K-theory for crossed products by Bernoulli shifts

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Motivation (Wreath products)

Let G, H be discrete groups. The wreath product $H \wr G$ is defined as the semidirect product

$$\big(\oplus_{g\in G} H\big) \rtimes G \quad \text{w.r.t} \quad g\cdot\big((h_s)_{s\in G}\big) = (h_{g^{-1}s})_{s\in G}.$$

Aim: Want to compute $K_*(C_r^*(H \wr G))$.

Observation: $C_r^*(H \wr G) \cong C_r^*(\bigoplus_G H) \rtimes_r G \cong C_r^*(H)^{\otimes G} \rtimes_r G$ since

$$C_r^*(\oplus_G H) = \lim_{F\subseteq G} C_r^*(\oplus_F H) = \lim_{F\subseteq G} C_r^*(H)^{\otimes F} = C_r^*(H)^{\otimes G}$$

and the action transforms to the Bernoulli action on $C_r^*(H)^{\otimes G}$ given by shifting the tensor factors!

More generally: A unital C^*-algebra, $A^{\otimes G} := \lim_{F\subseteq G} A^{\otimes F}$ w.r.t.

$$A^{\otimes F} \to A^{\otimes F'} : x \mapsto x \otimes 1_{A^{\otimes F'\setminus F}} \quad \forall F \subseteq F'.$$

Problem: Compute $K_*(A^{\otimes G} \rtimes_r G)$!

Previous results

- Bratteli, Kishimoto, Rørdam, Størmer, 1993 For $\mathcal{B} = M_2^{\otimes \mathbb{Z}} \rtimes \mathbb{Z}$ they show $K_0(\mathcal{B}) \cong \mathbb{Z}[\frac{1}{2}] = K_1(\mathcal{B})$.
- Ohhashi, 2015 $A^{\otimes \mathbb{Z}} \rtimes \mathbb{Z}$ if A is in the bootstrap class and $\mathbb{Z} \to K_0(A); n \mapsto n[1_A]$ is split injective.
- Flores, Pooya, Valette '17: $H \wr \mathbb{Z}$ with H finite.
- Pooya '19 $H \wr \mathbb{F}_2$ with H finite, $\mathbb{F}_2$ free group in 2 generators.
- Xin Li '19 $H \wr G$ with H finite and G satisfies the Baum-Connes conjecture with coefficients.

Indeed, if H is finite, $C^*(H) \cong \mathbb{C} \oplus B$ for some finite dim. B.

Xin Li computed $K_*(A^{\otimes G} \rtimes_r G)$ for all fin. dim. $A \cong \mathbb{C} \oplus B$, motivated by some previous work on K-theory of $C_0(\Omega) \rtimes_r G$ with Ω totally disconnected by Cuntz-E-Li.

(if $A \cong \mathbb{C}^n$, then $A^{\otimes G} = C(\Omega)$ with $\Omega = \{1, \ldots, n\}^G$ Cantor set.)

The Baum-Connes conjecture

Definition The group G satisfies the Baum-Connes conjecture with coefficients (BCC) if for all G-C^*-algebras A, the assembly map

$$\mu_A : K_*^G(\underline{E}(G); A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Theorem (Chabert-E-Oyono-Oyono '04, Meyer-Nest '06)
G satisfies BCC if and only if the following holds true:

If $\phi \in KK^G(A, B)$ for a pair of G-algebras A, B such that

$$\phi \rtimes H : K_*(A \rtimes H) \stackrel{\cong}{\to} K_*(B \rtimes H) \quad \forall \text{ finite } H \subseteq G.$$

Then

$$\phi \rtimes_r G : K_*(A \rtimes_r G) \stackrel{\cong}{\to} K_*(B \rtimes_r G).$$

This is the main tool for our computations!

The first step

The next major tool is the following result due to Izumi (in case A, B nuclear, $H = \mathbb{Z}/2\mathbb{Z}$) and Szabo (general A, B).

