THE CUNTZ SEMIGROUP

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1. Introduction

Throughout, A, B will usually denote C^*-algebras, X will usually be a locally compact, Hausdorff space. If H is a Hilbert space, then $\mathbb{B}(H)$ denotes the C^*-algebra of bounded, linear operators on H. We let $\mathcal{K}$ denote the C^*-algebra of compact operators on an infinite-dimensional, separable Hilbert space.

1.1. **Recapitulation: The Murray-von Neumann semigroup.**

1.1. Given projections $p, q \in A$, recall that p is *Murray-von Neumann subequivalent* to q, denoted $p \precsim_{\mathrm{MvN}} q$, if there exists $v \in A$ with $p = vv^*$ and $v^*v \leq q$. We say that p and q are *Murray-von Neumann equivalent*, denoted $p \sim_{\mathrm{MvN}} q$, if there exists $v \in A$ with $p = vv^*$ and $v^*v = q$.

The *Murray-von Neumann semigroup* $V(A)$ of A is defined as the set of equivalence classes of projections in matrices over A. Every projection in $A \otimes \mathcal{K}$ is equivalent to a projection in some matrix over A. Therefore, we have $V(A) = \mathrm{Proj}(A \otimes \mathcal{K})/\sim_{\mathrm{MvN}}$. The class of a projection $p \in A \otimes \mathcal{K}$ in $V(A)$ is denoted by $[p]$.

Given projections $p \in A \otimes M_m$ and $q \in A \otimes M_n$, we consider $\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) \in A \otimes M_{m+n}$. This induces an addition on $V(A)$ by setting $[p] + [q] := \left[\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)\right]$, for p and q projections in matrices over A. We let 0 denote the class of the zero projection in $V(A)$. Then $(V(A), +, 0)$ is an abelian monoid.

We can endow $V(A)$ with the pre-order induced by $\precsim_{\mathrm{MvN}}$, that is, we set $[p] \leq [q]$ if and only if $p \precsim_{\mathrm{MvN}} q$. The pre-order $\leq$ is compatible with addition: If $[p_1] \leq [q_1]$ and $[p_2] \leq [q_2]$, then $[p_1] + [p_2] \leq [q_1] + [q_2]$. Moreover, we have $[0] \leq [p]$ for all p.

1.2. Let M be an abelian monoid with a partial order (a pre-order) $\leq$. We call M a *positively ordered monoid* (a *positively pre-ordered monoid*) if the order is compatible with addition, that is, $s_1 \leq t_1$ and $s_2 \leq t_2$ imply $s_1 + s_2 \leq t_1 + t_2$, and if every element is positive, that is, $0 \leq s$ for every $s \in M$.

If M is an abelian monoid, we define the *algebraic pre-order* $\leq_{\mathrm{alg}}$ on M by setting $s \leq_{\mathrm{alg}} t$ if and only if there exists $x \in M$ with $s + x = t$. Then M together with $\leq_{\mathrm{alg}}$ is a positively pre-ordered monoid, but in general $\leq_{\mathrm{alg}}$ is not a partial order.

We say that monoid with a (pre-)order $\leq$ is *algebraically (pre-)ordered* if $\leq = \leq_{\mathrm{alg}}$.

Recall that A is called *finite* if every projection in A is finite, that is, if $q \precsim p \leq q$ implies $p = q$ for all projections $p, q \in A$. If A is unital, then A is finite if and only if every isometry in A is a unitary, that is, if $v^*v = 1$ implies $vv^* = 1$, for every $v \in A$. We say that A is *stably finite* if $A \otimes M_n$ is finite for every $n \geq 1$, or equivalently, if $A \otimes \mathcal{K}$ is finite.

Proposition 1.3. *The Murray-von Neumann semigroup $V(A)$ is algebraically pre-ordered. (It is thus a positively pre-ordered monoid.) Moreover, $V(A)$ is partially ordered (and hence a positively ordered monoid) if and only if A is stably finite.*

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1.4. If A is unital, then $K_0(A)$ is the Grothendieck group of $V(A)$. The elements of $K_0(A)$ can be considered as formal differences $[p]-[q]$, for projections $p,q \in A\otimes\mathcal{K}$. We usually consider $K_0(A)$ not only as an abelian group, but also with its natural (pre-)order defined by the positive cone given by the elements in $K_0(A)$ that come from $V(A)$.

If $V(A)$ has cancellation (that is, if $s+x\leq t+x$ implies $s\leq t$, for $s,t,x\in V(A)$), then $V(A)$ is naturally isomorphic to the positive cone of $K_0(A)$ and can thus be recovered from $K_0(A)$. For instance, if A has stable rank one, then $V(A)$ has cancellation.

1.2. Motivation: K-theory vs. Cuntz semigroup.

K-theory is easy to compute:

- Six-term exact sequence for extensions.
- Pimsner-Voiculescu exact sequence for crossed products by $\mathbb{Z}$ and $\mathbb{R}$.

K-theory may contain very little information about A:

- It does not record the ideal structure, simplicity, or topological dimension (of the primitive ideal space): $K_*(C([0,1]^n))$ does not depend on n.
- K-theory is not even enough to distinguish 'very nice' simple C^*-algebras. (One needs also information about traces.)

The problem is that a C^*-algebra may contain very few (if any) projections. Naturally, if there are very few projections, then $V(A)$ and $K_0(A)$ contain little information about A. The idea of Cuntz was to consider positive elements instead of projections and to build an ordered semigroup $\mathrm{Cu}(A)$ out of $(A\otimes\mathcal{K})_+$ analogous to how $V(A)$ is constructed from $\mathrm{Proj}(A\otimes\mathcal{K})$; see Definition 2.4. The advantage is that every C^*-algebra contains an abundance of positive elements!

The Cuntz semigroup contains much more information than $V(A)$ or $K_0(A)$. The following properties of A are encoded in its Cuntz semigroup $\mathrm{Cu}(A)$:

- The lattice of ideals of A.
- The space of (quasi)traces of A.
- The Cuntz semigroup of all ideals and quotients of A.

The drawback is that the Cuntz semigroup is in general much more difficult to compute.

1.3. Motivation: Applications of the Cuntz semigroup.

1.3.1. *Existence of (quasi)traces on stably finite, simple C^*-algebras.* The Cuntz semigroup was introduced by Cuntz in 1978, [Cun78], in order to prove the existence of dimension functions and (quasi)traces on stably finite, unital, simple C^*-algebras. Let A be a unital, simple C^*-algebra. Recall that a tracial state on A is a state $\tau\colon A\to\mathbb{C}$ that satisfies $\tau(xx^*)=\tau(x^*x)$ for all $x\in A$. If there exists a tracial state on A, then A is finite: Indeed, let $v\in A$ satisfy $v^*v=1$. Consider the projection $q:=1-vv^*$. Then

$$\tau(q)=\tau(1-vv^*)=1-\tau(vv^*)=1-\tau(v^*v)=0.$$

Since A is simple, every tracial state on A is faithful, and thus $\tau(q)=0$ implies that $q=0$, as desired.

More generally, if τ is a tracial state on A, then $\tau\otimes\mathrm{tr}_n$ is a tracial state on $A\otimes M_n$. Thus, if A is a unital, simple C^*-algebra with a tracial state, then A is stably finite. It is natural to ask whether the converse holds. This is known to be the case under the additional assumption that A is exact. The proof proceeds in two steps. First, it was shown by Cuntz that a unital, simple C^*-algebra is stably finite if and only it has a 2-quasitracial state. This uses the Cuntz semigroup, since - as we will see later in detail - 2-quasitracial states on A correspond to so-called functionals on $\mathrm{Cu}(A)$. Second, it was shown by Haagerup that every 2-quasitracial state on an exact C^*-algebra is already a tracial state.

1.3.2. *The Elliott classification program.* In 1976, Elliott classified AF-algebras (C^*-algebras that are inductive limits of finite-dimensional C^*-algebras) in terms of their K-theory: If A and B are separable AF-algebras, then $A \cong B$ if and only if $K_0(A) \cong K_0(B)$, as scaled, ordered groups. This was subsequently generalized to the classification of (simple) C^*-algebras that are inductive limits of more general building blocks. To obtain these more general classifications, the K_0-group is not sufficient and thus the invariant had to be enlarged: Given a (simple) C^*-algebra A, its *Elliott invariant*, denoted $\mathrm{Ell}(A)$, is the tuple consisting of the scaled, ordered K_0-group $K_0(A)$, the K_1-group $K_1(A)$, the trace simplex $T(A)$, and the pairing between $T(A)$ and $K_0(A)$ given by a natural map $r_A \colon T(A) \to \mathrm{St}(K_0(A))$.

The successful classification results lead Elliott to formulate his famous conjecture:

Conjecture 1.5 (Elliott conjecture)**.** Let A, B be unital, separable, simple, nonelementary, nuclear C^*-algebras. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

This conjecture was a driving force for decades of intensive research on the classification of simple, nuclear C^*-algebras, known as the *Elliott classification program.* The conjecture was eventually disproved by Rørdam and shortly after by Toms.

Theorem 1.6 (Toms, [Tom08], Counterexample to the Elliott conjecture)**.** *There exist unital, separable, simple, nonelementary, nuclear C^*-algebras A and B such that* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$*, yet* $\mathrm{Cu}(A) \not\cong \mathrm{Cu}(B)$ *and therefore $A \not\cong B$.*

This is a crucial instance where the Cuntz semigroup records strictly more information about a C^*-algebra than just K-theoretic and tracial data. It proved the Cuntz semigroup to be a sensitive invariant and a powerful tool.

After the setback to the classification program, it was proposed by Toms and Winter that one has to assume additional regularity properties to obtain classification. They considered three properties that are very different in spirit. They conjectured that these properties are equivalent ('the regularity conjecture', often called the 'Toms-Winter conjecture') and that these properties ensure classification by the Elliott invariant ('the adjusted Elliott conjecture').

Conjecture 1.7 (Toms-Winter conjecture)**.** Let A be a separable, simple, nonelementary, nuclear C^*-algebra. Then the following are equivalent:

(1) A has finite nuclear dimension.
(2) A is $\mathcal{Z}$-stable, that is, A tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$, $A \cong A \otimes \mathcal{Z}$.
(3) $\mathrm{Cu}(A)$ is almost unperforated.

There has been substantial progress on verifying this conjecture for large classes of C^*-algebras.

Additionally, we have the following interesting result:

Theorem 1.8 (Winter, [Win12, Corollary 7.4])**.** *Let A be a separable, simple, nonelementary, unital C^*-algebra with locally finite nuclear dimension. Then $A \cong A \otimes \mathcal{Z}$ and only if* $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Z})$*.*

While the Toms-Winter conjecture has not been settled completely, the classification has been obtained by an astounding recent tour-de-force by Elliott-Gong-Lin-Niu, culminating the work over decades involving dozens of people.

Theorem 1.9 (..., Kirchberg-Phillips, Elliott-Gong-Lin-Niu, Tikuisis-White-Winter)**.** *Let A, B be unital, separable, simple, nonelementary C^*-algebras with finite nuclear dimension. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$*.*

1.3.3. *Construction of *-homomorphisms.* A remarkable result of Robert allows us to lift maps between Cuntz semigroups to *-homomorphisms:

Theorem 1.10 (Robert, [Rob12, Theorem 1.0.1])**.** *Let A and B be separable, unital C^*-algebras with stable rank one, and let $\alpha\colon$* $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$ *be a* Cu*-morphism with $\alpha([1_A]) = [1_B]$.*

Assume that A is an inductive limit of one-dimensional NCCW-complexes with trivial K_1-groups (for example, an AF-algebra, or $C([0,1])$, or the Jiang-Su algebra $\mathcal{Z}$). Then there exists a unital *-homomorphism $\varphi\colon A \to B$ (unique up to approximate unitary equivalence) such that $\mathrm{Cu}(\varphi) = \alpha$.

Corollary 1.11 (Robert, [Rob12, Corollary 5.2.3])**.** *Let A and B be unital, separable C^*-algebras that are inductive limits of one-dimensional NCCW-complexes with trivial K_1-groups. Then $A \cong B$ if and only if* $(\mathrm{Cu}(A), [1_A]) \cong (\mathrm{Cu}(B), [1_B])$.

It follows from Robert's results that the Jiang-Su algebra $\mathcal{Z}$ embeds unitally in the reduced free group C^*-algebra $C^*_\lambda(\mathbb{F}_\infty)$. Indeed, one can show that $\mathcal{Z}$ and $C^*_\lambda(\mathbb{F}_\infty)$ have isomorphic Cuntz semigroups. Then, the identity map $\mathrm{Cu}(\mathcal{Z}) \to \mathrm{Cu}(C^*_\lambda(\mathbb{F}_\infty))$ lifts to a unital *-homomorphism $\mathcal{Z} \to C^*_\lambda(\mathbb{F}_\infty)$. This is the only known proof of this (nontrivial!) result.

1.4. **Overview of material, literature.** The specific material covered in this class can be found here:

- [APT14]: Antoine, Perera, Thiel. *Tensor products and regularity properties of Cuntz semigroups*, arXiv:1410.0483 (2014).
- [APT11]: Ara, Perera, Toms. *K-theory for operator algebras. Classification of C^*-algebras*, Contemp. Math. 543 (2011), arXiv:0902.3381 (2009).

Books with basic material on C^*-algebras and lattice theory:

- [GHK+03]: Gierz, Hofmann, Keimel, Lawson, Mislove, Scott. *Continuous lattices and domains*, CUP 2003.
- [Bla06]: Blackadar. *Operator algebras*, Springer 2006.
- [Ped79]: Pedersen. *C^*-algebras and their automorphism groups*, AP 1979.

2. The Cuntz semigroup via positive elements

2.1. **Definition and first properties.** We define the Cuntz semigroup of a C^*-algebra A as a set of equivalence classes of positive elements in the stabilization $A \otimes \mathcal{K}$.

Definition 2.1. Let $a, b \in A_+$. We say that a is *Cuntz subequivalent* to b, denoted $a \precsim b$, if there exists a sequence $(r_n)_n$ in A with $a = \lim_n r_n b r_n^*$. We say that a is *Cuntz equivalent* to b, denoted $a \sim b$, if $a \precsim b$ and $b \precsim a$.

Remark 2.2. We have $a \precsim b$ if and only if for every $\varepsilon > 0$ there exists $r \in A$ with $\|a - rbr^*\| \leq \varepsilon$.

Lemma 2.3 (Exercise)**.** *(1) We have $a \sim a$ for every $a \in A_+$.*
(2) If $a, b, c \in A_+$ satisfy $a \precsim b \precsim c$, then $a \precsim c$.

It follows that $\sim$ is an equivalence relation on A_+, which justifies the following definition.

Definition 2.4. The *Cuntz semigroup* of A is defined as $\mathrm{Cu}(A) := (A \otimes \mathcal{K})_+ / \sim$. We denote the class of $a \in (A \otimes \mathcal{K})_+$ in $\mathrm{Cu}(A)$ by $[a]$.

Remark 2.5. We define a binary relation $\leq$ on $\mathrm{Cu}(A)$ by setting $[a] \leq [b]$, for $a, b \in A_+$, if and only if $a \precsim b$. It follows from Lemma 2.3 that $\leq$ is a well-defined partial order. The class of the zero element $0 \in A_+$ is the minimal element with respect to the partial order on $\mathrm{Cu}(A)$.

In Paragraph 2.22, we will define an addition on $\mathrm{Cu}(A)$.

Recall that the *support* of a function $f \in C_0(X)$ is defined as $\mathrm{supp}(f) := \{x \in X : f(x) \neq 0\}$.

Proposition 2.6. *Let X be a locally compact, Hausdorff space, and let $f, g \in C_0(X)_+$. Then $f \precsim g$ if and only if* $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$.

Proof. To show the forward implication, assume $f \precsim g$. Choose a sequence $(r_n)_n$ in A with $f = \lim_n r_n g r_n^*$. Let $x \notin \mathrm{supp}(g)$. Then $g(x) = 0$, and therefore

$$f(x) = \lim_n r_n(x) g(x) \overline{r_n(x)} = \lim_n 0 = 0.$$

Thus, $f(x) = 0$, and hence $x \notin \mathrm{supp}(f)$. This shows that $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$, as desired.

To show the backward implication, assume $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$. Let $\varepsilon > 0$. We will find $r \in C_0(X)$ such that $\|f - rgr^*\| \leq \varepsilon$. Set $K := \{x \in X : f(x) \geq \varepsilon\}$. Then K is a compact set with $K \subseteq \mathrm{supp}(g)$. Since g is continuous and strictly positive on K, there is $\delta > 0$ with $g(x) \geq \delta$ for all $x \in K$. Set $U := \{x \in X : g(x) > \frac{\delta}{2}\}$. Then U is open and $K \subseteq U$. By Urysohn's lemma, there is a continuous function $h\colon X \to [0,1]$ that takes the value 1 on K and vanishes outside U. Define $s\colon X \to \mathbb{R}_+$ by

$$s(x) = \begin{cases} \frac{f(x)}{g(x)}h(x), & \text{if } x \in U \\ 0, & \text{otherwise.} \end{cases}$$

Then $s \in C_0(X)$, and $\|f - sg\| \leq \frac{1}{n}$. Thus $r := s^{1/2}$ has the desired properties. ☐

We denote the spectrum of an operator a by $\sigma(a)$.

Corollary 2.7. *(1) Let $a \in A_+$ and let $f\colon [0,\infty) \to [0,\infty)$ be a continuous function with $f(0) = 0$. Then $f(a) \precsim a$. If, moreover, $f(t) > 0$ for all $t \in \sigma(a) \setminus \{0\}$, then $f(a) \sim a$.*

In particular, $a \sim a^t$ for every $t \in (0,\infty)$, and $a \sim \lambda a$ for every $\lambda \in (0,\infty)$.

(2) We have $xx^* \sim x^*x$, for every $x \in A$.

Proof. (1). Let $f\colon [0,\infty) \to [0,\infty)$ be a continuous function with $f(0) = 0$. The continuous functional calculus gives a natural isomorphism $C^*(a) \cong C_0(\sigma(a))$. Under this isomorphism, a and $f(a)$ correspond to the restriction of the functions id and f to $\sigma(a)$, respectively. We have $\mathrm{supp}(f_{|\sigma(a)}) \subseteq \sigma(a) \setminus \{0\} = \mathrm{supp}(\mathrm{id}_{|\sigma(a)})$, and therefore $f_{|\sigma(a)} \precsim \mathrm{id}_{|\sigma(a)}$ in $C_0(\sigma(a))$ by Proposition 2.6. Thus, $f(a) \precsim a$ in $C^*(a)$, and therefore also in A.

If $f(t) > 0$ for all $t \in \sigma(a) \setminus \{0\}$, then $\mathrm{supp}(f_{|\sigma(a)}) = \mathrm{supp}(\mathrm{id}_{|\sigma(a)})$, and consequently $f(a) \sim a$ in $C^*(a)$, and therefore also in A.

