, under these assumptions all homotopy groups of S_a vanish for all soft elements $a \in A_+$. Does there exist A such that path connectedness of S_a fails for some (soft) $a \in A_+$?

(3) Posed also in 2021, can one determine the possible values of $\dim \mathrm{Cu}(A)$ (in the sense of [4, Definition 3.1]) for (separable, nuclear) simple C*-algebras A? Are there examples of infinite dimension? Is the situation the same as the nuclear dimension, where only the values 0, 1 and ∞ can occur?

(4) What is the correct version of the Cuntz semigroup of a T*-algebra (that is, a tracially complete C*-algebra)? Should Cuntz subequivalence be defined via the C*-norm $\|\cdot\|$ or the uniform 2-norm $\|\cdot\|_{2,X}$? Given a metrisable Choquet simplex X, let $\mathcal{R}_X$ denote the unique ($\|\cdot\|_{2,X}$-separable) factorial hyperfinite II$_1$ T*-algebra over X. Compute $\mathrm{Cu}(\mathcal{R}_X)$ (in particular when X is a Bauer simplex).

(5) Formulate and prove the right version of the Toms–Winter Conjecture for T*-algebras.

(6) Come up with a notion of *stable* radius of comparison that allows for the addition of classes of projections. Obtain a lower bound for commutative C*-algebras.

(7) Can the Cuntz semigroup be used to detect Hausdorffised unitary algebraic K_1 (for morphisms)? In particular, is there a version of the Thomsen exact sequence [6, Corollary 3.3] stated in the language of Cu?

(8) For certain C*-algebras, such as the one constructed in [2], it may be more natural to view the radius of comparison as a function of the trace space. Suggested also in 2021, the problem is to investigate the behaviour of comparison functions and the range of possible functions that can arise.

(9) Posed also in 2021, can one obtain an upper bound for $\mathrm{rc}(C(X,A))$ in terms of $\mathrm{rc}(A)$ and $\dim(X)$? If A is simple and unital (and of stable rank one), is it true that

$$\mathrm{rc}(C(X,A)) \leq \frac{1}{2}\dim(X) + \mathrm{rc}(A) + 1?$$

(10) Does there exist a simple, stably finite C*-algebra A (of stable rank greater than one) such that $\mathrm{Cu}(A)$ does not have weak cancellation?

(11) Can one provide a meaningful computation of the Cuntz semigroup for any simple, separable, nuclear, nonelementary, non-$\mathcal{Z}$-stable C*-algebras?

(12) Decide whether *-homomorphisms $\mathcal{Z} \to A$ are unique up to approximate unitary equivalence. Is there some non-$\mathcal{Z}$-stable A (perhaps one of the Villadsen algebras) that admits distinct morphisms $\alpha, \beta\colon \mathrm{Cu}(\mathcal{Z}) \to \mathrm{Cu}(A)$?

(13) Is there a simple, nuclear C*-algebra of stable rank one that does not admit an embedding of $\mathcal{Z}$?

(14) Does $\mathrm{Cu}(\mathrm{C}^*_\mathrm{r}(\mathrm{G}))$ have strict comparison if G is a C*-simple group? Can one actually compute $\mathrm{Cu}(\mathrm{C}^*_\mathrm{r}(\mathrm{G}))$, even in the case $G = \mathbb{F}_2$?

(15) Develop the theory of crossed products and Rokhlin properties for group actions on Cu-semigroups.

References

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