Theorem (Izumi, CEKN)
Let H be a finite group, Z a finite H-space. Then every KK-equivalence $\phi \in KK(A, B)$ induces an H-equivariant KK-equivalence

$$\phi^{\otimes Z} \in KK^H(A^{\otimes Z}, B^{\otimes Z}).$$

Corollary (CEKN) Suppose that G satisfies BCC and A and B are unital C^*-algebras. Let $\phi : A \to B$ be a unital $*$-homomorphism which induces a KK-equivalence. Then

$$\phi^{\otimes G} \rtimes_r G_* : K_*(A^{\otimes G} \rtimes_r G) \xrightarrow{\cong} K_*(B^{\otimes G} \rtimes_r G).$$

Proof Show that $\phi^{\otimes G} \rtimes H_* : K_*(A^{\otimes G} \rtimes H) \xrightarrow{\cong} K_*(B^{\otimes G} \rtimes H)$ for every finite $H \subseteq G$! But if Z runs through H-inv finite sets in G:
$A^{\otimes G} \rtimes H = \lim_Z (A^{\otimes Z} \rtimes H) \sim_{KK} \lim_Z (B^{\otimes Z} \rtimes H) = B^{\otimes G} \rtimes H$

The finite dimensional case

Lemma Let $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C})$ be a finite dimensional C^*-algebra such that $\gcd\{n_0, \ldots, n_l\} = 1$. Then there exists a unital $*$-homomorphism $\phi : \mathbb{C}^{l+1} \to A$ which induces a KK-equivalence.

Idea of proof One can show that for every tupel $(n_0, \ldots, n_l)$ with $\gcd\{n_0, \ldots, n_l\} = n$, there exist a matrix $X \in \mathrm{GL}(l+1, \mathbb{Z})$ with positive entries such that $X(n, \ldots, n)^t = (n_0, \ldots, n_l)^t$.

Then use the fact that unital $*$-homomorphism $\mathbb{C}^{l+1} \to A$ are classified by the maps they induce on the K_0-groups. If $n = 1$, this gives the result! □

Corollary (CEKN) Let A be as above and let G sat. BCC. Then

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*((\mathbb{C}^{l+1})^{\otimes G} \rtimes_r G)$$

$$\overset{X.Li}{\cong} K_*(C_r^*(G)) \oplus \bigoplus_{[C]\in\mathcal{C}} \bigoplus_{[X]\in N_C \backslash F(C)} \bigoplus_{[S]\in C\backslash\{1,\ldots,l\}^{CX}} K_*(C_S)$$

[$\mathcal{C}$ = conj cl of fin $H \subseteq G$, N_C = normalizer of C, $C_S = C \cap G_S$.]

The general finite dimensional case

Let $A = M_{n_0}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C})$ be a finite dimensional C^*-algebra with $\gcd\{n_0, \ldots, n_l\} = n$. Then

$$A \cong M_n(\mathbb{C}) \otimes B \quad \text{with} \quad B = M_{m_0}(\mathbb{C}) \oplus \cdots \oplus M_{m_l}(\mathbb{C})$$

and $\gcd\{m_0, \cdots, m_l\} = 1$.

Theorem (Kranz-Nishikawa '22)
Suppose G satisfies BCC and $|G| = \infty$. Let B be a unital C^*-algebra and let $A = M_n(\mathbb{C}) \otimes B$. Then

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*(B^{\otimes G} \rtimes_r G)\left[\frac{1}{n}\right].$$

A similar result holds for finite G if we replace $M_n(\mathbb{C})$ by $M_{n^\infty} := M_n(\mathbb{C})^{\otimes \mathbb{N}}$.

Corollary $K_*(M_2^{\otimes \mathbb{Z}} \rtimes \mathbb{Z}) \cong K_*(\mathbb{C} \rtimes \mathbb{Z})[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}]$.

Proof in case G torsion free

The result follows from the going-down principle:

Let G act trivially on M_{n^∞} and consider the G-inclusions

$$(B \otimes M_n)^{\otimes G} \overset{(1)}{\hookrightarrow} (B \otimes M_n)^{\otimes G} \otimes M_n^\infty; z \mapsto z \otimes 1$$

$$B^{\otimes G} \otimes M_n^\infty \overset{(2)}{\hookrightarrow} (B \otimes M_n)^{\otimes G} \otimes M_n^\infty; x \otimes b \mapsto (x \otimes 1) \otimes b.$$

Since M_{n^∞} is strongly self absorbing, both maps induce KK-equivalences when restricted to the trivial group.

Thus, if G is torsion free and satisfies BCC, we get

$$\begin{aligned} K_*((B \otimes M_n)^{\otimes G} \rtimes_r G) &\overset{(1)}{\cong} K_*\left(((B \otimes M_n)^{\otimes G} \otimes M_n^\infty) \rtimes_r G\right) \\ &\overset{(2)}{\cong} K_*\left((B^{\otimes G} \otimes M_n^\infty) \rtimes_r G\right) \\ &\cong K_*\left((B^{\otimes G} \rtimes G) \otimes M_{n^\infty}\right) \\ &= K_*(B^{\otimes G} \rtimes_r G)\left[\frac{1}{n}\right]. \end{aligned}$$

A more general strategy – the algebra $\mathcal{J}_B$.