(2). Using (1) at the first step, we deduce that

$$xx^* = xx^*xx^* = x(x^*x)x^* \precsim x^*x.$$

Analogously, one obtains that $x^*x \precsim xx^*$. ☐

2.8 (Polar decomposition in $\mathbb{B}(H)$ and A^{**})**.** Let $x \in \mathbb{B}(H)$. Recall that the *left support projection* of x, denoted q_x, is the orthogonal projection onto the closed subspace generated by the range of x. The *right support projection* of x, denoted p_x, is the orthogonal projection onto the complement of the kernel of x. Note that q_x and p_x are the smallest orthogonal projections satisfying $q_x x = x$ and $xp_x = x$, respectively. It is easy to check that $p_x = p_{x^*x}$, and $q_x = q_{xx^*}$, and $p_{x^*} = q_x$. (Exercise.) If a is self-adjoint, then $q_a = p_a$, which we simply call the *support projection* of a.

Recall that $|x|$ is defined as $|x| := (x^*x)^{1/2}$. (This definition makes sense in any C^*-algebra.) There exists a unique partial isometry $v_x \in \mathbb{B}(H)$ with $v_x^*v_x = p_x$ and $v_xv_x^* = q_x$, and such that $x = v_x|x|$. The decomposition $x = v_x|x|$ is called the *left polar decomposition* of x. We also have $x = |x^*|v_x$, which is called the *right polar decomposition* of x.

One can show that p_x, q_x, v_x and $|x|$ all belong to the von Neumann algebra generated by x. It follows that in a von Neumann algebra, every operator has a polar decomposition. In particular, every $x \in A$ has a polar decomposition in A^{**}: There exists a unique partial isometry $v_x \in A^{**}$ with $x = v_x|x|$, $x = |x^*|v_x$, $v_x^*v_x = p_x$ and $v_xv_x^* = q_x$.

2.9 (Well-supported elements)**.** Recall that $a \in A_+$ is said to be *well-supported* if $\sigma(a) \setminus \{0\}$ is closed. In other words, 0 is an isolated point in the spectrum, and thus there exists $\varepsilon > 0$ with $\sigma(a) \subseteq \{0\} \cup [\varepsilon,\infty)$.

Assume $a \in A_+$ is well-supported. We claim that the support projection of a belongs to A. Indeed, the characteristic function of $(0,\infty)$, denoted $\mathbb{1}_{(0,\infty)}$, is continuous on $\sigma(a)$. We have $p_a = \mathbb{1}_{(0,\infty)}(a)$, which belongs to A using continuous functional calculus. Moreover, applying Corollary 2.7, we obtain:

Proposition 2.10. *Let $a \in A_+$ be well-supported. Then $a \sim p_a$.*

For elements in a general C^*-algebra we only have a weaker form of the polar decomposition; see Proposition 2.13. To prove it, we first need the following generalization of the polar decomposition for operators in $\mathbb{B}(H)$; compare [Bla06, Proposition I.5.2.4, p.22].

Proposition 2.11. *Let $x, y \in B(H)$ with $x^*x \leq y^*y$. Then there exists a unique $w \in B(H)$ such that $x = wy$ and $p_w \leq q_y$. Moreover, w is contractive.*

Proof. Exercise. □

The proof of the following result is standard. It can for instance be found in the proof of [Bla06, Proposition III.5.2.16, p.321].

Lemma 2.12. *Let B be a C^*-algebra, and let $A \subseteq B$ be a sub-C^*-algebra. Let $v \in B$, and let $a \in A_+$ such that $va \in A$. Then $vf(a) \in A$, for every continuous function $f\colon \mathbb{R} \to \mathbb{C}$ with $f(0) = 0$.*

Proof. We first show that $vp(a)$ belongs to A for every polynomial p with vanishing constant term. Indeed, given such a polynomial p, there exists a polynomial p_0 such that $p(x) = xp_0(x)$. Note that the constant term of p_0 need not vanish. Therefore, we do not necessarily have $p_0(a) \in A$. However, we may assume that B is unital, and we may extend the inclusion $A \subseteq B$ to an inclusion of $\widetilde{A} \subseteq B$, where $\widetilde{A}$ denotes the minimal unitization of A. Then $p_0(a)$ belongs to $\widetilde{A}$. We have $vp(a) = vap_0(a)$. Since va belongs to A, and since $p_0(a)$ belongs to $\widetilde{A}$, we deduce that their product belongs to A.

Let $f \in \mathbb{R} \to \mathbb{C}$ be a continuous function with $f(0) = 0$. By Stone-Weierstrass, we can choose a sequence of polynomials $(p_n)_n$ that converges to f uniformly on $[0, \|a\|]$. Since $f(0) = 0$, we may assume that each p_n has vanishing constant term. Then $vp_n(a)$ belongs to A for every n. We have

$$vf(a) = \lim_n vp_n(a),$$

which implies that $vf(a)$ belongs to A, as desired. □

The next result provides the polar decomposition of an element in a general C^*-algebra; compare [Bla06, Proposition II.3.2.1, p.67].

Proposition 2.13. *Let $x \in A$, let $a \in A_+$, and let $t \in (0, 1)$. If $x^*x \leq a^2$, then there exists $r \in A$ such that $x = ra^t$. In particular, $x = r|x|^t$ for some $r \in A$.*

Proof. Represent A on a Hilbert space H. Then $x^*x \leq a^*a$ in $\mathbb{B}(H)$. By Proposition 2.11, there exists $w \in B(H)$ such that $x = wa$. Set $r := wa^{1-t}$. Then $x = ra^t$. By Lemma 2.12, we have $r \in A$, as desired.

The last statement follows by applying the result for $a = |x|2$. □

Lemma 2.14. *Let $x, y \in A$. Then $y \in \overline{x^*A}$ if and only if $y = \lim_n |x|^{1/n}y$. Analogously, $y \in \overline{Ax}$ if and only if $y = \lim_n y|x|^{1/n}$. Further, $y \in \overline{x^*Ax}$ if and only if $y = \lim_n |x|^{1/n}y|x|^{1/n}$.*

Proof. Let $x = v|x|$ be the polar decomposition of x in A^{**}. Then $x^* = |x|v^*$. We deduce that

$$\lim_n |x|^{1/n}x^* = \lim_n |x|^{1/n}|x|v^* = |x|v^* = x^*.$$

It follows that $y = \lim_n |x|^{1/n}y$ for every $y \in x^*A$, and consequently also for every $y \in \overline{x^*A}$ using an $\frac{\varepsilon}{3}$-type argument.

For the converse direction, it is enough to show that $|x|^{1/n} \in \overline{x^*A}$ for every $n \geq 1$. It is easy to check that $B := \overline{x^*Ax}$ is a sub-C^*-algebra of A. We have $x^*x \in B$, and therefore $|x|^{1/n} = (x^*x)^{1/2n} \in B \subseteq \overline{x^*A}$, as desired.

The second and third statements are shown analogously. (The second also follow from the first by taking adjoints.) □

Recall that a sub-C^*-algebra $B \subseteq A$ is called *hereditary* if for all $a, b \in A_+$ with $a \leq b \in B$ we have $a \in B$. Given $x \in A$, the next result shows that $\overline{x^*Ax}$ is the hereditary sub-C^*-algebra generated by x^*x; compare [Bla06, Proposition II.3.4.2, p.75].

Proposition 2.15. *Let $x \in A$. Then $\overline{x^*Ax} = \overline{|x|A|x|}$, and $\overline{x^*Ax}$ is a hereditary sub-C^*-algebra of A.*

Proof. Given $y \in A$, it is easy to see that $y \in \overline{|x|A|x|}$ if and only if $y = \lim_n |x|^{1/n}y|x|^{1/n}$. Using Lemma 2.14 we deduce $\overline{x^*Ax} = \overline{|x|A|x|}$.

To show that $\overline{x^*Ax}$ is hereditary, let $a, b \in A_+$ with $a \leq b$ and $b \in \overline{x^*Ax}$. By Proposition 2.13, there exists $r \in A$ such that $a^{1/2} = rb^{1/4}$. Then $a = b^{1/4}r^*rb^{1/4}$. We have $b^{1/4} \in \overline{x^*Ax}$, and therefore $b^{1/4} = \lim_n |x|^{1/n}b^{1/4} = \lim_n b^{1/4}|x|^{1/n}$ by Lemma 2.14. It follows that

$$\lim_n |x|^{1/n}a|x|^{1/n} = \lim_n |x|^{1/n}b^{1/4}r^*rb^{1/4}|x|^{1/n} = b^{1/4}r^*rb^{1/4} = a,$$

which shows that $a \in \overline{x^*Ax}$, as desired. □

Notation 2.16. Given $a \in A_+$, we let A_a denote the hereditary sub-C^*-algebra of A generated by a, that is $A_a := \overline{aAa}$.

Proposition 2.17. *Let $a, b \in A_+$. If $a \in A_b$, then $a \precsim b$. In particular, if $a \leq b$, then $a \precsim b$.*

Proof. Assume that $a \in A_b$. To show that $a \precsim b$, let $\varepsilon > 0$. By Lemma 2.14, we have $a = \lim_n b^{1/n}ab^{1/n}$. Thus, we may choose n such that $\|a - b^{1/n}ab^{1/n}\| \leq \frac{\varepsilon}{2}$. Using Corollary 2.7 at the second and fourth step, we deduce that

$$b^{1/n}ab^{1/n} = (b^{1/n}a^{1/2})(a^{1/2}b^{1/n}) \sim a^{1/2}b^{2/n}a^{1/2} \precsim b^{2/n} \sim b.$$

Thus, there exists $r \in A$ with $\|b^{1/n}ab^{1/n} - rbr^*\| \leq \frac{\varepsilon}{2}$. It follows that

$$\|a - rbr^*\| \leq \|a - b^{1/n}ab^{1/n}\| + \|b^{1/n}ab^{1/n} - rbr^*\| \leq \varepsilon,$$

as desired. □

The next result shows that Cuntz comparison does not change when passing to a hereditary sub-C^*-algebra.

Proposition 2.18. *Let $B \subseteq A$ be a hereditary sub-C^*-algebra, and let $a, b \in B_+$. Then $a \precsim b$ with respect to B if and only if $a \precsim b$ with respect to A.*

Proof. The forward implication is obvious. To show the backward implication, let $\varepsilon > 0$, and let $r \in A$ such that $\|a - rbr^*\| \leq \varepsilon$. Using that B is a hereditary sub-C^*-algebra, first choose $e \in B$ contractive such that $\|a - eae^*\| \leq \varepsilon$. Then choose $f \in B$ contractive such that $\|b - fbf^*\| \leq \|er\|^{-2}\varepsilon$. Set $s := erf$. Then $s \in B$ and

$$\begin{aligned}
\|a - sbs^*\| &= \|a - erfbf^*r^*e^*\| \\
&\leq \|a - eae^*\| + \|eae^* - erbr^*e^*\| + \|erbr^*e^* - erfbf^*r^*e^*\| \\
&\leq \|a - eae^*\| + \|e\|\|a - rbr^*\|\|e^*\| + \|er\|\|b - fbf^*\|\|r^*e^*\| \leq 3\varepsilon,
\end{aligned}$$

as desired. □

Given a unital C^*-algebra A, we let $\mathrm{Gl}(A)$ and $\mathcal{U}(A)$ denote the invertible and unitary elements in A, respectively. Recall that two elements a, b in a (unital) C^*-algebra A are said to be *unitarily equivalent*, denoted $a \sim_{\mathrm{u}} b$, if there exists $u \in \mathcal{U}(A)$ with $a = ubu^*$. Further, a and b are said to be *approximately unitarily equivalent*, denoted $a \sim_{\mathrm{au}} b$ if there exists a sequence $(u_n)_n$ in $\mathcal{U}(A)$ such that $a = \lim_n u_nbu_n^*$.

Lemma 2.19. *Let A be unital, and let $a, b \in A_+$. Then $a \sim_u b$ implies $a \sim_{au} b$, and $a \sim_{au} b$ implies $a \sim b$.*

Proof. Exercise. □

The next result generalizes this to the case that A is nonunital and the unitaries are from the minimal unitalization $\widetilde{A}$, or the multiplier algebra $M(A)$.

Corollary 2.20. *Let $a, b \in A_+$, and assume a is approximately unitarily equivalent to b with unitaries from $M(A)$. Then $a \sim b$ in A.*

Proof. Assume $a \sim_{\mathrm{au}} b$ in $M(A)$. Then a is Cuntz equivalent to b in $M(A)$, by Lemma 2.19. Since A is an ideal in $M(A)$, and thus in particular a hereditary sub-C^*-algebra of $M(A)$, it follows from Proposition 2.18 that a is Cuntz equivalent to b in A. □

We now proceed to define an addition on the Cuntz semigroup $\mathrm{Cu}(A)$. Given $a, b \in A_+$, we use $a \oplus b$ to denote the element $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ in $A \otimes M_2$.

Lemma 2.21. *(1) If $a_1 \precsim b_1$ and $a_2 \precsim b_2$ in A, then $(a_1 \oplus a_2) \precsim (b_1 \oplus b_2)$ in $A \otimes M_2$.*
(2) If $a, b \in A_+$, then $(a \oplus b) \sim (b \oplus a)$ in $A \otimes M_2$.

Proof. Exercise. □

2.22. Choose a *-isomorphism $\psi\colon \mathcal{K} \otimes M_2 \to \mathcal{K}$. This isomorphism is unique up to unitary equivalence. More precisely, for every automorphism $\varphi\colon \mathcal{K} \to \mathcal{K}$, there exists a unitary $u \in \mathbb{B}(H)$ such that $\varphi(a) = uau^*$ for every $a \in \mathcal{K}$. Thus, if $\psi_1, \psi_2\colon \mathcal{K} \otimes M_2 \to \mathcal{K}$ are two *-isomorphisms, then there exists a unitary $u \in \mathbb{B}(H)$ such that $\psi_1(a) = u\psi_2(a)u^*$ for every $a \in \mathcal{K} \otimes M_2$.

Fix a *-isomorphism $\psi\colon \mathcal{K} \otimes M_2 \to \mathcal{K}$. We obtain a *-isomorphism $\mathrm{id}_A \otimes \psi\colon A \otimes \mathcal{K} \otimes M_2 \to A \otimes \mathcal{K}$. Given $a, b \in (A \otimes \mathcal{K})_+$, we consider $a \oplus b := \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ in $A \otimes \mathcal{K} \otimes M_2$ and we define

$$[a] + [b] := [(\mathrm{id}_A \otimes \psi)(a \oplus b)].$$

First, note that this definition is independent of the choice of ψ. Indeed, given another *-isomorphism $\psi'\colon \mathcal{K} \otimes M_2 \to \mathcal{K}$, by the above remarks there is a unitary $u \in \mathbb{B}(H)$ with $\psi(\_) = u\psi'(\_)u^*$. It follows that $(\mathrm{id}_A \otimes \psi)(a \oplus b)$ and $(\mathrm{id}_A \otimes \psi')(a \oplus b)$ are unitarily equivalent, via the unitary $1_{\widetilde{A}} \otimes u$ in $M(A \otimes \mathcal{K})$. By Corollary 2.20, we obtain $[(\mathrm{id}_A \otimes \psi)(a \oplus b)] = [(\mathrm{id}_A \otimes \psi')(a \oplus b)]$.

It follows from statements (1) and (2) in Lemma 2.21 that the definition of $[a] + [b]$ is independent of the representatives a and b, and that $[a] + [b] = [b] + [a]$. Thus, we have defined an abelian operation (an addition) on $\mathrm{Cu}(A)$. We denote the class of $0 \in A_+$ in $\mathrm{Cu}(A)$ by 0. As the notation suggests, 0 is a zero element for the addition on $\mathrm{Cu}(A)$, that is, we have $s + 0 = s$ for every $s \in \mathrm{Cu}(A)$.

Even more, the order and addition on $\mathrm{Cu}(A)$ are compatible in the following sense: If $s_1, s_2, t_1, t_2 \in \mathrm{Cu}(S)$ satisfy $s_1 \leq t_1$ and $s_2 \leq t_2$, then $s_1 + s_2 \leq t_1 + t_2$.

Proposition 2.23. *The Cuntz semigroup $\mathrm{Cu}(A)$ is a positively ordered monoid.*

Recall that $M(A)$ denotes the multiplier algebra of A. If A is unital, then $A = M(A)$. Otherwise, $M(A)$ is a unital C^*-algebra containing A is a closed, two-sided ideal. The statement and proof of the following result are from [APT11, Lemma 2.21].

Lemma 2.24. *Let $a \in A_+$, and let p be a projection in $M(A)$. Then*

$$a \precsim pap + (1-p)a(1-p).$$

Proof. Since A is an ideal in $M(A)$, it is enough to show the Cuntz subequvialence in $M(A)$, by Proposition 2.18. Set $s = p-(1-p)$. The $a \leq a + sas$, and hence $a \precsim a + sas$ by Proposition 2.17. Then

$$\begin{aligned} a \precsim a + sas &= [p + (1-p)]a[p + (1-p)] + [p - (1-p)]a[p - (1-p)] \\ &= 2(pap + (1-p)a(1-p)) \\ &\sim pap + (1-p)a(1-p), \end{aligned}$$

as desired. □

Recall that two elements x, y in a C^*-algebra are said to be *orthogonal*, denoted $x \perp y$, if $xy = x^*y = xy^* = x^*y^* = 0$. If x and y are self-adjoint, then x and y are orthogonal if and only if $xy = 0$.

Lemma 2.25. *Let $a, b \in A_+$. Then $a + b \precsim a \oplus b$ in $A \otimes M_2$. If $a \perp b$, then $a + b \sim a \oplus b$ in $A \otimes M_2$.*

Proof. As usual, we embed A into the upper-left corner of $A \otimes M_2$. Set $x := \left(\begin{smallmatrix} a^{1/2} & b^{1/2} \\ 0 & 0 \end{smallmatrix}\right)$ in $A \otimes M_2$. Using Corollary 2.7 at the third step, we obtain that

$$a+b = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix} = xx^* \sim x^*x = \begin{pmatrix} a & b^{1/2}a^{1/2} \\ a^{1/2}b^{1/2} & b \end{pmatrix}.$$

Next, we show that $x^*x \precsim a \oplus b$. We consider the projection $p := 1_{M(A)} \otimes e_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $M(A) \otimes M_2$. Then $1-p = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Applying Lemma 2.24, we deduce that

$$x^*x \precsim px^*xp + (1-p)x^*x(1-p) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a \oplus b.$$

If $a \perp b$, then for x as above, we obtain that

$$a+b = x^*x \sim xx^* = \begin{pmatrix} a & b^{1/2}a^{1/2} \\ a^{1/2}b^{1/2} & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a \oplus b,$$

as desired. □

Proposition 2.26. *Let $x \in A$, and let $x = v|x|$ be the polar decomposition in A^{**}. Then:*

1. *If $z \in \overline{x^*A}$, then $vz \in \overline{xA}$ and $v^*vz = z$.*
2. *If $z \in \overline{Ax}$, then $zv^* \in \overline{Ax^*}$ and $zv^*v = z$.*
3. *If $z \in \overline{x^*Ax}$, then $vzv^* \in \overline{xAx^*}$ and $v^*vzv^*v = z$.*

Proof. Claim: For each $n \geq 1$, the element $v|x|^{1/n}$ belongs to $\overline{xA}$. To prove the claim, let n be given. Choose a sequence $(f_k)_k$ of continuous functions $f_k \colon [0,\infty) \to [0,\infty)$ with $f_k(0) = 0$ such that $|x|^{1/n} = \lim_k |x|f_k(|x|)$. Using that $v|x| = |x^*|v$ at the second step, we obtain

$$v|x|^{1/n} = \lim_k v|x|f_k(|x|) = \lim_k |x^*|vf_k(|x|).$$

By Lemma 2.12, we have $vf_k(|x|) \in A$ for each k, and therefore $v|x|^{1/n} \in \overline{|x^*|A}$. Applying Proposition 2.15, we have $\overline{|x^*|A} = \overline{xA}$, and consequently $v|x|^{1/n} \in \overline{xA}$.

