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W*-superrigidity of group von Neumann algebras

Noncommutativity in the North – MikaelFest

Gothenburg – 17-20 June 2024

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Group von Neumann algebras

Definition

Let G be a discrete group. Define $u_g \in \mathcal{U}(\ell^2(G))$ by $u_g \delta_h = \delta_{gh}$. The **group von Neumann algebra** $L(G)$ is generated by $(u_g)_{g \in G}$.

- Unique tracial von Neumann algebra generated by unitaries $(u_g)_{g \in G}$ with $\tau(u_g) = 0$ for all $g \neq e$ and $u_g u_h = u_{gh}$.
- $L(G)$ is a factor if and only if G has infinite conjugacy classes (icc).

Flexibility

- Whenever G is an amenable icc group, $L(G) \cong R$.
- Whenever $G_1, \ldots, G_n$ are infinite amenable groups, $L(G_1 * \cdots * G_n) \cong L(\mathbb{F}_n)$.

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W*-superrigidity

Definition

A discrete group G is called **W*-superrigid** if the following holds: if Λ is **any** discrete group and $L(G) \cong L(\Lambda)$, then $G \cong \Lambda$. In other words: G can be recovered from $L(G)$.

Connes' rigidity conjecture

Lattices in higher rank simple Lie groups are W*-superrigid.

- (Ioana-Popa-V, 2010) First constructions of W*-superrigid groups as generalized wreath products $(\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes \Gamma$.
- (Berbec-V, 2012) W*-superrigidity for left-right wreath products $(\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ whenever Γ is a free group, or a hyperbolic group, or ...
- (Chifan-Ioana-Osin-Sun, 2021) First W*-superrigid groups **with property (T)**.

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A new degree of W*-superrigidity

Joint work with Milan Donvil (2024)

- Allow for **2-cocycle twists:** twisted group von Neumann algebras $L_\mu(G)$.
- Prove W*-superrigidity up to **virtual isomorphism:** from bifinite bimodules between group von Neumann algebras to virtual isomorphism between the groups.

Corollary: the first II_1 factors M for which no amplification M^t is a twisted group von Neumann algebra.

- (Connes 1975 and Jones 1979) II_1 factors M that cannot be written as $L(\Lambda)$.
- (Ioana 2010) II_1 factors M such that no pMp is of the form $L_\omega(\Lambda)$.

⇝ We first introduce some of these concepts.

Twisted group von Neumann algebras

Let G be a discrete group.

- A **projective representation** is a map $\pi : G \to \mathcal{U}(H)$ such that $\pi(g)\pi(h) \in \mathbb{T}\pi(gh)$.
- We get a **2-cocycle** $\mu : G \times G \to \mathbb{T} : \pi(g)\pi(h) = \mu(g,h)\pi(gh)$.
- Abelian group $H^2(G,\mathbb{T})$ of 2-cocycles modulo coboundaries $\mu(g,h) = \varphi(g)\varphi(h)\overline{\varphi(gh)}$.
- For every $\mu \in H^2(G,\mathbb{T})$, the regular $\lambda_\mu : G \to \mathcal{U}(\ell^2(G)) : \lambda_\mu(g)\delta_h = \mu(g,h)\delta_{gh}$.
- This generates $L_\mu(G)$, the **twisted group von Neumann algebra**.

Examples

- Every **bicharacter** $\mu : \Gamma \times \Gamma \to \mathbb{T}$ is also a 2-cocycle.
- We have $\mu \in H^2(\mathbb{Z}^2,\mathbb{T})$ by $\mu((a,b),(a',b')) = \exp(2\pi i\theta ab')$, with irrational θ. Then $L_\mu(\mathbb{Z}^2) \cong R$.

Cocycle W*-superrigidity

Class $\mathcal{C}$: nonamenable, weakly amenable, biexact groups in which the centralizer of a nontrivial element is amenable.

Examples: free groups, free products of amenable groups, torsion free hyperbolic groups.

Theorem (Donvil-V, 2024)

Let Γ be a discrete group in class $\mathcal{C}$. Consider $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$.

If Λ is **any** discrete group and $\mu \in H^2(G, \mathbb{T})$ and $\omega \in H^2(\Lambda, \mathbb{T})$ are **any** 2-cocycles such that $L_\mu(G) \cong L_\omega(\Lambda)$, then $(G, \mu) \cong (\Lambda, \omega)$.

This means: there exists an isomorphism $\delta : G \to \Lambda$ such that $\omega \circ \delta = \mu$ in $H^2(G, \mathbb{T})$.

Note: for these groups G, the cohomology $H^2(G, \mathbb{T})$ is always nontrivial.

Bifinite bimodules and virtual isomorphisms

Let M and P be II_1 factors.