Want to consider the following situation:

Let A be a unital C^*-algebra and let $\iota : \mathbb{C} \to A; \lambda \mapsto \lambda 1_A$. Assume that B is any C^*-algebra and $\phi \in KK(B, A)$ such that

$$\iota \oplus \phi \in KK(\mathbb{C} \oplus B, A) \quad \text{is a } KK\text{-equivalence.}$$

We want to compute $K_*(A^{\otimes G} \rtimes_r G)$ in terms of (a substitute of) "$(\mathbb{C} \oplus B)^{\otimes G} \rtimes_r G$" using Izumi's result for finite groups!

Notice: $(\mathbb{C} \oplus B)^{\otimes G}$ does not exist if $\mathbb{C} \oplus B$ is not unital. Even if $\mathbb{C} \oplus B$ is unital, $\iota \oplus \phi$ is not a unital $*$-homomorphism, so the previous results do not apply!

Definition For a group G and a C^*-algebra B we define

$$\mathcal{J}_B := \bigoplus_{F \in \mathrm{FIN}(G)} B^{\otimes F} \quad \text{with } B^{\emptyset} := \mathbb{C},$$

where $\mathrm{FIN}(G)$ denotes the collection of finite subsets of G, equipped with G-action sending $B^{\otimes F}$ to $B^{\otimes gF}$, $g \in G$.

The algebra $\mathcal{J}_B$

Lemma We have $\quad \mathcal{J}_B = \lim_{S \in \mathrm{FIN}(G)} (\mathbb{C} \oplus B)^{\otimes S}.$

Proof Simply observe that $\forall S \in \mathrm{FIN}(G)$:

$$(\mathbb{C} \oplus B)^{\otimes S} = \bigoplus_{F \subseteq S} \mathbb{C}^{\otimes S \setminus F} \otimes B^{\otimes F} = \bigoplus_{F \subseteq S} B^{\otimes F}.$$

The discrete G-space $\mathrm{FIN}(G)$ decomposes into the G-orbits $\{G \cdot F : [F] \in G \backslash \mathrm{FIN}(G)\}$ and hence $\mathcal{J}_B$ decomposes as

$$\mathcal{J}_B = \bigoplus_{[F] \in G \backslash \mathrm{FIN}(G)} \Big(\bigoplus_{[g] \in G/G_F} B^{\otimes gF} \Big) = \bigoplus_{[F] \in G \backslash \mathrm{FIN}(G)} \mathrm{Ind}_{G_F}^{G} B^{\otimes F}$$

where $G_F = \{g \in G : gF = F\}$ and $\mathrm{Ind}_{G_F}^{G} B^{\otimes F} = C_0(G \times_{G_F} B^{\otimes F})$ denotes the G-algebra induced from the G_F-algebra $B^{\otimes F}$.

Main theorem

Main theorem (CEKN) Suppose G satisfies BCC, A is unital, $\iota : \mathbb{C} \to \mathbb{C}1_A \subseteq A$ the unital inclusion, and $\phi \in KK(B, A)$ such that $\iota \oplus \phi \in KK(\mathbb{C} \oplus B, A)$ is a KK-equivalence. Then

$$K_*(A^{\otimes G} \rtimes_r G) \overset{(1)}{\cong} K_*(\mathcal{J}_B \rtimes_r G) \overset{(2)}{\cong} \bigoplus_{[F]\in G\backslash \mathrm{FIN}(G)} K_*(B^{\otimes F} \rtimes_r G_F).$$

Remark For $F = \emptyset$ we get $B^{\otimes\emptyset} \rtimes_r G_\emptyset = \mathbb{C} \rtimes_r G = C_r^*(G)$.
If $F \neq \emptyset$ we always have G_F finite (trivial if G torsion free)! Hence

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F]\in G\backslash \mathrm{FIN}^+(G)} K_*(B^{\otimes F} \rtimes G_F).$$

and if G is torsion free, we get

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F]\in G\backslash \mathrm{FIN}^+(G)} K_*(B^{\otimes F}).$$

Main theorem

Main theorem (CEKN) Suppose G satisfies BCC, A is unital, $\iota : \mathbb{C} \to \mathbb{C}1_A \subseteq A$ the unital inclusion, and $\phi \in KK(B, A)$ such that $\iota \oplus \phi \in KK(\mathbb{C} \oplus B, A)$ is a KK-equivalence. Then