(1). Let $z \in \overline{x^*A}$. By Lemma 2.14, we have $z = \lim_n |x|^{1/n}z$. Thus, $vz = \lim_n v|x|^{1/n}z$. Using the claim, we deduce that vz belongs to $\overline{xA}$.

To show $v^*vz = z$, choose a sequence $(a_n)_n$ in A with $z = \lim_n x^*a_n$. We have $v^*vx^* = x^*$. (Since v^*v is the left support projection of x^*.) Thus $v^*vx^*a_n = x^*a_n$ for each n, and passing to the limit we obtain the desired equality.

(2). This follows from (1) by passing to adjoints.

(3). Let $z \in \overline{x^*Ax}$. Since $\overline{x^*Ax}$ is a C^*-algebra, we can choose $z_1, z_2 \in \overline{x^*Ax}$ such that $z = z_1z_2$. We have $z_1 \in \overline{x^*Ax} \subseteq \overline{x^*A}$. Therefore, statement (1) shows that vz_1 belongs to $\overline{xA}$ and that $v^*vz_1 = z_1$. Analogously, we obtain that z_2v^* belongs to $\overline{Ax^*}$ and that $z_2v^*v = z_2$. We have $vzv^* = (vz_1)(z_2v^*)$, which implies that vzv^* belongs to $\overline{xAx^*}$. Moreover, we have

$$v^*vzv^*v = (v^*vz_1)(z_2v^*v) = z_1z_2 = z,$$

as desired. □

If $a, b \in A$ are self-adjoint elements satisfying $a \leq b$, then $xax^* \leq xbx^*$ for every $x \in A$. Indeed, an element $d \in A$ is positive if and only if there exists $y \in A$ with $d = yy^*$. By definition, $a \leq b$ means that $b - a \geq 0$. Thus, there exists $y \in A$ with $b - a = yy^*$. Then

$$xbx^* - xax^* = x(b-a)x^* = xyy^*x^* = xy(xy)^*,$$

which shows that $xax^* \leq xbx^*$, as claimed. This observation will be used throughout.

2.27. Let $\varepsilon > 0$, and let $a, b \in A_+$ with $\|a - b\| < \varepsilon$. We claim that $(a-\varepsilon)_+ \precsim b$. Indeed, the assumption implies that $a - \varepsilon \leq b$. Multiplying on both sides by $(a-\varepsilon)_+$, we obtain that $(a-\varepsilon)_+(a-\varepsilon)(a-\varepsilon)_+ \leq (a-\varepsilon)_+b(a-\varepsilon)_+$. Using this at the third step, and using Corollary 2.7 at the first step, we obtain that

$$(a-\varepsilon)_+ \sim (a-\varepsilon)_+^3 = (a-\varepsilon)_+(a-\varepsilon)(a-\varepsilon)_+ \leq (a-\varepsilon)_+b(a-\varepsilon)_+ \precsim b,$$

as desired.

Thus, there exists a sequence $(r_n)_n$ in A such that $(a-\varepsilon)_+ = \lim_n r_n b r_n^*$. In Lemma 2.29 we substantially improve this result.

Lemma 2.28. *Let $a \in A_+$ and $y \in A$ satisfy $a \leq y^*y$. Then there exists $x \in A$ such that $a = x^*x$ and $xx^* \leq yy^*$.*

Proof. By Proposition 2.15, we have $a \in \overline{y^*Ay}$. Let $y = v|y|$ be the polar decomposition of y in A^{**}. Set $x := va^{1/2}$. Therefore, by Proposition 2.26, we have $x \in A$ and $x^*x = a^{1/2}a^{1/2} = a$. Further,

$$xx^* = vav^* \leq vy^*yv^* = yy^*,$$

as desired. □

Lemma 2.29. *Let $\varepsilon > 0$, and let $a, b \in A_+$ with $\|a - b\| < \varepsilon$. Then there exists $r \in A$ such that $(a-\varepsilon)_+ = rbr^*$*

Proof. We proceed in three steps.

1. Choose $s > 1$ (close enough to 1) such that $\|a - b^s\| < \varepsilon$. Set $b_0 := b^s$.

Choose ε_0 such that $\|a - b_0\| < \varepsilon_0 < \varepsilon$. Let $f\colon \mathbb{R}_+ \to [0,1]$ be given by

$$f(t) = \begin{cases} \left(\frac{t-\varepsilon}{t-\varepsilon_0}\right)^{1/2}, & \text{if } t \geq \varepsilon, \\ 0, & \text{otherwise} \end{cases}.$$

Set $e := f(a)$. Then $\|e\| \leq 1$. Moreover, we have $(a-\varepsilon)_+ = e(a-\varepsilon_0)e$ and thus $(a-\varepsilon)_+ \leq eb_0e$.

2. Set $y := b_0^{1/2}e$. Then $(a-\varepsilon)_+ \leq eb_0e = y^*y$. By Lemma 2.28, there exists $x \in A$ such that $(a-\varepsilon)_+ = x^*x$ and $xx^* \leq yy^*$.

3. Using that e is contractive at the third step, we obtain that

$$xx^* \leq yy^* = b_0^{1/2}e^2b_0^{1/2} \leq b_0 \leq b^s.$$

By Proposition 2.13, there exists $r \in A$ such that $x^* = rb^{1/2}$. Then

$$(a-\varepsilon)_+ = x^*x = rb^{1/2}b^{1/2}r^* = rbr^*,$$

as desired. □

Theorem 2.30 (Rørdam's lemma)**.** *Let A be a C^*-algebra, and let $a, b \in A_+$. Then the following are equivalent:*

(1) We have $a \precsim b$.
(2) For every $\varepsilon > 0$, we have $(a-\varepsilon)_+ \precsim b$.
(3) For every $\varepsilon > 0$, there is $\delta > 0$ such that $(a-\varepsilon)_+ \precsim (b-\delta)_+$.
(4) For every $\varepsilon > 0$, there are $\delta > 0$ and $x \in A$ such that $(a-\varepsilon)_+ = x^*x$ and $xx^* \in A_{(b-\delta)_+}$.
(5) For every $\varepsilon > 0$, there are $\delta > 0$ and $r \in A$ such that $(a-\varepsilon)_+ = r(b-\delta)_+r^*$.

Proof. **1.** We show that (2) implies (1). To verify $a \precsim b$, let $\varepsilon > 0$. By assumption, we have $(a-\frac{\varepsilon}{2})_+ \precsim b$. Choose $r \in A$ with $\|(a-\frac{\varepsilon}{2})_+ - rbr^*\| < \frac{\varepsilon}{2}$. We have $\|a - (a-\frac{\varepsilon}{2})_+\| \leq \frac{\varepsilon}{2}$ and therefore

$$\|a - rbr^*\| \leq \|a - (a-\tfrac{\varepsilon}{2})_+\| + \|(a-\tfrac{\varepsilon}{2})_+ - rbr^*\| \leq \varepsilon,$$

as desired.

2. We show that (3) implies (2). For every $\delta > 0$, we have $(b-\delta)_+ \leq b$, and therefore $(b-\delta)_+ \precsim b$ by Proposition 2.17. Thus, (3) implies (2).

3. We show that (4) implies (3). Let $\varepsilon > 0$. By assumption, there is $\delta > 0$ and $x \in A$ such that $(a-\varepsilon)_+ = x^*x$ and $xx^* \in A_{(b-\delta)_+}$. Using Corollary 2.7 at the second step, and using Proposition 2.17 at the third step, we obtain that

$$(a-\varepsilon)_+ = x^*x \sim xx^* \precsim (b-\delta)_+,$$

as desired.

4. We show that (5) implies (4). Let $\varepsilon > 0$. By assumption, there is $\delta > 0$ and $r \in A$ such that $(a-\varepsilon)_+ = r(b-\delta)_+r^*$. Set $x := (b-\delta)_+^{1/2}r^*$. Then $(a-\varepsilon)_+ = x^*x$ and $xx^* = (b-\delta)_+^{1/2}r^*(b-\delta)_+^{1/2}$, which belongs to $A_{(b-\delta)_+}$, as desired.

5. We show that (1) implies (5). Let $\varepsilon > 0$. Choose $c \in A$ with $\|a - cbc^*\| < \varepsilon$. Choose $\delta > 0$ such that $\|a - c(b-\delta)_+c^*\| < \varepsilon$. By Lemma 2.29, there exists $d \in A$ such that $(a-\varepsilon)_+ = dc(b-\delta)_+c^*d^*$. Then $r := dc$ has the desired properties. □

2.2. **Cuntz comparison with projections.** As a first application of Rørdam's lemma, we obtain:

Proposition 2.31. *Let $p \in A$ be a projection, and let $b \in A_+$. Then $p \precsim b$ if and only if there exists a partial isometry $v \in A$ such that $p = v^*v$ and $vv^* \in A_b$.*

Proof. If there exists $v \in A$ with $p = vv^*$ and $v^*v \in A_b$, then $p \precsim b$ by Corollary 2.7 and Proposition 2.17. Let us show the converse. Using condition (4) of Rørdam's lemma for $\varepsilon = \frac{1}{2}$, we can choose $x \in A$ such that $(p - \frac{1}{2})_+ = x^*x$ and $xx^* \in A_b$. Note that $(p - \frac{1}{2})_+ = \frac{1}{2}p$. Therefore, $v := \sqrt{2}x$ has the desired properties. □

Proposition 2.32. *Let $p, q \in A$ be projections. Then $p \precsim_{MvN} q$ if and only if $p \precsim q$.*

Proof. It is clear that $p \precsim_{\mathrm{MvN}} q$ implies $p \precsim q$. Conversely, assume $p \precsim q$. By Proposition 2.31, there exists a partial isometry $v \in A$ with $p = v^*v$ and $vv^* \in A_q$. Since q is a projection, we have $A_q = qAq$, and therefore $vv^* = qvv^*q$. This implies $vv^* \leq q$, as desired. □

For every C^*-algebra A, the inclusion $\mathrm{Proj}(A \otimes \mathcal{K}) \subseteq (A \otimes \mathcal{K})+$ induces a natural map

$$V(A) \to \mathrm{Cu}(A),$$

sending the Murray-von Neumann equivalence class of projection to its Cuntz equivalence class. This map is easily seen to be a morphism of positively pre-ordered monoids, that is, it preserves addition, order and the zero element.

Recall that a map $f\colon S \to T$ between partially ordered sets is called an *order-embedding* if for all $x, y \in S$ we have $x \leq y$ if and only if $f(x) \leq f(y)$. It is easy to see that an order-embedding is injective.

Corollary 2.33. *If A is stably finite, then the natural map $V(A) \to \mathrm{Cu}(A)$ is an injective order-embedding.*

Proof. Since A is stably finite, we have that $V(A)$ is partially ordered. (See exercises.) It follows directly from Proposition 2.32 that the map $V(A) \to \mathrm{Cu}(A)$ is an order-embedding. □

Remarks 2.34. (1) If A is not stably finite, then the map $V(A) \to \mathrm{Cu}(A)$ need not be injective. Consider for example $\mathcal{O}_\infty$, the Cuntz algebra with infinitely many generators. (We will study such algebras in more detail later.) One can show that $V(\mathcal{O}_\infty) \cong \{0\} \sqcup \mathbb{Z}$, the disjoint union of 0 (the class of the zero projection) and a copy of $\mathbb{Z}$ whose elements we denote by $0', 1', -1', 2', \ldots$ for clarity. The addition in $\mathbb{Z}$ is the usual, and 0 is the zero element of the monoid. Note that $0 \neq 0'$ and $0 + 0' = 0'$. The order is degenerate: We have $0 \leq n'$ for every $n \in \mathbb{Z}$, but also $n' \leq k'$ for every $n, k \in \mathbb{Z}$.

On the other hand, we will see that $\mathrm{Cu}(\mathcal{O}_\infty) \cong \{0, \infty\}$. The map $V(\mathcal{O}_\infty) \to \mathrm{Cu}(\mathcal{O}_\infty)$ is identified with the map $\{0\} \sqcup \mathbb{Z} \to \{0, \infty\}$ that sends every nonzero element to ∞.

(2) Conversely, the map $V(A) \to \mathrm{Cu}(A)$ may be injective even if A is not stably finite. Consider for example $\mathcal{O}_2$, the Cuntz algebra with two generators. Then $V(\mathcal{O}_\infty) \cong \{0\} \sqcup \{0'\}$ and $\mathrm{Cu}(\mathcal{O}_2) \cong \{0, \infty\}$. The map $V(\mathcal{O}_2) \to \mathrm{Cu}(\mathcal{O}_2)$ sends 0 to 0 and $0'$ to ∞. It is therefore even bijective.

Example 2.35. We compute the Cuntz semigroup of $\mathcal{K}$. Set $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\} = \{0, 1, 2, \ldots, \infty\}$. For $a \in \mathcal{K}_+$, let $\mathrm{rk}(a) \in \overline{\mathbb{N}}$ denote the rank of a, that is, the Hilbert space dimension of the range of a. Given $a, b \in \mathcal{K}_+$, we will show that $a \precsim b$ if and only if $\mathrm{rk}(a) \leq \mathrm{rk}(b)$. It follows that $\mathrm{Cu}(\mathcal{K}) \cong \overline{\mathbb{N}}$.

Let $a \in \mathcal{K}_+$. By the spectral theorem for self-adjoint compact operators, there exist pairwise orthogonal finite-rank projections p_k, for $k \geq 1$, and a decreasing sequence $(\lambda_k)_k$ (of Eigenvalues of a) in $[0, \infty)$ such that $a = \sum_{k=1}^{\infty} \lambda_k p_k$, and such that either $(\lambda_k)_k$ is strictly decreasing with limit 0, or such that $(\lambda_k)_k$ is eventually 0. Accordingly, we distinguish two cases:

(a) If $(\lambda_k)_k$ is eventually zero then a is well-supported. Let $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ be the nonzero Eigenvalues of a. Then $a = \sum_{k=1}^{n} \lambda_k p_k$, and a is Cuntz equivalent to its support projection, which is $p_a = \sum_{k=1}^{n} p_k$. Moreover, $\mathrm{rk}(a) = \mathrm{rk}(p_a) = \sum_{k=1}^{n} \mathrm{rk}(p_k)$ is finite.

(b) If $(\lambda_k)_k$ is strictly decreasing, then a is not well-supported. Indeed, 0 is not isolated in $\sigma(a)$ since $(\lambda_k)_k$ converges to 0. In this case, we have $\mathrm{rk}(a) = \infty$.

For finite-rank projections $p, q \in \mathcal{K}$, it is well-known that $p \precsim_{\mathrm{MvN}} q$ if and only if $\mathrm{rk}(p) \leq \mathrm{rk}(q)$. Consequently, if $a, b \in \mathcal{K}_+$ are well-supported, then

$$a \precsim b \quad \Leftrightarrow \quad p_a \precsim p_b \quad \Leftrightarrow \quad \mathrm{rk}(p_a) \leq \mathrm{rk}(p_b) \quad \Leftrightarrow \quad \mathrm{rk}(a) \leq \mathrm{rk}(b),$$

as desired.

Next, let $a, b \in \mathcal{K}_+$, and assume that b is not well-supported. We claim that $a \precsim b$. By Rørdam's lemma, it is enough to show that $(a - \varepsilon)_+ \precsim b$ for every $\varepsilon > 0$. Let $\varepsilon > 0$. Using the spectral theorem for a, we obtain that $(a - \varepsilon)_+$ is well-supported and that the support projection of $(a - \varepsilon)_+$ has finite rank. The assumption on b implies that A_b contains projections of arbitrarily high rank. It follows that $(a - \varepsilon)_+ \precsim b$, as desired.

Thus, any two non-well-supported elements in $\mathcal{K}_+$ are Cuntz equivalent. Moreover, there exists such an element: Let $(e_k)_{k \geq 1}$ be an orthonormal basis for H. For each k, let q_k be the rank-one projection onto the space spanned by e_k. Then $a := \sum_{k=1}^{\infty} \frac{1}{k} q_k$ is a positive compact operator with spectrum $\{0\} \cup \{\frac{1}{k} : k \geq 1\}$.

It follows that the map $\mathrm{rk}\colon \mathcal{K}_+ \to \overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ induces a bijection between the Cuntz equivalence classes of elements in $\mathcal{K}_+$ and $\overline{\mathbb{N}}$. Moreover, rk is order-preserving and additive on orthogonal elements.

2.3. **Blackadar (sub)equivalence.** Given $a \in A_+$, recall that A_a denotes the hereditary sub-C^*-algebra of A generated by a. It will be convenient to introduce some notation.

Notation 2.36. Let $a, b \in A_+$.

- We write $a \approx b$ if there exists $x \in A$ with $a = x^*x$ and $b = xx^*$. (This is the direct translation of Murray-von Neumann equivalence to positive elements.)
- We write $a \subseteq b$ if $A_a \subseteq A_b$.
- We write $a \cong b$ if $A_a = A_b$.

Let $a \in A_+$. Recall that a is called *strictly positive* (in A) if $A = A_a$, that is, if a generates A as a hereditary sub-C^*-algebra.

Given $a, b \in A_+$, we have $a \in A_b$ if and only if $A_a \subseteq A_b$. Moreover, an element $b \in (A_a)_+$ satisfies $A_a = A_b$ if and only if b is strictly positive in A_a.

