Coronas and strongly self-absorbing C^*–algebras

Ilijas Farah (with Gabór Szabó)

Mikaelfest, June 20, 2024

I. Notation and definitions

A, B, C: C^*–algebras.
$B \leq A$: B is a C^*–subalgebra of A.
$\mathcal{D}$: separable strongly self-absorbing C^*–algebra.
$\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$.

Definition

A C^*–algebra A is called

1. *(tensorially) $\mathcal{D}$-absorbing* if $A \otimes \mathcal{D} \cong A$.
2. *locally (tensorially) $\mathcal{D}$-absorbing* if for every separable $B \leq A$ there is separable C, $B \leq C \leq A$, such that $C \otimes \mathcal{D} \cong C$.
3. *$\mathcal{D}$-saturated* if for every separable $C \leq A$ we have
   3.1 If A is unital, $\mathcal{D}$ embeds unitally into $A \cap C'$
   3.2 If A is not unital, $\mathcal{D}$ embeds unitally into $(A \cap C')/C^{\perp}$.

Some of my favourite things: Massive C^*–algebras

$$\text{(norm) ultrapower: } A_{\mathcal{U}} = \ell_\infty(A)/\{(a_n) \mid \lim_{n\to\mathcal{U}} \|a_n\| = 0\}$$

$$\text{asymptotic sequence algebra: } A_\infty = \ell_\infty(A)/c_0(A)$$

$$\text{path algebra: } A_{\blacktriangleright\!\blacktriangleright} = C_b([0,\infty))(A)/C_0([0,\infty))(A)$$

$$\text{corona: } \mathcal{Q}(A) = \mathcal{M}(A)/A.$$

Theorem (classical)

For every separable,~~separable~~ σ-unital A, TFAE:

1. *A is locally $\mathcal{D}$-absorbing.*
2. *$A_{\mathcal{U}}$ is $\mathcal{D}$-saturated.*
3. *A_∞ is $\mathcal{D}$-saturated.*
4. *(F., Szabó, 2022) $A_{\blacktriangleright\!\blacktriangleright}$ is $\mathcal{D}$-saturated.*

Proof by standard reflection arguments. Sketching a similar argument earned me a golden transparency some years ago and I am not doing it again in public.

My original motivation (for what follows)

Question (Sakai)

Assume A, B are simple, separable C^*–algebras. Does $\mathcal{Q}(A) \cong \mathcal{Q}(B)$ imply $A \cong B$?

Some related results.

1. L. Brown (1977): $\mathcal{M}(A) \cong \mathcal{M}(B)$ implies $A \cong B$, for A, B separable.
2. S. Ghasemi (2017): There are separable type I C^*–algebras A, B such that the assertion $\mathcal{Q}(A) \cong \mathcal{Q}(B)$ is independent from ZFC.
3. A. Vignati (2019): A positive answer to Sakai's question for stabilizations of unital C^*–algebras, using additional set-theoretic axioms.
4. F. (2022): $\mathcal{Q}(\mathcal{K}) \ncong \mathcal{Q}(\mathcal{K} \otimes A)$ for any separable, unital, $\mathcal{Z}$-absorbing A.

The main result

Theorem (F.–Szabó, 2024)

For every σ-unital A and every $\mathcal{D}$, TFAE:

1. *A is locally tensorially $\mathcal{D}$-absorbing.*
2. *$\mathcal{Q}(\mathcal{K} \otimes A)$ is locally tensorially $\mathcal{D}$-absorbing.*
3. *$\mathcal{Q}(\mathcal{K} \otimes A)_\infty$ is locally tensorially $\mathcal{D}$-absorbing.*
4. *$\mathcal{M}(A)$ is locally tensorially $\mathcal{D}$-absorbing.*
5. *$\mathcal{Q}(\mathcal{K} \otimes A)$ is $\mathcal{D}$-saturated.*
6. *$\mathcal{Q}(\mathcal{K} \otimes A)_\infty$ is $\mathcal{D}$-saturated.*
7. *$\mathcal{M}(A)_\infty$ is locally tensorially $\mathcal{D}$-absorbing.*
8. *$\mathcal{M}(A)_\infty$ is $\mathcal{D}$-saturated.*

By classical results one can add other equivalent statements.

II. Some remarks about the proof

For unital B, C, C is called *weakly B-saturated* if there is a unital *-homomorphism from B into $C \cap A'$, for every separable $A \leq C$.