- A **bifinite** M-P-bimodule is a Hilbert M-P-bimodule ${}_M H_P$ such that H is finitely generated as a left Hilbert M-module and finitely generated as a right Hilbert P-module.
- We say that M and P are **virtually isomorphic** if there exists a bifinite M-P-bimodule.
- The **amplification** M^t is defined as $p(M_n(\mathbb{C}) \otimes M)p$ where $(\mathrm{Tr} \otimes \tau)(p) = t$.
- Bifinite M-P-bimodule ${}_M H_P$ is the same as ${}_{\varphi(M)} p(\mathbb{C}^n \otimes L^2(P))_P$ where $\varphi : M \to P^t$ is a **finite index** embedding.

Virtual isomorphisms and (twisted) group von Neumann algebras

A first source of canonical virtual isomorphisms

- Discrete groups G and Λ are called **virtually isomorphic** if there exist finite index subgroups $G_0 < G$, $\Lambda_0 < \Lambda$ and finite normal subgroups $\Sigma \lhd G_0$, $\Sigma' \lhd \Lambda_0$ such that $G_0/\Sigma \cong \Lambda_0/\Sigma'$.
- Then there exists a nonzero bifinite $L(G)$-$L(\Lambda)$-bimodule.

A second source of canonical virtual isomorphisms

- A 2-cocycle $\mu \in H^2(G, \mathbb{T})$ is said to be **of finite type** if there is a finite-dimensional projective representation $\pi : G \to \mathcal{U}(d)$ with $\pi(g)\pi(h) = \mu(g,h)\pi(gh)$.
- Then $L_\mu(G) \to M_d(\mathbb{C}) \otimes L(G) : u_g \mapsto \pi(g) \otimes u_g$ defines a bifinite $L_\mu(G)$-$L(G)$-bimodule.

Virtual isomorphism W*-superrigidity

Theorem (Donvil-V, 2024)

Let Γ be a discrete group in class $\mathcal{C}$. Consider $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$.

Let Λ be **any** discrete group and $\mu \in H^2(G, \mathbb{T})$ and $\omega \in H^2(\Lambda, \mathbb{T})$ **any** 2-cocycles. Then the following are equivalent.

- There exists a nonzero bifinite $L_\mu(G)$-$L_\omega(\Lambda)$-bimodule.
- (G, μ) and (Λ, ω) are virtually isomorphic: there exist a finite index $\Lambda_0 < \Lambda$ and a group homomorphism $\delta : \Lambda_0 \to G$ such that $\operatorname{Ker} \delta$ is finite, $\delta(\Lambda_0) < G$ has finite index and $\omega|_{\Lambda_0} \cdot \overline{\mu \circ \delta}$ is of finite type.

⇝ Such a virtual isomorphism W*-superrigidity is also new without the 2-cocycle twists.

Decomposability as twisted group von Neumann algebra

Theorem (Donvil-V, 2024)

Let (A_0, τ_0) be a nontrivial amenable tracial von Neumann algebra. Let Γ be a discrete group in class $\mathcal{C}$.

Construct $(A, \tau) = \overline{\bigotimes}_{g \in \Gamma}(A_0, \tau_0)$. Define the II_1 factor $M = A \rtimes (\Gamma \times \Gamma)$.

Then the following are equivalent.

- There exists a $t > 0$, a discrete group Λ and $\omega \in H^2(\Lambda, \mathbb{T})$ with $M^t \cong L_\omega(\Lambda)$.
- There exists a discrete group Λ_0, $\omega_0 \in H^2(\Lambda_0, \mathbb{T})$ and a trace preserving $A_0 \cong L_{\omega_0}(\Lambda_0)$.

$\rightsquigarrow$ With $A_0 = \mathbb{C}^2$ and τ_0 not uniform, no amplification M^t is a twisted group von Neumann algebra.

Approach: comultiplications (Popa-V 2009, Ioana-Popa-V & Ioana 2010)

Let G be a "very specific" group, e.g. $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$. Write $M = L(G)$.

- If $M \cong L(\Lambda)$, generated by $(v_s)_{s\in\Lambda}$, we have: $\Delta : M \to M \mathbin{\overline{\otimes}} M : v_s \mapsto v_s \otimes v_s$ for $s \in \Lambda$.
- If $M \cong L_\omega(\Lambda)$, generated by $(v_s)_{s\in\Lambda}$, we have:
  $\Delta : M \to M \mathbin{\overline{\otimes}} M^{\mathrm{op}} \mathbin{\overline{\otimes}} M : v_s \mapsto v_s \otimes \overline{v_s} \otimes v_s$ for $s \in \Lambda$.
- If $P = L_\omega(\Lambda)$ and if ${}_M H_P$ is a bifinite bimodule, we have Δ as composition of

$$M \xrightarrow{\text{by } H} P^r \xrightarrow{\text{by } \Lambda} (P \mathbin{\overline{\otimes}} P^{\mathrm{op}} \mathbin{\overline{\otimes}} P)^r \xrightarrow{\text{by } \overline{H}} (M \mathbin{\overline{\otimes}} M^{\mathrm{op}} \mathbin{\overline{\otimes}} M)^t$$

⇝ In all cases, we obtain an embedding $\Delta : M \to (M_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} M_k)^t$ in which M and M_i are "specific and known", while Δ is unknown.