Lemma 2.37. *Let $x \in A$. Let $x = v|x|$ be the polar decomposition of x in A^{**}. Then, for each $z \in A_{x^*x}$, the element vzv^* belongs to A_{xx^*} and the map $A_{x^*x} \to A_{xx^*}$ given by $z \mapsto vzv^*$ is a *-isomorphism, with inverse $A_{xx^*} \to A_{x^*x}$ given by $z \mapsto v^*zv$.*

Proof. We have $A_{x^*x} = \overline{x^*Ax}$ and $A_{xx^*} = \overline{xAx^*}$. Thus, given $z \in \overline{x^*Ax}$, it follows from Proposition 2.26 that vzv^* belongs to $\overline{xAx^*}$. Thus, we obtain a well-defined map $\varphi\colon A_{x^*x} \to A_{xx^*}$ by setting $\varphi(z) := vzv^*$ for $z \in A_{x^*x}$. Clearly, φ is linear and *-preserving. To show that

φ is multiplicative, let $z_1, z_2 \in A_{x^*x}$. By Proposition 2.26 we have $z_1 v^* v = z_1$. Using this at the second step, we obtain

$$\varphi(z_1)\varphi(z_2) = vz_1v^*vz_2v^* = vz_1z_2v^* = \varphi(z_1z_2),$$

as desired. Thus, φ is a *-homomorphism.

By Proposition 2.26, we have $v^*vzv^*v = z$ for every $z \in \overline{x^*Ax}$. This implies that φ is bijective and thus a *-isomorphism, with inverse map given by $z \mapsto v^*zv$. □

Following [ORT11, Definition 2.1], we define:

Definition 2.38. Let $a, b \in A_+$. We say that a and b are *Blackadar equivalent*, denoted $a \sim_s b$, if there exists $x \in A$ such that $a \cong x^*x$ and $xx^* \cong b$. We say that a is *Blackadar subequivalent* to b, denoted $a \precsim_s b$ if $a \sim_s a' \subseteq b$ for some a'.

Lemma 2.39 (Ortega, Rørdam, T, [ORT11, Lemma 4.2])**.** *Let $a, b \in A_+$. Then the following are equivalent:*

(1) We have $a \sim_s b$.
(2) We have $a \approx a' \cong b$, for some a'.
(3) We have $a \cong b' \approx b$, for some b'.

Proof. It is clear that (2) implies (1), and that (3) implies (1). To prove that (1) implies (2), let $x \in A$ satisfy $a \cong x^*x$ and $xx^* \cong b$. Let $x = v|x|$ be the polar decomposition of x in A^{**}. Since $a^{1/2}$ belongs to $\overline{x^*A}$, it follows from Proposition 2.26 that $va^{1/2}$ belongs to A and $v^*va^{1/2} = a^{1/2}$. Set $y := va^{1/2}$. Then

$$y^*y = a^{1/2}v^*va^{1/2} = a^{1/2}a^{1/2} = a.$$

Let us show $yy^* \cong b$. By Lemma 2.37, we have a *-isomorphism $\varphi\colon A_{x^*x} \to A_{xx^*}$ given by $\varphi(z) := vzv^*$ for $z \in A_{x^*x}$. Note that a is a strictly positive element in A_{x^*x}. Since φ is a *-isomorphism, $\varphi(a)$ is strictly positive in A_{xx^*} and therefore $\varphi(a) \cong b$. We have

$$yy^* = va^{1/2}a^{1/2}v^* = vav^* = \varphi(a),$$

and therefore $yy^* \cong b$, as desired.

Analogously, by switching the role of a and b, one shows that (1) implies (3). □

Corollary 2.40. *Let $a, b \in A_+$. Then the following are equivalent:*

(1) We have $a \precsim_s b$.
(2) We have $a \approx a' \subseteq b$, for some a'.

Lemma 2.41. *Let $a, b, c, d \in A_+$. Then:*

(1) If $a \sim_s b \sim_s c$, then $a \sim_s c$.
(2) If $a \subseteq b \precsim_s c \subseteq d$, then $a \precsim_s d$.
(3) If $a \precsim_s b \precsim_s c$, then $a \precsim_s c$.

Proof. Exercise. □

Proposition 2.42. *Blackadar equivalence $\sim_s$ is an equivalence relation on A_+. Further, Blackadar subequivalence $\precsim_s$ is a reflexive, transitive relation on A_+.*

The next result shows that Blackadar and Cuntz (sub)equivalence are closely connected.

Theorem 2.43. *Let $a, b \in A_+$. Then $a \sim_s b$ implies $a \sim b$. Further, $a \precsim_s b$ implies $a \precsim b$. Moreover, Rørdam's Lemma shows that the following are equivalent:*

(1) We have $a \precsim b$.
(2) For every $\varepsilon > 0$ we have $(a-\varepsilon)_+ \precsim_s b$.
(3) For every $\varepsilon > 0$ there is $\delta > 0$ such that $(a-\varepsilon)_+ \precsim_s (b-\delta)_+$.

Proof. Exercise. □

2.4. **Unitary implementation of Cuntz comparison.** If A is a unital C^*-algebra, then we let $\mathrm{Gl}(A)$ and $\mathcal{U}(A)$ denote the invertible and unitary elements in A, respectively. We let $\widetilde{A}$ denote the minimal unitization of A. If A is unital, then $\widetilde{A}=A$. Given $a,b\in A_+$, recall that a and b are *unitarily equivalent*, denoted $a\sim_{\mathrm{u}} b$, if there exists $u\in\mathcal{U}(\widetilde{A})$ with $b=uau^*$; further, a and b are *approximately unitarily equivalent*, denoted $a\sim_{\mathrm{au}} b$, if there exists a sequence $(u_n)_n$ in $\mathcal{U}(\widetilde{A})$ such that $b=\lim_n u_nau_n^*$.

In this subsection, we study the relationship between Cuntz (sub)equivalence and approximate unitary equivalence of positive elements. Let $a,b\in A_+$. Then $a\sim_{\mathrm{u}} b$ implies $a\sim_{\mathrm{au}} b$. By Lemma 2.19, $a\sim_{\mathrm{au}} b$ implies $a\sim b$. It is easy to see that $a\sim_{\mathrm{u}} b$ implies $a\approx b$, and we have shown in Corollary 2.7 that $a\approx b$ implies $a\sim b$. Furthermore, in Corollary 2.53 we show that $a\approx b$ implies $a\sim_{\mathrm{au}} b$ if A has weak stable rank one (see Definition 2.50). Every stable C^*-algebra has weak stable rank one; see Lemma 2.51. We therefore have the following implications:

$$\begin{array}{ccccc} a\sim_{\mathrm{u}} b & & \Longrightarrow & & a\sim_{\mathrm{au}} b \\ \Downarrow & & \nearrow & & \Downarrow \\ a\approx b & \Longrightarrow & a\sim_s b & \Longrightarrow & a\sim b. \end{array}$$

Of course, we cannot expect that $a\sim b$ implies $a\sim_{\mathrm{au}} b$ for all elements in a (nonzero) C^*-algebra. For instance, $a\sim_{\mathrm{au}} b$ implies $\|a\|=\|b\|$, while for every nonzero $a\in A_+$ we have $a\sim 2a$ but $\|a\|\neq\|2a\|$. For the same reason, we cannot expect that $a\sim_s b$ implies $a\approx b$.

It will be convenient to introduce the following relation:

Notation 2.44. Let $a,b\in A_+$. We write $a\subseteq_u b$ if there exists $u\in\mathcal{U}(\widetilde{A})$ such that $uau^*\subseteq b$.

Remark 2.45. We have $a\subseteq_u b$ if and only if $a\sim_{\mathrm{u}} a'\subseteq b$ for some a'. This shows that we have the following implications:

$$a\subseteq_u b\Longrightarrow a\precsim_s b\Longrightarrow a\precsim b.$$

By Theorem 2.43, we have $a\precsim b$ if and only if $(a-\varepsilon)_+\precsim_s b$ for every $\varepsilon>0$. We will consider the analog condition with $\subseteq_u$ in place of $\precsim_s$:

Definition 2.46. Let A be a C^*-algebra. We say that Cuntz comparison in A is *unitarily implemented* if $a,b\in A_+$ satisfy $a\precsim b$ if and only if for every $\varepsilon>0$ we have $(a-\varepsilon)_+\subseteq_u b$.

Proposition 2.47. *Let $a,b\in A$. Assume that Cuntz comparison in A is unitarily implemented. Then the following are equivalent:*

1. *We have $a\precsim b$.*
2. *For every $\varepsilon>0$ we have $(a-\varepsilon)_+\subseteq_u b$.*
3. *For every $\varepsilon>0$ there exists $\delta>0$ such that $(a-\varepsilon)_+\subseteq_u (b-\delta)_+$.*

Proof. It is clear that (3) implies (2). By Rørdams's Lemma, (2) implies (1). To show that (1) implies (3), let $\varepsilon>0$. Using Rørdams's Lemma for $a\precsim b$ and $\frac{\varepsilon}{2}$, there exists $\delta>0$ such that $(a-\frac{\varepsilon}{2})_+\precsim(b-\delta)_+$. Using that Cuntz comparison in A is unitarily implemented, we obtain

$$\big((a-\tfrac{\varepsilon}{2})_+-\tfrac{\varepsilon}{2}\big)_+\subseteq_u (b-\delta)_+.$$

The conclusion follows using that the left hand side agrees with $(a-\varepsilon)_+$. □

Our goal in this section is to show that Cuntz comparison in stable C^*-algebras is unitarily implemented; see Corollary 2.56.

Definition 2.48. A unital C^*-algebra A is said to have *stable rank one*, denoted $\mathrm{sr}(A)=1$, if the set of invertible elements of A is dense. We say that a nonunital C^*-algebra A has stable rank one if $\widetilde{A}$ has stable rank one.

Lemma 2.49. *We collect some permanence properties of stable rank one:*

1. *Given a compact, Hausdorff space X, we have* $\mathrm{sr}(C(X))=1$ *if and only if* $\dim(X)\leq 1$.

(2) If $\mathrm{sr}(A) = 1$*, then* $\mathrm{sr}(A \otimes M_n) = 1$ *for each* $n \geq 1$*.*
(3) If $A = A_1 \oplus A_2$ *and* $\mathrm{sr}(A_1) = \mathrm{sr}(A_2) = 1$*, then* $\mathrm{sr}(A) = 1$*.*
(4) If $J \lhd A$ *is a closed ideal and* $\mathrm{sr}(A) = 1$*, then* $\mathrm{sr}(J) = \mathrm{sr}(A/J) = 1$*.*
(5) If $A = \varinjlim_j A_j$ *is an inductive limit and* $\mathrm{sr}(A_j) = 1$ *for all* j*, then* $\mathrm{sr}(A) = 1$*.*
(6) We have $\mathrm{sr}(A) = 1$ *if and only if* $\mathrm{sr}(A \otimes \mathcal{K}) = 1$*.*
(7) If $B \subseteq A$ *is a hereditary sub-*C^**-algebra and* $\mathrm{sr}(A) = 1$*, then* $\mathrm{sr}(B) = 1$*.*

Note that $\mathrm{sr}(A) = 1$ if and only if $\widetilde{A} \subseteq \overline{\mathrm{Gl}(\widetilde{A})}$. We will frequently consider the following weaker condition:

Definition 2.50. We say that A has *weak stable rank one* if $A \subseteq \overline{\mathrm{Gl}(\widetilde{A})}$.

If A is unital, then A has weak stable rank one if and only if $\mathrm{sr}(A) = 1$. However, for nonunital A the seemingly minor change makes a big difference: In Lemma 2.51 we show that every stable C^*-algebra has weak stable rank one. This implies that in every stable C^*-algebra, Cuntz comparison is unitarily implemented (see Theorem 2.55), which ultimately is the main ingredient for our proof that the Cuntz semigroup has suprema of increasing sequences; see Theorem 2.60.

In Lemma 2.49, we saw that a C^*-algebra has stable rank one if and only if its stabilization does. Thus, for a unital C^*-algebra A that does not have stable rank one (for example, $A = C([0,1]^2)))$ we have that $B := A \otimes \mathcal{K}$ has weak stable rank one, but not stable rank one, that is, $\overline{\mathrm{Gl}(\widetilde{B})}$ contains B but not $\widetilde{B}$. This also shows that weak stable rank is not as well behaved as stable rank one since it enjoys fewer permanence properties.

The following result and its proof appeared in [BRT+12, Lemma 4.3.2].

Lemma 2.51. *Let* A *be a stable* C^**-algebra. Then* A *has* A *has weak stable rank one, that is,* $A \subseteq \overline{\mathrm{Gl}(\widetilde{A})}$*.*

Proof. We may assume that $A = B \otimes \mathcal{K}$ for some C^*-algebra B. We consider the usual upper-left corner embeddings $B \otimes M_m \subseteq B \otimes M_n \subseteq B \otimes \mathcal{K}$ for $m \leq n$.

1. We claim that it is enough to show that for every n, we have $B \otimes M_n \subseteq \overline{\mathrm{Gl}(\widetilde{A})}$. Indeed, this follows easily since $\bigcup_n B \otimes M_n$ is dense in $B \otimes \mathcal{K}$.

2. Let $n \geq 1$, and let $x \in B \otimes M_n$. We claim that x is the product of two nilpotent elements in $B \otimes M_n \otimes M_2$. To prove the claim, choose $y, z \in B \otimes M_n$ such that $x = yz$. (In general, every element a of a C^*-algebra is the product of two elements. This can for instance be obtained using the polar decomposition: By Proposition 2.13 there exists an element r such that $a = r|a|^{1/2}$.) Then

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}.$$

in $B \otimes M_n \otimes M_2$. Clearly, both $\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ z & 0 \end{smallmatrix}\right)$ are nilpotent, as desired.

3. We claim that every nilpotent element x in a C^*-algebra D is the limit of invertible elements in $\widetilde{D}$. Indeed, $x + \varepsilon 1_{\widetilde{D}}$ belongs to $\mathrm{Gl}(\widetilde{D})$ for every $\varepsilon > 0$, and $x = \lim_{\varepsilon \to 0} x + \varepsilon 1_{\widetilde{D}}$. □

Let $x \in A$. If $A \subseteq \overline{\mathrm{Gl}(\widetilde{A})}$, then $xx^* \sim_{\mathrm{au}} x^*x$, as observed in [BRT+12, Lemma 4.3.3]. One can obtain a stronger result by appealing to [Ped87, Corollary 8]; see also [APT11, Theorem 2.15].

In the next result, $\mathrm{dist}(x, \mathrm{Gl}(\widetilde{A}))$ denotes the distance of x to the invertible elements in $\widetilde{A}$.

Theorem 2.52. *Let* $x \in A$*, and let* $\delta > \mathrm{dist}(x, \mathrm{Gl}(\widetilde{A}))$*. Let* $f \colon [0, \infty) \to [0, \infty)$ *be a continuous function that vanishes on* $[0, \delta]$*. Then* $f(x^*x) \sim_u f(xx^*)$*.*

Corollary 2.53. *Let* $x \in A$ *with* $x \in \overline{\mathrm{Gl}(\widetilde{A})}$*. Then* $(x^*x - \varepsilon)_+ \sim_u (xx^* - \varepsilon)_+$ *for every* $\varepsilon > 0$*. In particular,* $x^*x \sim_{au} xx^*$*.*

Proof. By assumption, we have $\mathrm{dist}(x, \mathrm{Gl}(\widetilde{A})) = 0$. Let $\varepsilon > 0$. Let $f_\varepsilon \colon [0, \infty) \to [0, \infty)$ be the function given by $f_\varepsilon(t) = \max\{t - \varepsilon, 0\}$ for $t \in [0, \infty)$. Then f_ε is continuous and vanishes on

$[0,\varepsilon]$. Therefore, applying Theorem 2.52 at the second step, we deduce

$$(x^*x-\varepsilon)_+ = f_\varepsilon(x^*x) \sim_{\mathrm{u}} f_\varepsilon(xx^*) = (xx^*-\varepsilon)_+,$$

as desired. For each $n\geq 1$ let u_n be a unitary in $\widetilde{A}$ such that $u_n(x^*x-\frac{1}{n})_+u_n^* = (xx^*-\frac{1}{n})_+$. Then

$$\|u_n(x^*x)u_n - xx^*\| \leq \|u_n(x^*x)u_n - u_n(x^*x-\tfrac{1}{n})_+u_n^*\| + \|(xx^*-\tfrac{1}{n})_+ - xx^*\| \leq \tfrac{2}{n},$$

which verifies $x^*x \sim_{\mathrm{au}} xx^*$. □

Corollary 2.54. *Let $a,b\in A_+$ satisfy $a\approx b$. Assume that A has weak stable rank one. Then $(a-\varepsilon)_+ \sim_u (b-\varepsilon_+)$ for every $\varepsilon>0$.*

Theorem 2.55. *If A has weak stable rank one, then Cuntz comparison in A is unitarily implemented.*

Proof. To show that Cuntz comparison in A is unitarily implemented, let $a,b\in A_+$ satisfy $a\precsim b$, and let $\varepsilon>0$. Applying Rørdam's Lemma for $\frac{\varepsilon}{2}$, there exists $a'\in A_+$ such that $(a-\frac{\varepsilon}{2})_+ \approx a' \subseteq b$. Using Corollary 2.54 at the second step, we deduce

$$(a-\varepsilon)_+ = \big((a-\tfrac{\varepsilon}{2})_+ - \tfrac{\varepsilon}{2}\big)_+ \sim_{\mathrm{u}} (a'-\tfrac{\varepsilon}{2})_+ \subseteq b,$$

and thus $(a-\varepsilon)_+ \subseteq_u b$, as desired. □

By Lemma 2.51, every stable C^*-algebra has weak stable rank one. By Theorem 2.55, if A has weak stable rank one, then Cuntz comparison in A is unitarily implemented. We thus obtain:

Corollary 2.56. *In every stable C^*-algebra, Cuntz comparison is unitarily implemented.*

2.5. **Existence of suprema of increasing sequences.** In this subsection, we show that every increasing sequence in $\mathrm{Cu}(A)$ has a supremum.

First, we prove several lemmas about the existsence of suprema of special sequences. We formulate the results for Cuntz comparison in A. By passing to the stabilization, we obtain the same results for $\mathrm{Cu}(A)$.

The first lemma shows that for every convergent, increasing sequence $(a_n)_n$ in A_+, we have that the class of $\lim_n a_n$ is the supremum of $([a_n])_n$ in $A_+/\sim$.

Lemma 2.57. *Let $(a_n)_n$ be a sequence in A_+ with $a_1\leq a_2\leq a_3\leq\ldots$. Assume that there is $a\in A_+$ with $a=\lim_n a_n$. Then $[a]=\sup_n[a_n]$ in $A_+/\sim$.*

In particular, if $(\varepsilon_n)_n$ is a decreasing sequence in $[0,\infty)$ that converges to zero, then $[a]=\sup_n[(a-\varepsilon_n)_+]$ in $A_+/\sim$.

Proof. For each n, we have $a_n\leq a$ and therefore $[a_n]\leq[a]$. Thus, $[a]$ is an upper bound for the sequence $([a_n])_n$. To show that $[a]$ is the smallest upper bound, let $b\in A_+$ satisfy $[a_n]\leq[b]$ for all n. By Rørdam's Lemma, it is enough to verify that $(a-\varepsilon)_+\precsim b$ for every $\varepsilon>0$.