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Proof of 2nd iso: By decomposing $\mathcal{J}_B$ over G-orbits, we get

$$\mathcal{J}_B \rtimes_r G \cong \bigoplus_{G \backslash \mathrm{FIN}(G)} \left(\mathrm{Ind}_{G_F}^G B^{\otimes F}\right) \rtimes_r G \sim_{Morita} \bigoplus_{G \backslash \mathrm{FIN}(G)} B^{\otimes F} \rtimes G_F,$$

hence $\quad K_*(\mathcal{J}_B \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \mathrm{FIN}(G)} K_*(B^{\otimes F} \rtimes G_F).$

Proof of main theorem

Proof of 1st Iso: Need to show $K_*(\mathcal{J}_B \rtimes_r G) \cong K_*(A^{\otimes G} \rtimes_r G)$.
Using $\mathcal{J}_B \cong \bigoplus_{G \backslash \mathrm{FIN}(G)} \mathrm{Ind}_{G_F}^G B^{\otimes F}$, we get

$$KK^G(\mathcal{J}_B, A^{\otimes G}) = \prod_{G \backslash \mathrm{FIN}(G)} KK^G(\mathrm{Ind}_{G_F}^G B^{\otimes F}, A^{\otimes G})$$
$$\cong \prod_{G \backslash \mathrm{FIN}(G)} KK^{G_F}(B^{\otimes F}, A^{\otimes G}).$$

Let $\psi_F := \left[B^{\otimes F} \hookrightarrow (\mathbb{C} \oplus B)^{\otimes F} \xrightarrow{(\iota \otimes \phi)^{\otimes F}} A^{\otimes F} \right] \in KK^{G_F}(B^{\otimes F}, A^{\otimes G})$
and $\psi := (\psi_F)_{[F] \in G \backslash \mathrm{FIN}(G)} \in KK^G(\mathcal{J}_B, A^{\otimes G})$.

Now let $H \subseteq G$ be a finite subgroup. Then, for every finite H-invariant set $S \subseteq G$, ψ restricts to the KK^H-equivalence

$$\psi_S : \bigoplus_{F \subseteq S} B^{\otimes F} \cong (\mathbb{C} \oplus B)^{\otimes S} \xrightarrow{\sim_{KK}} A^{\otimes S}$$

Taking limits over S gives $\psi \rtimes H : K_*(\mathcal{J}_B \rtimes H) \xrightarrow{\cong} K_*(A^{\otimes G} \rtimes H)$.

Applications of the main theorem

Corollary Suppose that A is unital with UCT such that $\iota : \mathbb{C} \to \mathbb{C}1_A \subseteq A$ induces a split injection $\iota_* : K_*(\mathbb{C}) \to K_*(A)$. Let B be any C^*-algebra with UCT s.t. $K_*(B) \cong \mathrm{cokern}(\iota_*)$. Then

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F]\in G\backslash \mathrm{FIN}^+(G)} K_*(B^{\otimes F} \rtimes_r G_F).$$

Proof: By the assumption $\exists \phi \in KK(B, A)$ such that $\iota \oplus \phi \in KK(\mathbb{C} \oplus B, A)$ is a KK-equivalence.

Corollary $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C})$ with $\gcd\{n_0, \ldots, n_l\} = 1$. Then $\iota_* : K_*(\mathbb{C}) \to K_*(A)$ is split injective and with $B = \mathbb{C}^{l-1}$:

$$\begin{aligned} K_*(A^{\otimes G} \rtimes_r G) &\cong \bigoplus_{[F]\in G\backslash \mathrm{FIN}(G)} K_*((\mathbb{C}^l)^{\otimes F} \rtimes G_F) \\ &\cong K_*(C_r^*(G)) \oplus \bigoplus_{[F]\in G\backslash \mathrm{FIN}^+(G)} K_*(C(\{1, \ldots, l\}^F) \rtimes G_F) \end{aligned}$$

which easily gives the formula of Xin Li we saw before!

Wreath products

Consider a wreath product $H \wr G$ such that H is a-T-menable (or, more generally, H satisfies the strong Baum-Connes conjecture) and G satisfies BCC.

Then Jean-Luis Tu showed that $C_r^*(H)$ satisfies the UCT and H is K-amenable (hence $C_r^*(H) \sim_{KK} C^*(H)$). It follows that

$$\mathbb{C} \xrightarrow{\iota} \mathbb{C}e_H \subseteq C^*(H) \xrightarrow{1_H} \mathbb{C}$$

induces a split injection $\iota_* : K_*(\mathbb{C}) \to K_*(C^*(H)) \cong K_*(C_r^*(H))$.