Lemma

For all unital B, C, TFAE:

1. *C_∞ is weakly B-saturated.*
2. *$(C_\infty)_\infty$ is weakly B-saturated.*
3. *$(((C_\infty)_\infty)_\infty$ is weakly B-saturated.*
4. *etc. (can throw in an ultrapower as well).*

Proof: C_∞ is practically the same as $(C \otimes C(\text{Cantor set}))_{\mathcal{U}}$ (F., 2022).

Using a quotient of $\mathrm{C}^*(\mathrm{SL}_3(\mathbb{Z}))$ or such

Lemma

If A is a nonzero C^*–algebra, then there are a separable $C \leq \mathcal{Q}(\mathcal{K} \otimes A)$ and a u.c.p. map

$$\Psi\colon \mathcal{Q}(\mathcal{K} \otimes A) \to \mathcal{M}(A)_\infty$$

such that $\mathcal{Q}(\mathcal{K} \otimes A) \cap C'$ is included in the multiplicative domain of Ψ.

(F. 2023, following S. Wassermann: The case when $A = \mathbb{C}$.)

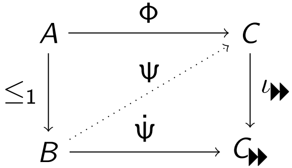
Corollary

If A is a nonzero C^*–algebra, then for every unital separable B (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

1. *$\mathcal{Q}(\mathcal{K} \otimes A)_\infty$ is weakly B-saturated.*
2. *$(\mathcal{M}(A)_\infty)_\infty$ is weakly B-saturated.*
3. *$\mathcal{M}(A)_\infty$ is weakly B-saturated.*

The path algebra, $C_{\blacktriangleright\!\blacktriangleright} = C_b([0,\infty))(C)/C_0([0,\infty))(C)$

A unital C has the *folding property* if for all separable unital $A \leq_1 B$, an injective unital $*$-homomorphism $\Phi\colon A \to C$ and a unital $*$-homomorphism $\dot\Psi\colon B \to C_{\blacktriangleright\!\blacktriangleright}$ such that $\iota_{\blacktriangleright\!\blacktriangleright} \circ \Phi = \dot\Psi \restriction A$ there is a unital $*$-homomorphism $\Psi\colon B \to C$ such that $\Phi = \Psi \restriction A$ and $\iota_{\blacktriangleright\!\blacktriangleright} \circ \Psi = \dot\Psi$.



Lemma (Manuilov–Thomsen, Phillips–Weaver, 2007)

The corona of every σ-unital C^*–algebra has the folding property.

Lemma

If C has the folding property, then

1. *(Gabe?) Every two unital copies of $\mathcal{D}$ in C are unitarily*

Lemma

If A is σ-unital, and localy tensorially $\mathcal{D}$-absorbing, then so is $\mathcal{M}(A)$.

Proof: The case when A is separable follows by Toms–Winter (preservation under extensions). Use reflection.

III. Applications

Corollary

The Calkin algebra is not elementarily equivalent to a nuclear C^*–algebra.

Proof: Every nuclear, purely infinite C^*–algebra is $\mathcal{O}_\infty$-stable.

Corollary (A very partial positive answer to Sakai's question)

If A and B are separable, unital C^*–algebras and there is $\mathcal{D}$ such that A is $\mathcal{D}$-absorbing and B is not, then $\mathcal{Q}(\mathcal{K} \otimes A) \ncong \mathcal{Q}(\mathcal{K} \otimes B)$.

Theorem

If A is σ-unital and locally $\mathcal{D}$-absorbing then so is $\mathcal{Q}(A)$.

(The converse is true if A has a full projection.)

The following is most interesting in case when H is the trace-kernel ideal and $\mathcal{D} = \mathcal{Z}$.

Corollary

Assume A is unital and locally $\mathcal{D}$-absorbing and H is a hereditary C^*–subalgebra of $A_{\mathcal{U}}$. Then H is an inductive limit of separable $E \leq H$ such that $\mathcal{Q}(E)$ is $\mathcal{D}$-absorbing.

Corollary

If A is σ-unital and locally tensorially $\mathcal{D}$-absorbing then $\mathcal{M}(A)$ has strict comparison.

Proof: $\mathcal{Z}$-stability implies strict comparison.

Corollary

The Calkin algebra is not isomorphic to the corona of a $\mathcal{Z}$-stable C^*–algebra.

(Conjecturally, it is not isomorphic to the corona of a separable simple C^*–algebra other than $\mathcal{K}$; true assuming forcing axioms (Vignati).)
For additional implications, ask Hannes.

Tillykke med fødselsdagen, Mikael!

Happy Birthday, Mikael!