Let $\varepsilon>0$. Since $a=\lim_n a_n$, we can choose n with $\|a-a_n\|\leq\varepsilon$. Using Paragraph 2.27 at the first step, we obtain that

$$(a-\varepsilon)_+ \precsim a_n \precsim b,$$

as desired.

The last statement follows by setting $a_n := (a-\varepsilon_n)_+$. Then $(a_n)_n$ is increasing since $(\varepsilon_n)_n$ is decreasing. Moreover, $a=\lim_n a_n$ since $0=\lim_n\varepsilon_n$. □

Lemma 2.58. *Let $(a_n)_n$ be a sequence in A_+ satisfying $a_1\subseteq a_2\subseteq a_3\subseteq\ldots$. For each n set $b_n := \sum_{k=1}^n \frac{1}{2^k\|a_k\|}a_k$, and set $b:=\sum_{k=1}^\infty \frac{1}{2^k\|a_k\|}a_k$. Then $(b_n)_n$ is an increasing sequence in A_+ with limit b. Moreover, we have $a_n\sim b_n$ for each n, and $[b]=\sup_n[a_n]$ in $A_+/\sim$.*

Proof. It is easy to see that the sequence $(b_n)_n$ is increasing and that $b = \lim_n b_n$. Let $n \geq 1$. We have

$$a_n \sim \tfrac{1}{2^n\|a_n\|} a_n \leq b_n.$$

By assumption, we have $a_1, a_2, \ldots, a_n \subseteq a_n$. This implies that b_n belongs to the hereditary sub-C^*-algebra generated by a_n, and therefore $b_n \precsim a_n$ by Proposition 2.17. Thus, $a_n \sim b_n$, and therefore $[a_n] = [b_n]$.

It follows from Lemma 2.57 that $[b] = \sup_n [b_n]$, and thus also $[b] = \sup_n [a_n]$. □

Lemma 2.59. *Let $(a_n)_n$ be a sequence in A_+ satisfying $a_1 \subseteq_u a_2 \subseteq_u a_3 \subseteq_u \ldots$. Then there exists a convergent, increasing sequence $(c_n)_n$ in A_+ such that $a_n \sim c_n$ for each n. For $c := \lim_n c_n$ we have $[c] = \sup_n [a_n]$ in $A_+/\sim$.*

Proof. For each n, choose a unitary u_n in $\widetilde{A}$ such that $u_n a_n u_n^* \subseteq a_{n+1}$. We will construct a sequence $(b_n)_n$ in A_+ such that $a_n \sim b_n$ for each n, and such that $b_1 \subseteq b_2 \subseteq b_3 \subseteq \ldots$.

We set $b_1 := a_1$ and

$$b_n := u_1^* u_2^* \ldots u_{n-1}^* a_n u_{n-1} \ldots u_2 u_1$$

for $n \geq 2$. For each n we have $a_n \sim_{\mathrm{u}} b_n$ and therefore $a_n \sim b_n$.

For each $u \in \mathcal{U}(\widetilde{A})$, the map $A \to A$ given by $x \mapsto u^* x u$ is a *-isomorphism. Therefore, if $x, y \in A_+$ satisfy $x \subseteq y$, then $u^* x u \subseteq u^* y u$. We will use this repeatedly below.

We have $u_1 a_1 u_1^* \subseteq a_2$ and therefore

$$b_1 = a_1 = u_1^*(u_1 a_1 u_1^*) u_1 \subseteq u_1^*(a_2) u_1 = b_2.$$

Similarly, for each n, it follows from $u_n a_n u_n^* \subseteq a_{n+1}$ that

$$a_n = u_n^*(u_n a_n u_n^*) u_n \subseteq u_n^* a_{n+1} u_n,$$

and consequently

$$b_n = u_1^* \ldots u_{n-1}^* a_n u_{n-1} \ldots u_1 \subseteq u_1^* \ldots u_{n-1}^* u_n^* a_{n+1} u_n u_{n-1} \ldots u_1 = b_{n+1}.$$

This shows that the sequence $(b_n)_n$ has the claimed properties.

Now, applying Lemma 2.58 to the seuqnce $(b_n)_n$ we obtain $c \in A_+$ and a sequence $(c_n)_n$ with the desired properties. □

Theorem 2.60. *Let A be a C^*-algebra such that Cuntz comparison in A is unitarily implemented. Let $(a_n)_{n\geq 1}$ be a sequence in A_+ satisfying $a_1 \precsim a_2 \precsim a_3 \precsim \ldots$.*

Then there exists a convergent, increasing sequence $(c_n)_n$ in A_+ such that $c_n \precsim a_n$ for each n, and such that for $c := \lim_n c_n$ we have $[c] = \sup_n [a_n]$ in $A_+/\sim$.

In particular, every increasing sequence in $\mathrm{Cu}(A)$ has a supremum.

Proof. We inductively choose $\varepsilon_k^{(n)} > 0$ for $n, k \geq 1$ and set $a_{n,k} := (a_n - \varepsilon_k^{(n)})_+$ such that:

(a) For each n, the sequence $(\varepsilon_k^{(n)})_k$ is decreasing and converges to zero.
(b) For each n and k, we have $a_{n,k} \subseteq_u a_{n+1,k}$.

We obtain the following scheme:

$$\begin{array}{ccccccc}
a_1 & \precsim & a_2 & \precsim & a_3 & \precsim & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \\
a_{1,3} & \subseteq_u & a_{2,3} & \subseteq_u & a_{3,3} & \subseteq_u & \ldots \\
\vee\!| & & \vee\!| & & \vee\!| & & \\
a_{1,2} & \subseteq_u & a_{2,2} & \subseteq_u & a_{3,2} & \subseteq_u & \ldots \\
\vee\!| & & \vee\!| & & \vee\!| & & \\
a_{1,1} & \subseteq_u & a_{2,1} & \subseteq_u & a_{3,1} & \subseteq_u & \ldots
\end{array}$$

We proceed by induction over n. First, set $\varepsilon_k^{(1)} := \frac{1}{k}$ for each $k \geq 1$. Clearly, the sequence $(\varepsilon_k^{(1)})_k$ is decreasing and converges to zero.

Next, let $n \geq 1$, and assume that we have chosen $\varepsilon_k^{(n)}$ for all $k \geq 1$. We will choose $\varepsilon_k^{(n+1)}$ for all $k \geq 1$. We have $a_n \precsim a_{n+1}$. Since Cuntz comparison in A is unitarily implemented, for each k we can choose $\delta_k^{(n+1)} > 0$ such that $(a_n - \varepsilon_k^{(n)})_+ \subseteq_u (a_{n+1} - \delta_k^{(n+1)})_+$. Set

$$\varepsilon_k^{(n+1)} := \min\{\delta_1^{(n+1)}, \delta_2^{(n+1)}, \ldots, \delta_k^{(n+1)}, \tfrac{1}{k}\}.$$

Then $(\varepsilon_k^{(n+1)})_k$ is a decreasing sequence that converges to zero, which verifies condition (a). Moreover, for each k we have $\varepsilon_k^{(n+1)} \leq \delta_k^{(n+1)}$ and therefore

$$a_{n,k} = (a_n - \varepsilon_k^{(n)})_+ \subseteq_u (a_n - \delta_k^{(n+1)})_+ \subseteq (a_n - \varepsilon_k^{(n+1)})_+ = a_{n+1,k},$$

which verifies condition (b).

For each $n \geq 1$ set $b_n := a_{n,n}$. Then $b_1 \subseteq_u b_2 \subseteq_u b_3 \subseteq_u \ldots$. Applying Lemma 2.59 to the sequence $(b_n)_n$, we obtain $c \in A_+$ and an increasing sequence $(c_n)_n$ in A_+ with $b_n \sim c_n$ for each n, and such that $c := \lim_n c_n$ and $[c] = \sup_n [b_n]$ in $A_+/\sim$.

For each n, we have $c_n \sim b_n \subseteq a_n$. It remains to show that $[c] = \sup_n [a_n]$.

First, we show that $[a_n] \leq [c]$ for each n. Let $n \geq 1$. Using condition (a), it follows from Lemma 2.57 that $[a_n] = \sup_k [a_{n,k}]$. Thus, in order to show $[a_n] \leq [b]$, it is enough to verify $[a_{n,k}] \leq [b]$ for all $k \geq 1$. Since the sequence $([a_{n,k}])_k$ is increasing, it is enough to show $[a_{n,k}] \leq [b]$ for all $k \geq n$. Given such k, we apply condition (b) several times to obtain

$$a_{n,k} \subseteq_u a_{n+1,k} \subseteq_u \ldots \subseteq_u a_{k,k} = b_k,$$

which shows that $[a_{n,k}] \leq [b]$, as desired.

We have shown that $[c]$ is an upper bound for the sequence $([a_n])_n$. To show that it is the smallest upper bound, let $d \in A_+$ satisfy $[a_n] \leq [d]$ for all $n \geq 1$. Then, for each n, we have

$$[c_n] = [a_{n,n}] \leq [a_n] \leq [c].$$

Since $[c] = \sup_n [c_n]$, we deduce that $[c] \leq [d]$, as desired. □

Proposition 2.61. *Let A be a C^*-algebra such that Cuntz comparison in A is unitarily implemented. Let $a \in A_+$, and let $\varepsilon > 0$. Further, let $(b_n)_n$ be a sequence in A_+ satisfying $b_1 \precsim b_2 \precsim b_3 \precsim \ldots$, and let $b \in A_+$ such that $[b] = \sup_n [b_n]$ in $A_+/\sim$. Assume that $a \precsim b$. Then there exists n such that $(a - \varepsilon)_+ \precsim b_n$.*

Proof. Applying Theorem 2.60 to the sequence $(b_n)_n$, we obtain $c \in A_+$ and an increasing sequence $(c_n)_n$ in A_+ with $c_n \precsim b_n$ for each n, and such that $c = \lim_n c_n$ and $[c] = \sup_n [b_n]$. By assumption, we have $a \precsim b$, and therefore $a \precsim c$. Using Rørdam's Lemma, choose $\delta > 0$ such that $(a - \varepsilon)_+ \precsim (c - \delta)_+$. Since $c = \lim_n c_n$, we can choose n with $\|c - c_n\| \leq \delta$. Using Paragraph 2.27 at the second step, we obtain that

$$(a - \varepsilon)_+ \precsim (c - \delta)_+ \precsim c_n \precsim b_n,$$

as desired. □

Remark 2.62. Let $(s_n)_n$ be an increasing sequence in $\mathrm{Cu}(A)$. By Corollary 2.56, Cuntz comparison in $A \otimes \mathcal{K}$ is unitarily implemented. Therefore, we obtain from Theorem 2.60 that $(s_n)_n$ has a supremum in $\mathrm{Cu}(A)$.

Let $a \in (A \otimes \mathcal{K})_+$ and let $\varepsilon > 0$. Assume that $[a] \leq \sup_n s_n$. We obtain from Proposition 2.61 that there exists n such that $[(a - \varepsilon)_+] \leq s_n$.

In the next section we will see that these properties are instances of important concepts from the theory of partially ordered sets.

3. The category Cu of abstract Cuntz semigroups

In this section, we first recall basic notions from the theory of partially ordered sets and domains. For further details on this rich theory we refer to [GHK+03]. We then introduce the category Cu of abstract Cuntz semigroup. We show that the Cuntz semigroup of a C^*-algebra is an object in this category, and that every *-homomorphism (and more generally every order-zero c.p. map) induces a morphism in the category Cu between the involved Cuntz semigroups.

3.1. **Lattices and dcpos.** Let S be a set with a transitive relation $\prec$. Given a subset $F \subseteq S$ and $y \in S$, we write $F \prec y$ if and only if $x \prec y$ for every $x \in F$, and similarly for $y \prec F$. Recall that $D \subseteq S$ is called *upward directed* if for every finite subset $F \subseteq D$ there exists $y \in D$ such that $F \prec y$.

Recall that a partially ordered set $(S, \leq)$ is called an *inf-semilattice* if any two elements $x, y \in S$ have an infimum in S, denoted by $x \wedge y$ or $\inf\{x, y\}$. Equivalently, every finite subset $F \subseteq S$ has an infimum, denoted by $\bigwedge F$ or $\inf F$. Analogously, one defines sup-semilattices. We call S a *lattice* if it is both an inf-semilattice and a sup-semilattice. We say that S a *complete lattice* if every subset of S has an infimum and a supremum.

Examples 3.1. (1) Let X be a set. Then the powerset $\mathcal{P}(X) := \{S : S \subseteq X\}$ is a complete lattice. Given $D \subseteq \mathcal{P}(X)$, we have

$$\sup D = \bigcup D = \bigcup_{S \in D} S, \quad \text{and} \quad \inf D = \bigcap D = \bigcap_{S \in D} S.$$

(2) Let X be a topological space. We let $\mathcal{O}(X)$ denote the collection of open sets in X. By definition, $\mathcal{O}(X)$ is a sublattice of $\mathcal{P}(X)$, closed under passage to arbitrary suprema. Moreover, $\mathcal{O}(X)$ is a complete lattice, with supremum given by $\bigcup$, but the infimum in $\mathcal{O}(X)$ and $\mathcal{P}(X)$ need not agree. Given $D \subseteq \mathcal{O}(X)$, the infimum of D in $\mathcal{O}(X)$ is the interior of $\bigcap D$.

Similarly, the collection $\Gamma(X)$ of closed sets in X is a complete lattice, with infimum given by $\bigcap$.

(3) Let H be a Hilbert space. Then the family of closed subspaces of H, ordered by inclusion, forms a complete lattice.

Exercise 3.2. Let A be a C^*-algebra. Let $\mathrm{Her}(A)$ denote the collection of hereditary sub-C^*-algebras of A, ordered by inclusion. Show that $\mathrm{Her}(A)$ is a complete lattice, with infimum given by intersection $\bigcap$.

We consider the real numbers $\mathbb{R}$ with its usual ordering. We set $\mathbb{R}_+ := [0, \infty)$. We let $\overline{\mathbb{P}} := \mathbb{R}_+ \cup \{\infty\} = [0, \infty]$ denote the extended positive reals, with ∞ as largest element. We equip $\overline{\mathbb{P}}$ with the topology generated by the usual open sets in $\mathbb{R}_+$ and the sets $(t, \infty]$ for $t \in \mathbb{R}$. Then, any order-isomorphism of $\overline{\mathbb{P}}$ with a closed interval, say $[0, 1]$, is a homeomorphism.

Given a topological space X, a function $f \colon X \to \overline{\mathbb{P}}$ is *lower semicontinuous* if for every $t \in \overline{\mathbb{P}}$ the set $\{x : t < f(x)\}$ is open. We let $\mathrm{Lsc}(X, \overline{\mathbb{P}})$ denote the set of lower-semicontinuous functions $X \to \overline{\mathbb{P}}$. As usual, we let $C(X, \overline{\mathbb{P}})$ denote the collection of continuous functions $X \to \overline{\mathbb{P}}$.

Exercise 3.3 ([GHK+03, Exercise 0.2.10, p.17])**.** Let X be a topological space.

(1) Under the pointwise ordering, $C(X, \overline{\mathbb{P}})$ is a lattice. If X is compact, then $C(X, \overline{\mathbb{P}})$ is a complete lattice for the pointwise ordering if and only if X is extremally disconnected.

(2) Under the pointwise ordering, $\mathrm{Lsc}(X, \overline{\mathbb{P}})$ is a complete lattice.

Definition 3.4. Let S be a partially ordered set. We say that S is *bounded directed complete* if every upward directed subset of S that has an upper bound has a least upper bound, that is, a supremum. If every upward directed subset of S has a supremum, then we say that S is *directed complete*.

Analogously, we say that S is *sequentially complete* if every increasing sequence of S has a supremum. If we only require the existence of suprema of bounded increasing sequences, then we call S *bounded sequentially complete*.

As customary in domain theory, we abbreviate **d**irected **c**omplete **p**artially **o**rdered set to **dcpo**. Similarly, a ω-**dcpo** is a sequentially complete partially ordered set.

We let DCPO denote the category whose objects are **dcpo**s, and whose morphisms are order-preserving maps that moreover preserve suprema of directed subsets. Analogously, $\mathtt{DCPO}_\omega$ denotes the category whose objects are ω-**dcpo**s, and whose morphisms are order-preserving maps that preserve suprema of increasing sequences.

Remark 3.5. The terminology '(bounded) directed complete' is well-established in lattice and domain theory. The concept '(bounded) sequentially complete' has not been studied much in domain theory since the restriction to sequences is considered as less significant branch of the theory. We remark that instead of '(bounded) sequentially complete' one could also speak of '(bounded) sequentially directed complete'.

Of course, 'sequentially complete' does not mean that every sequence of elements has a supremum, a condition which is sometimes called 'countably complete' (or more precisely 'countably sup-complete').

Example 3.6. Let X be a topological space. The directed union of connected subsets of X is connected. Thus, the family of connected subsets of X, ordered by inclusion, forms a **dcpo**. In general, this **dcpo** is not a (complete) lattice.

Example 3.7. Let A be a C^*-algebra. If Cuntz comparison in A is unitarily implemented, then it follows from Theorem 2.60 that $A_+/\sim$ is a ω-**dcpo**. In particular, the Cuntz semigroup $\mathrm{Cu}(A)$ is a ω-**dcpo**.

If A is separable, we will see that $\mathrm{Cu}(A)$ is even directed complete and thus a **dcpo**. On the other hand, if A is not σ-unital, then we will see that $\mathrm{Cu}(A)$ is not directed complete.

3.2. **The way-below relation.** The non-sequential part of the following definition can be found in [GHK$^+$03, Definition I-1.1, p.49].

Definition 3.8. Let S be a partially ordered set, and let $x, y \in S$. We say that x is *way-below* y, or that x is *compactly contained* in y, denoted $x \ll y$, if for all directed subsets $D \subseteq S$ for which a supremum exists we have that $y \leq \sup D$ implies that $x \leq d$ for some $d \in D$.

We say that x is *sequentially way-below* y, or that x is *sequentially compactly contained* in y, denoted $x \ll_\omega y$, if for all increasing sequences $(d_n)_n$ for which a supremum exists we have that $y \leq \sup_n d_n$ implies that $x \leq d_n$ for some n.

The following result is [GHK$^+$03, Proposition I-1.2, p.50].

Proposition 3.9. *Let S be a partially ordered set, and let $u, x, y, z \in S$. Then:*

(1) If $x \ll y$, then $x \leq y$.
(2) If $u \leq x \ll y \leq z$, then $u \ll z$.
(3) If S contains a smallest element 0, then $0 \ll x$.

Analogous statements hold for $\ll_\omega$ in place of $\ll$. Moreover, $x \ll y$ implies $x \ll_\omega y$.