Thus, if B is any C^*-algebra with UCT s.t. $K_*(B) \cong \mathrm{cokern}(\iota_*)$ we get

$$K_*(C_r^*(H \wr G)) = K_*(C_r^*(H)^{\otimes G} \rtimes_r G) = \bigoplus_{[F] \in G \backslash \mathrm{FIN}(G)} K_*(B^{\otimes F} \rtimes_r G_F).$$

Example If $H = \mathbb{F}_n$, we can choose $B = \bigoplus_{j=1}^n C_0(\mathbb{R})$ and the groups $K_*(B^{\otimes F} \rtimes_r G_F)$ can be computed explicitly.

More examples

- Let $A = C(\mathbb{T}) \sim_{KK} \mathbb{C} \oplus C_0(\mathbb{R})$. Then

$$K_*(C(\mathbb{T})^{\otimes G} \rtimes_r G) \cong K_*(C^*(G)) \oplus \bigoplus_{[F] \in G \backslash \mathrm{FIN}^+(G)} K_*(C_0(\mathbb{R})^{\otimes F} \rtimes G_F)$$

  The groups $K_*(C_0(\mathbb{R})^{\otimes F} \rtimes G_F) = K_*^{G_F}(\mathbb{R}^{|F|})$ have been computed by Karoubi '02 and/or E-Pfante '09.

- Let $A = A_\theta$ (ir)rational rotation algebra. We have $A_\theta \sim_{KK} C(\mathbb{T}^2) \sim_{KK} \mathbb{C} \oplus B$ with $B = \mathbb{C} \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$.

$$K_*(A_\theta^{\otimes G} \rtimes_r G)$$
$$\cong K_*(C_r^*(G)) \oplus_{[F] \in G \backslash \mathrm{FIN}^+(G)} K_*(\mathbb{C} \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R}))^{\otimes F} \rtimes G_F).$$

  The algebras $(\mathbb{C} \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R}))^{\otimes F} \rtimes G_F$ decompose into direct sums of the form $C_0(\mathbb{R}^l) \rtimes H$ for some l and $H \subseteq G_F$, and are therefore all computable.

Cuntz algebras

Since $\mathcal{O}_\infty \sim_{KK} \mathbb{C}$ and $\mathcal{O}_2 \sim_{KK} \{0\}$ we get

$$K_*(\mathcal{O}_\infty^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \quad \text{and} \quad K_*(\mathcal{O}_2^{\otimes G} \rtimes_r G) = \{0\}.$$

For general $n \in \mathbb{N}$, one has $\mathcal{O}_{n+1} \otimes M_n^\infty \sim_{KK} \{0\}$, hence

$$K_*(\mathcal{O}_{n+1}^{\otimes G} \rtimes_r G)\left[\frac{1}{n}\right] = K_*((\mathcal{O}_{n+1} \otimes M_n^\infty)^{\otimes G} \rtimes_r G) = \{0\}.$$

Notice If B satsifies UCT and $K_*(B)$ is finitely generated, then

$$B \sim_{KK} \bigoplus_{i=1}^{m} B_i \quad \text{with} \quad B_i = \mathbb{C}, C_0(\mathbb{R}), \mathcal{O}_n, C_0(\mathbb{R}) \otimes \mathcal{O}_n.$$

Then (for $F \neq \emptyset$) $B^{\otimes F} \rtimes G_F$ decomposes into direct sums of crossed products of the form $D \rtimes H$, where D is a finite tensor product of the B_i and $H \subseteq G_F$ is finite.

Open problem: Compute

$$K_*(\mathcal{O}_n^{\otimes Z} \rtimes H) \quad \text{and} \quad K_*((C_0(\mathbb{R}) \otimes \mathcal{O}_n)^{\otimes Z} \rtimes H)$$

Further results

We can prove a quite technical formula if A is an AF-algebra. If A is AF and G is torsion-free, the formula reads

$$K_*(A^{\otimes G} \rtimes_r G) \cong \tilde{K}_*(C_r^*(G))[S^{-1}] \oplus K_*(A^{\otimes G})_G.$$

where

$$S := \{n \in \mathbb{N} \mid [1_A] \in K_0(A) \text{ divisible by } n\}$$

and $K_*(A^{\otimes G})_G$ denotes the coinvariants for $G \curvearrowright K_*(A^{\otimes G})$, the largest quotient of $K_*(A^{\otimes G})$ with trivial G-action!

In a recent preprint, Julian Kranz and Shintaro Nishikawa extended some of the methods presented here to the setting of the Farrel-Jones conjectures in algebraic topology.

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Happy birthday, Mikael!