⇝ When also $M = L_\mu(G)$ is twisted, we first take
$L(G) \to L_\mu(G) \mathbin{\overline{\otimes}} L_\mu(G)^{\mathrm{op}} : u_g \to u_g \otimes \overline{u_g}$ and then proceed as above.

Analysis of comultiplication maps

Let $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$.

- Any (virtual) isomorphism of an $L_\mu(G)$ with an $L_\omega(\Lambda)$ gives rise to an embedding $\Delta : L(G) \to (M_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} M_k)^t$ where each M_i is $L_\mu(G)$ or $L_\mu(G)^{\mathrm{op}}$.
- It would be "asking too much" to describe all possible embeddings of $L(G)$ into $(M_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} M_k)^t$.
- But these comultiplication embeddings have certain qualitative properties.
- We classify, for $L(G)$ with Γ in $\mathcal{C}$, all embeddings with these qualitative properties.

Coarse tensor embeddings

Recall: a Hilbert M-P-bimodule ${}_M H_P$ is called coarse if ${}_M H_P$ is contained in a multiple of ${}_{M \otimes 1} L^2(M \overline{\otimes} P)_{1 \otimes P}$.

Definition (Donvil-V, 2024)

$\psi : M \to M_1 \overline{\otimes} \cdots \overline{\otimes} M_k$ is called a **coarse tensor embedding** if all the bimodules

${}_{\psi(M)} L^2(M_1 \overline{\otimes} \cdots \overline{\otimes} M_k)_{M_1 \overline{\otimes} \cdots \overline{\otimes} M_{i-1} \overline{\otimes} 1 \overline{\otimes} M_{i+1} \overline{\otimes} \cdots \overline{\otimes} M_k}$ are coarse.

- All tensor embeddings given by (virtual) isomorphisms of $L_\mu(G)$ and $L_\omega(\Lambda)$ are coarse.
- When M and M_i are such twisted left-right wreath products, we classify all coarse tensor embeddings.

Classifying coarse tensor embeddings

Take groups Γ_i in $\mathcal{C}$ and amenable (A_i, τ_i). Put $B_i = \overline{\otimes}_{g \in \Gamma_i}(A_i, \tau_i)$ and $M_i = B_i \rtimes (\Gamma_i \times \Gamma_i)$.

- (Popa-V, 2021) Classification of all embeddings $M_1 \to M_2^t$.
- (Donvil-V, 2024) Classif. of all coarse tensor embeddings $\psi : M_0 \to (M_1 \overline{\otimes} \cdots \overline{\otimes} M_k)^t$.
  - We have $\psi(B_0) \prec B_1 \overline{\otimes} \cdots \overline{\otimes} B_k$ because $\Gamma_i \in \mathcal{C}$ by (Popa-V, 2012).
  - We essentially have $\psi(L(\Gamma_0 \times e)) \prec L(\Gamma_1 \times e) \overline{\otimes} \cdots \overline{\otimes} L(\Gamma_k \times e)$ by methods of (Popa 2003 and Ozawa 2003).
  - Using height in group von Neumann algebras (next slide), we arrive at $\psi(u_{(g,h)}) \sim u_{\alpha_1(g,h)} \otimes \cdots \otimes u_{\alpha_k(g,h)}$ with $\alpha_i : \Gamma_0 \times \Gamma_0 \to \Gamma_i \times \Gamma_i$.
  - Only with $A_0 = \mathbb{C}^2$, a complete description of $\psi|_{B_0}$ follows.

Height in group von Neumann algebras

Consider $L(G)$, generated by $(u_g)_{g\in G}$.

- For $a \in L(G)$, define $h_G(a) = \max\{|\tau(u_g^* a)| \mid g \in G\}$. (Largest Fourier coefficient.)
- For a subgroup $\Lambda \subset \mathcal{U}(L(G))$, define $h_G(\Lambda) = \inf\{h_G(v) \mid v \in \Lambda\}$.

Theorem (Ioana-Popa-V, 2010)

Let G be an icc group and assume that $L(G) = L(\Lambda)$, with unitaries $(u_g)_{g\in G}$ and $(v_s)_{s\in\Lambda}$. Then the following are equivalent.

- $h_G(\Lambda) > 0$.
- There exists a $W \in \mathcal{U}(L(G))$, iso $\delta : \Lambda \to G$ and $\gamma : \Lambda \to \mathbb{T}$ s.t. $W v_s W^* = \gamma(s) u_{\delta(s)}$.

$\rightsquigarrow$ We prove and use generalizations to $L_\omega(\Lambda) \hookrightarrow L_\mu(G)$.