Proof. Exercise. □

Examples 3.10. (1) Let X be a set, and let $S, T \in \mathcal{P}(X)$. Then $S \ll T$ if and only if $S \ll_\omega$, if and only if S is finite and $S \subseteq T$.

(2) Let X be a topological space, and let $U, V \in \mathcal{O}(X)$. Then:

(1) If there exists a (countably) compact subset $K \subseteq X$ with $U \subseteq K \subseteq V$, then $U \ll V$ ($U \ll_\omega V$) in $\mathcal{O}(X)$.
(2) Suppose that X is locally compact. Then $U \ll V$ if and only if there exists compact subset $K \subseteq X$ with $U \subseteq K \subseteq V$.

For proofs see [GHK$^+$03, Proposition I-1.4, p.53].

(3) Let H be a Hilbert space, and let $E, F \subseteq H$ be closed subspaces. Then $E \ll F$ if and only if $\ll_\omega F$, if and only if E is finite-dimensional and $E \subseteq F$.

Notation 3.11. Let S be a partially ordered set. Given a subset $D \subseteq S$, we set

$$\downarrow D := \{x \in S : x \leq d \text{ for some } d \in D\}, \quad \text{and} \quad \twoheaddownarrow D := \{x \in S : x \ll d \text{ for some } d \in D\},$$

and analogously for $\uparrow D$ and $\twoheaduparrow D$. We also consider the analogs $\twoheaddownarrow^\omega D$ and $\twoheaduparrow^\omega D$ with $\ll_\omega$ in place of $\ll$. Given $x \in S$, we set $\downarrow x := \downarrow\{x\}$, and analogously for $\uparrow x$, $\twoheaddownarrow x$, $\twoheaddownarrow^\omega x$, $\twoheaduparrow x$ and $\twoheaduparrow^\omega x$.

The non-sequential part of the following definition is [GHK+03, Definition I-1.6, p.54].

Definition 3.12. Let S be a partially ordered set. We say that S is *continuous* if for every $x \in S$, the set $\twoheaddownarrow x$ is upward directed and $x = \sup \twoheaddownarrow x$. We say that S is *ω-continuous* if for every $x \in S$ there exists an increasing sequence $(x_n)_n$ in S with $x_n \ll_\omega x$ for each n, and such that $x = \sup_n x_n$.

A continuous **dcpo** is called a *domain*. Analogously, a ω-continuous ω-**dcpo** is called a *ω-domain*. A continuous, complete lattice is called a *continuous lattice*.

We let $\mathtt{DOM}$ denote the full subcategory of $\mathtt{DCPO}$ whose objects are domains, and similarly for the full subcategory $\mathtt{DOM}_\omega$ of $\mathtt{DCPO}_\omega$.

We let $\mathtt{DOM}_\ll$ denote the category whose objects are domains, and whose morphisms are $\ll$-preserving $\mathtt{DCPO}$-morphisms. Similarly, $\mathtt{DOM}_{\ll,\omega}$ denotes the category whose objects are ω-domains and whose morphisms are $\ll_\omega$-preserving $\mathtt{DCPO}_\omega$-morphisms.

Lemma 3.13. *Let S be a partially ordered set. Then the following are equivalent:*

1. *S is ω-continuous.*
2. *For every $s \in S$, the set $\twoheaddownarrow^\omega s$ is upward directed, $s = \sup \twoheaddownarrow^\omega s$, and $\twoheaddownarrow^\omega s$ contains a countable cofinal subset (equivalently, a cofinal, increasing sequence).*
3. *For every $s \in S$, there is a sequence $s_1 \ll_\omega s_2 \ll_\omega s_3 \ll_\omega \ldots$ in S with $s = \sup_n s_n$.*

Proof. It is easy to see that (2) implies (1).

To show that (3) implies (2), let $s \in S$. Choose a $\ll_\omega$-increasing sequence $(s_n)_n$ in S with supremum s. To show that $\twoheaddownarrow^\omega s$ is upward-directed, let $y, z \in \twoheaddownarrow^\omega s$. Then

$$y \ll_\omega s = \sup_n s_n,$$

whence we can choose k such that $y \leq s_k$. Analogously, we choose l such that $z \leq s_l$. For $n := \max\{k, l\}$, we have

$$y, z \leq s_l \ll_\omega s,$$

as desired. Using that $s = \sup_n s_n$, it follows easily that $s = \sup \twoheaddownarrow^\omega s$. It is also straightforward to check that $(s_n)_n$ is cofinal in $\twoheaddownarrow^\omega s$.

To show that (1) implies (3), let $s \in S$. Choose a $\leq$-increasing sequence $(s_n)_n$ in S with $s = \sup_n s_n$ and such that $s_n \ll_\omega s$ for each n. We claim that for each n there exists $m > n$ such that $s_n \ll_\omega s_m$. It follows from the claim that a suitable subsequence has the desired properties to verify (3).

To prove the claim, it is enough to consider $n = 1$. That is, we need to find m such that $s_1 \ll_\omega s_m$. For each n, choose a $\leq$-increasing sequence $(s_{n,k})_k$ in S with $s_n = \sup_k s_{n,k}$ and such that $s_{n,k} \ll_\omega s_n$ for each n and k. Using a diagonal-type argument (as in the proof of Theorem 2.60), we can choose indices $k(n)$ for each n such that $s_{n,k(n)} \leq s_{n+1,k(n+1)}$, and such that $s = \sup_n s_{n,k(n)}$. Then $s_1 \ll_\omega s = \sup_n s_{n,k(n)}$, which allows us to obtain m such that $s_1 \leq s_{m,k(m)}$. We deduce

$$s_1 \leq s_{m,k(m)} \ll_\omega s_m,$$

as desired. □

Lemma 3.14. *Let A be a C^*-algebra such that Cuntz comparison in A is unitarily implemented, and let $a, b \in A_+$. Then $[a] \ll_\omega [b]$ in $A_+/\sim$ if and only if there exists $\varepsilon > 0$ with $a \precsim (b - \varepsilon)_+$.*

In particular, for every $\varepsilon > 0$ we have $[(a - \varepsilon)_+] \ll_\omega [a]$ in $A_+/\sim$.

Proof. It follows directly from Proposition 2.61 that $[(a-\varepsilon)_+] \ll_\omega [a]$ in $A_+/\sim$. This also shows that $[a] \ll_\omega [b]$ in $A_+/\sim$ whenever there exists $\varepsilon > 0$ with $a \precsim (b-\varepsilon)_+$. To show the forward implication, assume that $[a] \ll_\omega [b]$. For $n \geq 1$ set $b_n := (b - \frac{1}{n})_+$. By Lemma 2.57, we have $[b] = \sup_n [b_n]$. By definition of $\ll_\omega$, there exists n with $[a] \leq [b_n]$, in which case $\varepsilon = \frac{1}{n}$ has the desired properties. □

Proposition 3.15. *Let A be a C^*-algebra. If Cuntz comparison in A is unitarily implemented, then $A_+/\sim$ is a ω-domain. In particular, the Cuntz semigroup* $\mathrm{Cu}(A)$ *is a ω-domain.*

Proof. Assume that Cuntz comparison in A is unitarily implemented. By Theorem 2.60, $A_+/\sim$ is a ω-**dcpo**; see also Example 3.7,

To show that $A_+/\sim$ is ω-continuous, let $a \in A_+$. For $n \geq 1$ set $a_n := (a - \frac{1}{n})_+$. By Lemma 3.14, for each n we have $[a_n] \ll_\omega [a]$ in $A_+/\sim$. Moreover, by Lemma 2.57, we have $[a] = \sup_n [a_n]$, as desired. □

3.3. **The category** Cu. In [CEI08], Coward, Elliott and Ivanescu introduced a category Cu of abstract Cuntz semigroups. The objects in this category are ω-domains with a compatible structure as an abelian monoid. The morphisms in Cu are $\mathtt{DOM}_{\ll}$-morphisms that preserve the monoid structure.

Definition 3.16. A Cu-*semigroup*, also called *abstract Cuntz semigroup*, is a positively ordered monoid S that satisfies the following axioms (O1)-(O4):

- (O1) S is a ω-**dcpo**, that is, every increasing sequence in S has a supremum.
- (O2) S is a ω-continuous, that is, for every $s \in S$ there is a sequence $s_1 \ll_\omega s_2 \ll_\omega s_3 \ll_\omega \dots$ in S with supremum s.
- (O3) If $s' \ll_\omega s$ and $t' \ll_\omega t$ for $s', s', s, t \in S$, then $s' + t' \ll_\omega s + t$.
- (O4) If $(s_n)_n$ and $(t_n)_n$ are increasing sequences in S, then $\sup_n (s_n + t_n) = \sup_n s_n + \sup_n t_n$.

Given Cu-semigroups S and T, a Cu-*morphism* from S to T is a map $f\colon S \to T$ that preserves addition, order, the zero element, the sequential way-below relation and suprema of increasing sequences. A *generalized* Cu-*morphism* is a Cu-morphism that is not required to preserve the sequential way-below relation. We denote the set of Cu-morphisms by $\mathrm{Cu}(S,T)$; and we denote the set of generalized Cu-morphisms by $\mathrm{Cu}[S,T]$.

We let Cu be the category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms.

Theorem 3.17 (Coward, Elliott, Ivanescu, [CEI08, Theorem 1])**.** *Let A be a C^*-algebra. Then* $\mathrm{Cu}(A)$ *is a* Cu-*semigroup.*

Proof. We have already shown in Proposition 3.15 that $\mathrm{Cu}(A)$ is a ω-domain. To verify (O3), let $a', b', a, b \in (A \otimes \mathcal{K})_+$ satisfy $[a'] \ll_\omega [a]$ and $[b'] \ll_\omega [b]$. Without loss of generality, we may assume that $a' \perp b'$ and $a \perp b$. Use Lemma 3.14 to choose $\varepsilon_1, \varepsilon_2 > 0$ such that

$$a' \precsim (a - \varepsilon_1)_+, \quad \text{and} \quad b' \precsim (b - \varepsilon_2)_+.$$

Set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Using that a and b are orthogonal at the last step, we deduce that

$$a' + b' \precsim (a - \varepsilon_1)_+ + (b - \varepsilon_2)_+ \leq (a - \varepsilon)_+ + (b - \varepsilon)_+ = (a + b - \varepsilon)_+.$$

Using Lemma 3.14 at the second step, we deduce that

$$[a'] + [b'] = [a' + b'] \ll_\omega [a + b] = [a] + [b],$$

as desired.

To verify (O4), let $(a_n)_n$ and $(b_n)_n$ be $\precsim$-increasing sequences in $(A \otimes \mathcal{K})_+$. For each n, we have $[a_n] + [b_n] \leq \sup_n [a_n] + \sup_n [b_n]$, and consequently

$$\sup_n([a_n] + [b_n]) \leq \sup_n [a_n] + \sup_n [b_n].$$

Let us show the inverse inequality. Using Theorem 2.60, we choose convergent, $\leq$-increasing sequences $(a_n')_n$ and $(b_n')_n$ in $(A \otimes \mathcal{K})_+$ such that $a_n' \precsim a_n$ and $b_n' \precsim b_n$ for each n, and such that for $a' := \lim_n a'_n$ and $b' := \lim_n b'_n$ we have $[a'] = \sup_n [a_n]$ and $[b'] = \sup_n [b_n]$. It follows that $a' \oplus b' = \lim_n (a'_n \oplus b'_n)$. Using Lemma 2.57 at the second step, and using that $[a'_n] \leq [a_n]$ and $[b'_n] \leq [b_n]$ for each n at the last step, we deduce that

$$\sup_n [a_n] + \sup_n [b_n] = [a' \oplus b'] = \sup_n [a'_n \oplus b'_n] = \sup_n \big([a'_n] + [b'_n]\big) \leq \sup_n \big([a_n] + [b_n]\big),$$

as desired. □

Next we consider maps between C^*-algebras that induce maps between the respective Cuntz semigroups. First, we give a general criterion when a positive map induces a $\mathtt{DOM}_\omega$-morphism.

Lemma 3.18. *Let $\varphi\colon A \to B$ be bounded, positive, linear map such that for all $a, b \in A_+$ we have $\varphi(a) \sim \varphi(b)$ whenever $a \sim_u b$. Assume that Cuntz comparison in A and B is unitarily implemented. Then for all $a, b \in A_+$ we have $\varphi(a) \precsim \varphi(b)$ whenever $a \precsim b$.*

Moreover, the induced map $A_+/\sim \to B_+/\sim$ is a morphism in $\mathtt{DOM}_\omega$, that is, it preserves order and suprema of increasing sequence.

Proof. Let $a, b \in A_+$ satisfy $a \precsim b$. Let $\varepsilon > 0$. By Proposition 2.47, there is $\delta > 0$ and $x \in A_+$ such that $(a - \varepsilon)_+ \sim_u x$ and $x \subseteq (b - \delta_+)$.

We claim that there is $r \in \mathbb{R}_+$ such that $x \leq rb$. Indeed, there is a continuous function $f\colon [0, \infty) \to [0, 1]$ vanishing on $[0, \delta/2]$ such that $f(b)$ acts as a unit on $(b-\delta)_+$, that is, $(b-\delta)_+ = (b - \delta)_+ f(b)$. Then $f(b)^2 \leq \frac{2}{\delta} b$. Using that $x \subseteq (b - \delta)_+$ and applying Lemma 2.14, we obtain that $f(b)$ acts as a unit on x. Then

$$x = f(b)xf(b) \leq f(b)\|x\|f(b) \leq \tfrac{2\|x\|}{\delta} b,$$

which proves the claim.

Choose $r \in \mathbb{R}_+$ such that $x \leq rb$. We deduce that

$$\varphi((a - \varepsilon)_+) \sim \varphi(x) \leq \varphi(rb) = r\varphi(b) \sim \varphi(b).$$

For each $n \geq 1$, set $a_n := (a - \frac{1}{n})_+$. We have shown that $[\varphi(a_n)] \leq [\varphi(b)]$ in $B_+/\sim$ for every $n \geq 1$. Note that $(a_n)_n$ is an increasing sequence converging to a. Using that φ is bounded and positive, we obtain that $(\varphi(a_n))_n$ is an increasing sequence converging to $\varphi(a)$. Using Lemma 2.57 at the first step, we obtain that

$$[\varphi(a)] = \sup_n [\varphi(a_n)] \leq [\varphi(b)],$$

in $B_+/\sim$, as desired.

It follows that φ preserves Cuntz (sub)equivalence. The induced map $\bar{\varphi}\colon A_+/\sim \to B_+/\sim$ is order-preserving. To show that $\bar{\varphi}$ preserves suprema of increasing sequences, let $(a_n)_n$ be a $\precsim$-increasing sequence in A_+. Use Theorem 2.60 to choose a convergent, $\leq$-increasing sequence $(c_n)_n$ in A_+ such that $c_n \precsim a_n$ for each n, and such that for $c := \lim_n c_n$ we have $[c] = \sup_n [a_n]$ in $A_+/\sim$.

As above, we deduce that $[\varphi(c)] = \sup_n [\varphi(c_n)]$. For each n, we have $c_n \precsim a_n \precsim c$, which implies that

$$[\varphi(c)] = \sup_n [\varphi(c_n)] \leq \sup_n [\varphi(a_n)] \leq [\varphi(c)],$$

and consequently

$$\bar{\varphi}\left(\sup_n [a_n]\right) = \bar{\varphi}([c]) = [\varphi(c)] = \sup_n [\varphi(a_n)] = \sup_n \bar{\varphi}([a_n]),$$

as desired. □

Let A and B be C^*-algebras, and let $\varphi\colon A \to B$ be a linear map. For each $n \geq 2$, we consider the linear map $\varphi \otimes \mathrm{id}_n\colon A \otimes M_n \to B \otimes M_n$. We call $\varphi \otimes \mathrm{id}_n$ the *amplification* of φ to $n \times n$-matrices. Recall that φ is called *completely positive* if each amplification $\varphi \otimes \mathrm{id}_n$ is positive. Equivalently, the map $\varphi \otimes \mathrm{id}\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ is positive. By abuse of notation, we often denote the amplification of φ to matrices or to the stabilization also by φ.

Let $\varphi\colon A\to B$ be c.p.c. map such that for all $a,b\in(A\otimes\mathcal{K})_+$ we have $\varphi(a)\sim\varphi(b)$ whenever $a\sim_u b$. By Lemma 3.18, φ induces a $\mathtt{DOM}_\omega$-morphism $\bar{\varphi}\colon \mathrm{Cu}(A)\to\mathrm{Cu}(B)$.

It is a natural question to ask when $\bar{\varphi}$ preserves addition. A sufficient condition is that φ preserves orthogonality of positive elements. Such maps were studied in [WZ09]:

Definition 3.19 (Winter, Zacharias, [WZ09, Definition 2.3])**.** A c.p.c. map $\varphi\colon A\to B$ is said to have *order zero* if φ preserves orthogonality of positive elements: $a\perp b$ implies $\varphi(a)\perp\varphi(b)$, for $a,b\in A_+$.

Winter and Zacharias obtained a structure theorem for c.p.c. order-zero maps, which has many interesting applications. In particular, it implies that the amplification of a c.p.c. order-zero map is again a c.p.c. order-zero map; see [WZ09, Corollary 4.3]. We will use the structure theorem to show that every c.p.c. order-zero map $\varphi\colon A\to B$ satisfies the assumptions of Lemma 3.18. As a consequence, we deduce that φ induces a generalized Cu-morphism $\bar{\varphi}\colon \mathrm{Cu}(A)\to\mathrm{Cu}(B)$; see Proposition 3.22.

Recall that $\widetilde{A}$ denotes the minimal unitization of a C^*-algebra A.

Theorem 3.20 (Winter, Zacharias, [WZ09, Theorem 3.3])**.** *Let $\varphi\colon A\to B$ be a c.p.c. order-zero map. Set $C:=C^*(\varphi(A))$, the sub-C^*-algebra of B generated by the image of φ. Then there exist a unital *-homomorphism $\pi_\varphi\colon\widetilde{A}\to M(C)$ such that*

$$\varphi(ab)=\varphi(a)\pi_\varphi(b)=\pi_\varphi(a)\varphi(b),$$

for all $a,b\in\widetilde{A}$.

In particular, the element $h:=\varphi(1_{\widetilde{A}})$ is contractive, positive, it commutes with the image of π_φ, and we have $\varphi(a)=h\pi_\varphi(a)=\pi_\varphi(a)h$ for all $a\in A$.

Lemma 3.21. *Let $\varphi\colon A\to B$ be a c.p.c. order-zero map, and let $a,b\in A_+$ satisfy $a\sim_u b$. Then $\varphi(a)\sim\varphi(b)$.*

Proof. Set $C:=C^*(\varphi(A))$. Choose a unital *-homomorphism $\pi_\varphi\colon\widetilde{A}\to M(C)$ as in the structure Theorem 3.20. Let $a,b\in A_+$ satisfy $a\sim_u b$. Choose a unitary $u\in\mathcal{U}(\widetilde{A})$ such that $a=ubu^*$. Then $\pi_\varphi(u)$ is a unitary in $M(C)$. Then

$$\varphi(a)=\varphi(ubu^*)=\pi_\varphi(u)\varphi(bu^*)=\pi_\varphi(u)\varphi(b)\pi_\varphi(u)^*,$$

which implies that $\varphi(a)$ and $\varphi(b)$ are unitarily equivalent with a unitary from $M(C)$. Using Proposition 2.18, we obtain that $\varphi(a)$ and $\varphi(b)$ are Cuntz equivalent in C, and consequently also in B. ☐

Proposition 3.22. *Let $\varphi\colon A\to B$ be a c.p.c. order-zero map. Consider the amplification $\varphi\otimes\mathrm{id}\colon A\otimes\mathcal{K}\to B\otimes\mathcal{K}$, which is also a c.p.c. order-zero map. Then $\varphi\otimes\mathrm{id}$ preserves Cuntz (sub)equivalence. Moreover, the induced map $\mathrm{Cu}(\varphi)\colon\mathrm{Cu}(A)\to\mathrm{Cu}(B)$ is a generalized Cu-morphism.*

Proof. By Corollary 2.56, Cuntz comparison in $A\otimes\mathcal{K}$ and $B\otimes\mathcal{K}$ is unitarily implemented. Therefore, it follows directly from Lemma 3.18 and Lemma 3.21 that the amplification $\varphi\otimes\mathrm{id}$ induces a map $\mathrm{Cu}(\varphi)\colon\mathrm{Cu}(A)\to\mathrm{Cu}(B)$ that is a $\mathtt{DOM}_\omega$-morphism. It is clear that $\mathrm{Cu}(\varphi)$ perserves the zero element. Finally, to show additivity, let $a,b\in(A\otimes\mathcal{K})_+$. Using that φ (and its amplifications) preserves orthogonality at the third step, we deduce that

$$\begin{aligned}\mathrm{Cu}(\varphi)([a]+[b])=\mathrm{Cu}(\varphi)([a\oplus b])&=[\varphi(a\oplus b)]\\ &=[\varphi(a)\oplus\varphi(b)]=[\varphi(a)]+[\varphi(b)]=\mathrm{Cu}(\varphi)([a])+\mathrm{Cu}(\varphi)([b]),\end{aligned}$$

as desired. ☐

Every *-homomorphism is a c.p.c. order-zero map and therefore induces a generalized Cu-morphism between the respective Cuntz semigroups. Next, we show that it also preserves the way-below relation and therefore is a Cu-morphism.

Proposition 3.23. *Let A and B be C^*-algebra, and let $\varphi\colon A \to B$ be a *-homomorphism. Then the induced map $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ is a* Cu*-morphism.*

Proof. It only remains to show that $\mathrm{Cu}(\varphi)$ preserves the way-below relation. Thus, let $a, b \in (A \otimes \mathcal{K})_+$ satisfy $[a] \ll_\omega [b]$. By Lemma 3.14, we can choose $\varepsilon > 0$ such that $a \precsim (b-\varepsilon)_+$. A *-homomorphism preserves continuous functional calculus. Using this at the second step, we deduce that

$$\varphi(a) \precsim \varphi((b-\varepsilon)_+) = (\varphi(b)-\varepsilon)_+.$$

Applying Lemma 3.14 again, we obtain

$$\mathrm{Cu}(\varphi)([a]) = [\varphi(a)] \ll_\omega [\varphi(b)] = \mathrm{Cu}(\varphi)([b]),$$

as desired. □

3.24. We let C^* denote the category of C^*-algebras, whose objects are C^*-algebras and whose morphisms are *-homomorphisms. We can combine Theorem 3.17 with Proposition 3.23 to define a functor $C^* \to \mathrm{Cu}$. We map a C^*-algebra A to its Cuntz semigroup $\mathrm{Cu}(A)$, which is an object in Cu. Further, given C^*-algebras A and B, every *-homomorphism $\varphi\colon A \to B$ induces a Cu-morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$.

The functor $C^* \to \mathrm{Cu}$ has many desirable properties: it preserves inductive limits, short exact sequences, direct sums, direct products and ultraproducts. It is an interesting question to determine which Cu-semigroups can be realized as the Cuntz semigroup of a C^*-algebra. This question is asking to determine the range of the functor $C^* \to \mathrm{Cu}$. In general, this question is very hard. In the next subsection, we will present two additional axioms on Cu-semigroups that every Cuntz semigroup of a C^*-algebras satisfies.

3.4. **Additional axioms.** Recall that a partially ordered monoid S is *algebraically ordered* if for $a, b \in S$ we have $a \leq b$ if and only if there exists $x \in S$ (a 'complement') such that $a + x = b$. This property is rarely satisfied by Cuntz semigroups. However, a weaker version always holds:

Definition 3.25. Let S be a Cu-semigroup. We say that S has *almost algebraic order*, or that S satisfies (O5), if for all $a', a, b', b, c \in S$ satisfying

$$a + b \leq c, \quad \text{and} \quad a' \ll_\omega a, \quad \text{and} \quad b' \ll_\omega b,$$

there exists $x \in S$ (an 'almost complement') such that

$$a' + x \leq c \leq a + x, \quad \text{and} \quad b' \leq x.$$

Remark 3.26. A weaker form of (O5) has first been considered by Rørdam and Winter in [RW10, Lemma 7.1]. The version given in Definition 3.25 was introduced in [APT14, Definition 4.1]. It was shown in [APT14, Proposition 4.7, p.34] that the Cuntz semigroup of every C^*-algebra satisfies the stronger form of (O5).

Recall that a partially ordered monoid S is said to have *Riesz decomposition* if for $a, b, c \in S$ with $a \leq b + c$ there exist $a_1, a_2 \in S$ such that $a = a_1 + a_2$ and $a_1 \leq b$ and $a_2 \leq c$. Again, this property is rarely satisfied by Cuntz semigroups, but a weaker version always holds:

Definition 3.27. Let S be a Cu-semigroup. We say that S has *almost Riesz decomposition*, or that S satisfies (O6), if for all $a', a, b, c \in S$ satisfying

$$a' \ll_\omega a \leq b + c,$$

there exist $e, f \in S$ such that

$$a' \leq e + f, \quad \text{and} \quad e \leq a, b, \quad \text{and} \quad f \leq a, c.$$

Remark 3.28. Axiom (O6) was introduced by Robert in [Rob13]. It was shown in [Rob13, Proposition 5.1.1] that the Cuntz semigroup of every C^*-algebra satisfies (O6).

We omit the proof of the following result. It can be found in [APT14, Proposition 4.7, p.34] and [Rob13, Proposition 5.1.1].

Theorem 3.29. *Let A be a C^*-algebra. Then* $\mathrm{Cu}(A)$ *satisfies (O5) and (O6).*

Lastly, we consider a 'separability' condition for partially ordered sets, satisfied by Cuntz semigroups of separable C^*-algebras.

Definition 3.30. Let S be a ω-**dcpo**. We say that S is *countably based* if there exists a countable subset B such that for all $a', a \in S$ with $a' \ll_\omega a$ there exists $b \in B$ with $a' \leq b \ll_\omega a$. Equivalently, for every $a \in S$, the set $B \cap \downarrow^\omega a$ is cofinal in $\downarrow^\omega a$.

Lemma 3.31. *Let S be a countably based ω-**dcpo**. Then S is a domain if and only if S is a ω-domain.*

Proof. Let $B \subseteq S$ be a countable subset witnessing that S is countably based.

Claim 1: Assume that S is ω-continuous. Let $D \subseteq S$ be upward-directed. Then $B \cap \downarrow^\omega D$ is upward-directed.

To prove the claim, let $x, y \in B \cap \downarrow^\omega D$. Choose $d, e \in D$ such that $x \ll_\omega d$ and $y \ll_\omega e$. Since D is upward-directed, we can choose $f \in D$ with $d, e \leq f$. Then $x, y \ll_\omega f$. Since S is ω-continuous, the set $\downarrow^\omega f$ is upward-directed, and we can choose $f' \in D$ with $x, y \leq f' \ll_\omega f$. Using the assumption on B, we can choose $z \in B$ with $f' \leq z \ll_\omega f$. Then z has the desired properties.

Claim 2: Assume that S is ω-continuous. Let $D \subseteq S$ be upward-directed. Then $\sup D$ exists and agrees with the supremum of $B \cap \downarrow^\omega D$.

To prove the claim, let $D \subseteq S$ be upward-directed. By claim 1, the set $E := B \cap \downarrow^\omega D$ is upward-directed. Since B is countable, there exists a cofinal increasing sequence in E. Since S is a ω-**dcpo**, the supremum $\sup E$ exists. Let us show that $\sup E$ is the supremum of D.

To show that $\sup E$ is an upper bound for D, let $d \in D$. Since S is ω-continuous, the set $\downarrow^\omega d$ is upward-directed and has supremum d. By assumption on B, the set $B \cap \downarrow^\omega d$ is cofinal in $\downarrow^\omega d$. Using this at the first step, and using that $B \cap \downarrow^\omega d$ is a subset of $B \cap \downarrow^\omega D$ at the second step, we deduce that

$$d = \sup\left(B \cap \downarrow^\omega d\right) \leq \sup\left(B \cap \downarrow^\omega D\right) = \sup E.$$

Thus, $\sup E$ is an upper bound for D. Since $E \subseteq \downarrow D$, every upper bound for D is also an upper bound for E. It follows that $\sup E$ is the smallest upper bound for D, as desired.

Claim 3: The relations $\ll$ and $\ll_\omega$ agree.

In general, $\ll$ is stronger than $\ll_\omega$. To show the converse, let $a, b \in S$ satisfy $a \ll_\omega b$. To verify that $a \ll b$, let $D \subseteq S$ be an upward-directed set for which $\sup D$ exists (since we already showed that S is a **dcpo**, this assumption is actually superfluous), and such that $b \leq \sup D$. Set $E := B \cap \downarrow^\omega D$. By claim 2, we have $\sup D = \sup E$. Since B is countable, there exists a cofinal increasing sequence $(e_k)_k$ in E. Then

$$a \ll_\omega b \leq \sup D = \sup_k e_k,$$

which implies that we can choose k with $a \leq e_k$. Since e_k belongs to E, we can choose $d \in D$ with $e_k \ll_\omega d$. Then $a \leq d$, which verifies that $a \ll b$, as desired.

It follows directly from claim 2, the S is a **dcpo** whenever it is ω-continuous. Similarly, it follows from claim 3 that S is continuous whenever it is ω-continuous. Thus, a countably based ω-domain is a domain.

Conversely, assume that S is a domain. Then S is a ω-**dcpo**. Thus, to show that S is a ω-domain, it remains to show that S is ω-continuous. To verify statement (2) of Lemma 3.13, let $s \in S$. By claim 3, we have $\downarrow s = \downarrow^\omega s$. Using that S is continuous, it follows that $\downarrow^\omega s$ is upward directed and has supremum s. Using claim 2, the set $\downarrow^\omega s$ contains a cofinal increasing sequence, as desired. $\square$

Let $a, b \in A_+$ with $\|a - b\| < \varepsilon$. In Paragraph 2.27, we showed that this implies that $(a-\varepsilon)_+ \precsim b$. Using Lemma 3.14, we can improve this result.

Lemma 3.32. *Let A be a C^*-algebra such that Cuntz comparison in A is unitarily implemented, let $a, b \in A_+$, and let $\varepsilon > 0$. If $\|a - b\| < \varepsilon$, then $[(a-\varepsilon)_+] \ll_\omega [b]$ in $A_+/\sim$.*

Proof. Choose $\varepsilon' > 0$ with $\|a-b\| < \varepsilon' < \varepsilon$. We have $(a-\varepsilon)_+ = ((a-\varepsilon')_+ - (\varepsilon-\varepsilon'))_+$. Using this at the first step, using Lemma 3.14 at the second step, and using Paragraph 2.27 at the last step, we obtain that

$$[(a-\varepsilon)_+] = [((a-\varepsilon')_+ - (\varepsilon-\varepsilon'))_+] \ll_\omega [(a-\varepsilon')_+] \leq [b],$$

as desired. □

The following result has appeared as [APS11, Lemma 1.3].

Proposition 3.33. *Let A be a separable C^*-algebra. Then* $\mathrm{Cu}(A)$ *is countably based. It follows that* $\mathrm{Cu}(A)$ *is a domain.*

Proof. We may assume that A is stable. Then Cuntz comparison in A is unitarily implemented. Let M be a countable, dense subset of A_+. Set

$$B := \{[(a-\varepsilon)_+] : a \in M, \varepsilon \in \mathbb{Q}_+\}.$$

Then B is a countable subset of $\mathrm{Cu}(A)$. To show that B has the desired properties, let $s', s \in \mathrm{Cu}(A)$ satisfy $s' \ll_\omega s$. Choose $a \in A_+$ with $s = [a]$. By Lemma 3.14, we can choose $\varepsilon > 0$ such that $s' \leq [(a-\varepsilon)_+]$. We may assume that $\varepsilon \in \mathbb{Q}$.

Choose $b \in M$ with $\|a-b\| < \frac{\varepsilon}{2}$. We have

$$\|a - (b-\tfrac{\varepsilon}{2})_+\| \leq \|a-b\| + \|b-(b-\tfrac{\varepsilon}{2})_+\| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Using Lemma 3.32 at the second and third step, we deduce that

$$s' \leq [(a-\varepsilon)_+] \ll_\omega [(b-\tfrac{\varepsilon}{2})_+] \ll_\omega [a] = s.$$

The element $[(b-\frac{\varepsilon}{2})_+]$ belongs to B and therefore has the desired properties. □

4. The category W of pre-complete abstract Cuntz semigroups and inductive limits

In Section 3, we defined the functor $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$, from the category of C^*-algebras with *-homomorphisms to the category of abstract Cuntz semigroups (also called Cu-semigroups) and Cu-morphisms. For computations it is important to know how the functor behaves with respect to common constructions such as inductive limits, tensor products, passing to ideals or quotients. In this section, we will study the case of inductive limits. We will show that the category Cu has inductive limits (see Theorem 4.31), and that the functor $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$ preserves them (see Theorem 4.32), which means that an inductive system of C^*-algebras $(A_i, \varphi_{i,j})$ we have

$$\mathrm{Cu}(\varinjlim_i A_i) \cong \varinjlim_i \mathrm{Cu}(A_i).$$

4.1. Inductive limits in different categories.

4.1. Let $\mathcal{C}$ be a category. Recall that an *inductive system* $\mathcal{S}$ in $\mathcal{C}$ is a directed set I, together with a family $(A_i)_{i\in I}$ of objects in $\mathcal{C}$, and a family $(f_{i,j}\colon A_i \to A_j)_{i\leq j \in I}$ of morphisms in $\mathcal{C}$ such that the following coherence conditions are satisfied:

(1) We have $f_{i,i} = \mathrm{id}_{A_i}$, for each $i \in I$.
(2) We have $f_{i,k} = f_{j,k} \circ f_{i,j}$ whenever $i \leq j \leq k$ in I.

Given an object X in $\mathcal{C}$, a *morphism* from $\mathcal{S}$ to X is a family $(g_i\colon A_i \to X)_{i\in I}$ of morphisms in $\mathcal{C}$ such that $g_j \circ f_{i,j} = g_i$ whenever $i \leq j$ in I. An *inductive limit* of $\mathcal{S}$ is an object X in $\mathcal{C}$ together with a morphism $(g_i)_{i\in I}$ from $\mathcal{S}$ to X such that the following universal property holds: For every object Y and for every morphism $(h_i)_{i\in I}$ from $\mathcal{S}$ to Y, there exists a unique morphism $h\colon X \to Y$ such that $h_i = h \circ g_i$ for each $i \in I$. If it exists, the inductive limit of $\mathcal{S}$ is unique up to isomorphism and we denote it by $\varinjlim(A_i, f_{i,j})$, or simply $\varinjlim A_i$.

Very often, the construction of inductive limits in a specific category is based on the inductive limit of the underlying sets. We call the inductive limit of sets the *algebraic inductive limit.* Let us recall some details.

Example 4.2. Let $\mathcal{S} = ((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ be an inductive system of sets. We let $\bigsqcup_{i\in I} S_i$ denote the disjoint union. Define a relation $\sim$ on $\bigsqcup_{i\in I} S_i$, by setting $a \sim b$ for $a \in S_i$ and $b \in S_j$ if and only if there exists $k \geq i, j$ such that $\varphi_{i,k}(a) = \varphi_{j,k}(b)$. It is easy to check that $\sim$ is an equivalence relation. We set

$$\operatorname{alg-}\varinjlim S_i := \left(\bigsqcup_{i\in I} S_i \right) / \sim .$$

Given $a \in S_i$, we denote the equivalence class of a by $[a]$.

For each $i \in I$, we let $\varphi_{i,\infty} \colon S_i \to \operatorname{alg-}\varinjlim S_i$ be given by $\varphi_{i,\infty}(a) = [a]$ for $a \in S_i$. Then the family $(\varphi_{i,\infty})_{i\in I}$ is a morphism from $\mathcal{S}$ to $\operatorname{alg-}\varinjlim S_i$. It is easy to check that this is the inductive limit in the category of sets.

4.3. If $\mathcal{C}$ is a category whose objects are sets with operations (such as addition, multiplication, involution), and whose morphisms are maps that preserve all operations, then inductive limits in $\mathcal{C}$ are easy to describe: It is simply the algebraic inductive limit with induced operations.

Consider for example the category of semigroups: The objects are sets with one associative, binary operation (call it multiplication), and the morphisms are maps that preserve multiplication. Given an inductive system $\mathcal{S} = ((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ of semigroups, we let $S := \operatorname{alg-}\varinjlim S_i$ be the algebraic inductive limit. We define a binary operation on S by setting

$$[a][b] := [\varphi_{i,k}(a)\varphi_{j,k}(b)],$$

for any $a \in S_i$, $b \in S_j$ and any $k \geq i, j$. It is easy to check that this gives S the structure of a semigroup and that for each i the map $\varphi_{i,\infty} \colon S_i \to S$ is multiplicative. It follows that S is the inductive limit in the category of semigroups.

Analogous, one obtains inductive limits in the categories of commutative semigroups, of (commutative) monoids, of (commutative) groups, of rings, and many more: They are simply the algebraic inductive limit endowed with a natural structure as a commutative semigroup, (commutative) monoid, and so on. The same idea also works for the category of vector spaces over a field K (the morphisms are K-linear maps), and more generally for the category of modules over a ring.

The situation is different in categories whose objects carry a metric for which they are supposed to be complete, such as the category of C^*-algebras. In this case, the inductive limit is usually a certain completion of the algebraic inductive limit. Let us make this precise for the category of C^*-algebras.

Example 4.4. Let $((A_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ be an inductive system in C^*. The algebraic limit $\operatorname{alg-}\varinjlim A_i$ is a complex *-algebra. We define a pseudo-norm on $\operatorname{alg-}\varinjlim A_i$, by setting

$$\|a\| := \lim_{k\geq i} \|\varphi_{i,k}(a)\| = \inf_{k\geq i} \|\varphi_{i,k}(a)\|,$$

for $a \in A_i$. We let $C^*\text{-}\varinjlim A_i$, or simply $\varinjlim A_i$, denote the completion of $\operatorname{alg-}\varinjlim A_i$ with respect to this pseudo-norm. (That is, we first take the quotient by the subspace of elements of pseduo-norm zero, and then complete.) Then $C^*\text{-}\varinjlim A_i$ is a C^*-algebra. For each i, the map $A_i \to \operatorname{alg-}\varinjlim A_i$ to the algebraic limit induces a *-homomorphism $\varphi_{i,\infty} \colon A_i \to C^*\text{-}\varinjlim A_i$. This defines the inductive limit in the category C^*.

By definition, a C^*-algebra is called an *AF-algebra* (short for 'approximately finite-dimensional') if it is (isomorphic to) an inductive limit of finite-dimensional C^*-algebras.

Examples 4.5. (1) The C^*-algebra $\mathcal{K}$ of compact operators on a (separable) Hilbert space is an AF-algebra.

(2) If X is a compact, Hausdorff space, then $C(X)$ is an AF-algebra if and only if X is totally disconnected (equivalently, $\dim(X) = 0$).

Example 4.6. An interesting subclass of AF-algebras are the so-called *UHF-algebras* (short for 'uniformly hyperfinite'). Let $A_n := M_{2^n}$, and let $\varphi_{n,n+1} \colon A_n \to A_{n+1}$ be given by $\varphi_{n,n+1}(a) =$

$\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ for $a \in M_{2^n}$. We say that $\varphi_{n,n+1}$ is an embedding of multiplicity 2. The inductive limit $\varinjlim_n A_n$ is denoted by M_{2^∞} and is called the *UHF-algebra of type* 2^∞.

More generally, there is a UHF-algebra of type q, denote by M_q, for every supernatual number q. Let us recall some details. Let $p_0, p_1, p_2, \ldots$ be an enumeration of the prime numbers. Every natural number q can be written as a product $q = p_0^{m_0} p_1^{m_1} \ldots p_K^{m_K}$ for some $K \in \mathbb{N}$, and some exponents $m_k \in \mathbb{N}$. We call m_k the *multiplicity* of the prime p_k in q. Generalizing this factorization of natural numbers in powers of prime numbers, one defines a *supernatural number* q as a formal product

$$q = \prod_{k \in \mathbb{N}} p_k^{m_k},$$

where $m_k \in \{0, 1, 2, \ldots, \infty\}$ for each k. By definition, zero is not a supernatural number. The (formal) product of two supernatural numbers $q = \prod_k p_k^{m_k}$ and $r = \prod_k p_k^{n_k}$ is defined as $qr = \prod_k p_k^{m_k + n_k}$. Analogously, one can define the product of countably many (super)natural numbers.

We identify the nonzero natural numbers with the supernatural numbers of the form $\prod_{k \in \mathbb{N}} p_k^{m_k}$ where $\sum_{k \in \mathbb{N}} m_k < \infty$.

Let q be a supernatural number. We define the UHF-algebra of type q, denoted by M_q, as follows: If q is a natural number, then M_q is just the algebra of $q \times q$-matrices. Otherwise, choose a sequence $q_1, q_2, \ldots$ of prime numbers such that $q = \prod_k q_k$. For each n, set $A_n := M_{q_1} \otimes M_{q_2} \otimes \ldots \otimes M_{q_n}$. We have $A_{n+1} = A_n \otimes M_{q_{n+1}}$, which allows us to define $\varphi_{n,n+1} \colon A_n \to A_{n+1}$ by

$$\varphi_{n,n+1}(a) = a \otimes 1_{q_{n+1}} = \begin{pmatrix} a & 0 & \ldots & 0 \\ 0 & a & & \\ & & \ddots & \\ 0 & \ldots & 0 & a \end{pmatrix},$$

for $a \in A_n$. We set $M_q := \varinjlim_n A_n$. One can show that up to *-isomorphism the C^*-algebra M_q does not depend on the choice of the sequence $(q_k)_k$.

If q and r are supernatural numbers, then $M_q \otimes M_r \cong M_{qr}$.

If $q = q^2$, then each m_k is either 0 or ∞. In this case, we call M_q a UHF-algebra of infinite type. Such UHF-algebras have the property of being tensorially self-absorbing: They satisfy $M_q \otimes M_q \cong M_q$.

One goal of this section is to compute the Cuntz semigroup of UHF-algebras. Let us first recall how to compute their K_0-group and Murray-von Neumann semigroup.

4.7. Consider the UHF-algebra M_{2^∞}, obtained as the inductive limit of the C^*-algebras $A_n := M_{2^n}$, for $n \geq 1$, and with $\varphi_{n,n+1} \colon A_n \to A_{n+1}$ an embedding of multiplicity 2. For each n, we have $V(A_n) \cong \mathbb{N}$ and $K_0(A_n) \cong \mathbb{Z}$, by identifying a projection with its rank. Note that the map $\varphi_{n,n+1}$ doubles the rank of a projection. It follows that, after applying the above identifications, the map $V(\varphi_{n,n+1}) \colon \mathbb{N} \to \mathbb{N}$ is given by multiplication by 2. Similarly, the map $K_0(\varphi_{n,n+1}) \colon \mathbb{Z} \to \mathbb{Z}$ is given by $n \mapsto 2n$, for $n \in \mathbb{Z}$.

The inductive limit of the system

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \ldots$$

in the category of groups is the group $\mathbb{Z}[\frac{1}{2}]$ of dyadic numbers. Similarly, the inductive limit of the system

$$\mathbb{N} \xrightarrow{\cdot 2} \mathbb{N} \xrightarrow{\cdot 2} \mathbb{N} \to \ldots$$

in the category of monoids is the monoid $\mathbb{N}[\frac{1}{2}]$ of positive dyadic numbers.

We know that the K_0-group and the Murray-von Neumann semigroup are continuous functors, that is, they are compatible with inductive limits. We deduce that

$$K_0(M_{2^\infty}) \cong \mathbb{Z}[\tfrac{1}{2}], \quad \text{and} \quad V(M_{2^\infty}) \cong \mathbb{N}[\tfrac{1}{2}].$$

Next, let us consider the Cuntz semigroups. For each n, we have $A_n \otimes \mathcal{K} \cong \mathcal{K}$, and then $\mathrm{Cu}(A_n) \cong \overline{\mathbb{N}}$, with a finite number n in $\overline{\mathbb{N}}$ corresponding to the classes of projections in $A_n \otimes \mathcal{K}$ of rank n, and with ∞ in $\overline{\mathbb{N}}$ corresponding to the class of a strictly positive elements in $A_n \otimes \mathcal{K}$;

see Example 2.35. As above, the map $\varphi_{n,n+1}$ doubles the rank of a projection. Moreover, it maps a strictly positive element in $A_n \otimes \mathcal{K}$ to a strictly positive element in $A_{n+1} \otimes \mathcal{K}$. It follows that $\mathrm{Cu}(\varphi_{n,n+1})\colon \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ is given by $n \mapsto 2n$, for $n \in \overline{\mathbb{N}}$, with the convention that $2\infty = \infty$.

The algebraic inductive limit can be computed as

$$\text{alg}-\varinjlim\left(\overline{\mathbb{N}} \xrightarrow{\cdot 2} \overline{\mathbb{N}} \xrightarrow{\cdot 2} \overline{\mathbb{N}} \to \dots\right) \cong \mathbb{N}[\tfrac{1}{2}] \sqcup \{\infty\}.$$

There is a natural structure of $\mathbb{N}[\frac{1}{2}] \sqcup \{\infty\}$ as a positively ordered monoid, with the usual order and addition in $\mathbb{N}[\frac{1}{2}]$, and with ∞ as largest element that satisfies $s + \infty = \infty$ for every other element s. However, $\mathbb{N}[\frac{1}{2}] \sqcup \{\infty\}$ is not a Cu-semigroup. For instance, it does not satisfy (O1). (Exercise: Find an increasing sequence that does not have a supremum.)

The 'problem' is that Cu-semigroups are not just 'algebraic' objects since (O1) is an order-theoretic analogue of completeness. Therefore, the inductive limit in Cu is not simply the algebraic inductive limit, but a suitable completion.

4.2. **Predomains and the round ideal completion.** The material in this subsection is inspired by Keimel, [Kei16]. In particular, the (non-sequential versions of) Definition 4.10 and Theorem 4.16 appeared in [Kei16, Section 2.1].

We will show that the category $\texttt{DOM}_{\ll}$ has inductive limits. To obtain this, we will introduce the category $\texttt{PreDOM}$ of pre-complete domains (called predomains) and show that there is a well-behaved completion functor $\texttt{PreDOM} \to \texttt{DOM}_{\ll}$.

Definition 4.8. Let S be a set together with a binary relation $\prec$ on S. We say that $\prec$ has the *interpolation property* if for every finite subset $F \subseteq S$ and $a \in S$ with $F \prec a$ there exists $a' \in S$ such that $F \prec a' \prec a$.

As for the way-below relation, we write $\lightning s := \{x : x \prec s\}$; see Notation 3.11.

Remark 4.9. We think of the relation $\prec$ as an abstraction of the way-below relation. If $a' \prec a$, then we think of a' as an 'approximation' of a. If $x \prec y \prec a$, then we consider y a 'better approximation' of a than x. The interpolation property means that if we have finitely many elements approximating a, then there is an approximation of a that is better than each element in the finite set. Another way of rephrasing the interpolation property is to say that for every $a \in S$, the set $\lightning a$ is upward directed.

Definition 4.10. A *predomain* is a set S together with a binary relation $\prec$ that is transitive and satisfies the interpolation property. A *ω-predomain* is a predomain $(S, \prec)$ such that for every $s \in S$, there is a $\prec$-increasing sequence $(s_n)_n$ that is cofinal in $\lightning s$, that is, we have $s_1 \prec s_2 \prec s_3 \prec \ldots \prec s$ and for every s' with $s' \prec s$ there is n such that $s' \prec s_n$.

Let $f\colon S \to T$ be a map between two (ω-)predomains S and T. We say that f is *continuous* if for all $t \in T$ and $s \in S$ with $t \prec f(s)$ there exists $s' \in S$ such that $s' \prec s$ and $t \prec f(s')$. We say that f is *open* if $x \prec y$ implies $f(x) \prec f(y)$, for all $x, y \in S$.

We let $\texttt{PreDOM}$ denote the category whose objects are predomains, and whose morphisms are open, continuous maps. Analogously, we let $\texttt{PreDOM}_\omega$ denote the category whose objects are ω-predomains, and whose morphisms are open, continuous maps.

4.11. Let S be a domain. Then the way-below relation $\ll$ on S is transitive and satisfies the interpolation property. Thus, to every domain S we may associated the predomain $(S, \ll)$.

We define a functor $\iota\colon \texttt{DOM}_{\ll} \to \texttt{PreDOM}$ as follows: A domain S is mapped to the predomain $(S, \ll)$. A morphism $f\colon S \to T$ in $\texttt{DOM}_{\ll}$ is mapped to the same map, considered as a mapping from $(S, \ll)$ to $(T, \ll)$. The next result shows that this functor is well-defined and that it embeds $\texttt{DOM}_{\ll}$ is a full subcategory of $\texttt{PreDOM}$; see Proposition 4.13.

Lemma 4.12. *Let S and T be domains, and let $f\colon S \to T$ be an order-preserving map. Then f is a morphism in $\texttt{DOM}$ (that is, f preserves suprema of upward-directed sets) if and only if for every $s \in S$ and every $t \in T$ with $t \ll f(s)$ there exists $s' \in S$ with $s' \ll s$ and $t \leq \varphi(s')$.*

Moreover, f is a morphism in $\mathtt{DOM}_{\ll}$ if and only if $f\colon (S,\ll) \to (T,\ll)$ is a morphism in $\mathtt{PreDOM}$.

Similarly, an order-preserving map $f\colon S \to T$ between ω-domains is a morphism in $\mathtt{DOM}_\omega$ (that is, f preserves suprema of increasing sequences) if and only if for every $s \in S$ and every $t \in T$ with $t \ll_\omega f(s)$ there exists $s' \in S$ with $s' \ll_\omega s$ and $t \leq \varphi(s')$. Moreover, f is a morphism in $\mathtt{DOM}_{\ll,\omega}$ if and only if $f\colon (S,\ll) \to (T,\ll)$ is a morphism in $\mathtt{PreDOM}_\omega$.

Proof. We only prove the sequential versions. The non-sequential statements are proven analogously.

Assume first that f preserves suprema of increasing sequences. Let $s \in S$ and $t \in T$ satisfy $t \ll_\omega f(s)$. Choose a $\ll_\omega$-increasing sequence $(s_n)_n$ in S with supremum s. Then

$$t \ll_\omega f(s) = f(\sup_n s_n) = \sup_n f(s_n).$$

By definition of $\ll_\omega$, we can choose n with $t \leq f(s_n)$. Set $s' := s_n$, which has the desired properties.

To show the converse implication, let $(s_n)_n$ be an increasing sequence in S with supremum s. For each n, we have $f(s_n) \leq f(s)$ and therefore $\sup_n f(s_n) \leq f(s)$. Choose a $\ll_\omega$-increasing sequence $(t_m)_m$ in T with supremum $f(s)$. Given m, we have $t_m \ll_\omega f(s)$, which implies that there is $s' \in S$ with $s' \ll_\omega s$ and $t_m \leq f(s')$. Then there is k (depending on s', and hence also m) such that $s' \leq s_k$. It follows that

$$t_m \leq f(s') \leq f(s_k) \leq \sup_n f(s_n).$$

Since this holds for every m, we obtain that

$$f(s) = \sup_m t_m \leq \sup_n f(s_n),$$

as desired.

Lastly, it is straightforward to deduce that f is a $\mathtt{DOM}_{\ll,\omega}$-morphism if and only if $f\colon (S,\ll) \to (T,\ll)$ is a $\mathtt{PreDOM}_\omega$-morphism. ☐

Proposition 4.13. *The functor $\iota\colon \mathtt{DOM}_{\ll} \to \mathtt{PreDOM}$ from Paragraph 4.11 embeds $\mathtt{DOM}_{\ll}$ is a full subcategory of $\mathtt{PreDOM}$. Similarly, $\mathtt{DOM}_{\ll,\omega}$ is a full subcategory of $\mathtt{PreDOM}_\omega$.*

Next, we define a completion functor $\mathtt{PreDOM} \to \mathtt{DOM}_{\ll}$.

Definition 4.14. Let S be a set with a binary relation $\prec$. A *round ideal* in S is a subset $J \subseteq S$ satisfying the following two conditions:

(1) J is upward-directed with respect to $\prec$, that is, for every finite subset $F \subseteq J$ there is $x \in J$ such that $F \prec x$.
(2) J is downward-hereditary with respect to $\prec$, that is, for every $x, y \in S$ with $x \prec y$ and $y \in J$, we have $x \in J$.

A *round ω-ideal* in S is a round ideal $J \subseteq S$ that moreover contains a $\prec$-increasing cofinal sequence.

Remark 4.15. Let S be a subset with a transitive binary relation $\prec$. Then S is a (ω-)predomain if and only if $\downarrow\!\!s$ is a round (ω-)ideal for every $s \in S$.

The non-sequential version of following result has appeared as [Kei16, Proposition 2.4]; see also [APT14, Proposition 3.1.6].

Theorem 4.16. *Let S be a predomain. Then the collection $\mathfrak{RI}(S)$ of all round ideals in S, ordered by inclusion, is a domain. Given $I, J \in \mathfrak{RI}(S)$, we have $I \ll J$ if and only if there exists $b \in J$ such that $I \subseteq \downarrow\!\!b$.*

Similarly, if S is a ω-predomain, then the collection $\mathfrak{RI}_\omega(S)$ of all round ω-ideals in S, ordered by inclusion, is a ω-domain. Given $I, J \in \mathfrak{RI}_\omega(S)$, we have $I \ll_\omega J$ if and only if there exists $b \in J$ such that $I \subseteq \downarrow^\omega b$.

Proof. 1. Let us verify that $\mathfrak{RI}(S)$ is a **dcpo**. Let $D \subseteq \mathfrak{RI}(S)$ be an upward-directed subset. We claim that the union $K := \bigcup D$ is a round ideal which then clearly is the supremum of D in $\mathfrak{RI}(S)$. To show that K is upward-directed, let $F \subseteq K$ be a finite subset. Each element of F belongs to a member of D. Since D is upward-directed, there exists $I \in D$ such that $F \subseteq I$. Since I is a round ideal and thus upward-directed, there is $x \in I$ with $F \prec x$. The element x also belongs to K and has the desired properties.

To show that K is downward-hereditary, let $x, y \in S$ with $x \prec y \in K$. Choose $I \in D$ with $y \in I$. Since I is a round ideal and thus downward-hereditary, we deduce $x \in I$, and consequently $x \in K$, as desired.

2. Let $I, J \in \mathfrak{RI}(S)$. We show that $I \ll J$ if and only if there exists $b \in J$ such that $I \subseteq \downarrow b$.

In one direction, assume that there is $b \in J$ with $I \subseteq \downarrow b$. To verify $I \ll J$, let $D \subseteq \mathfrak{RI}(S)$ be upward-directed with $J \leq \bigvee D$. This means that $J \subseteq \bigcup D$. Thus, we can choose $K \in D$ with $b \in K$. It follows that $I \subseteq K$, as desired.

For the other direction, assume that $I \ll J$. Since S is a predomain, for each $s \in S$ the subset $\downarrow s$ is a round ideal. Set

$$D := \{\downarrow s : s \in J\}.$$

Then D is a collection of round ideals. Using that J is upward-directed, it follows easily that D is upward-directed. Moreover, we clearly have $\bigcup D \subseteq J$. To show that $J \subseteq \bigcup D$, let $s \in J$. Using that J is upward-directed, we can choose $\tilde{s} \in J$ with $s \prec \tilde{s}$. Then $s \in (\downarrow \tilde{s}) \in D$, and consequently $s \in \bigcup D$. We obtain that

$$I \ll J = \bigvee D.$$

By definition of the way-below relation, there exists an element of D dominating I. Thus, there is $b \in J$ such that $I \subseteq \downarrow b$, as desired.

3. To show that $\mathfrak{RI}(S)$ is continuous, let $I \in \mathfrak{RI}(S)$. Let D_I denote the collection of round-ideals way-below I. It follows from the characterization of the way-below relation in $\mathfrak{RI}(S)$ that

$$D_I = \{J \in \mathfrak{RI}(S) : \exists b \in I, J \subseteq \downarrow b\}.$$

As in step 2, it follows that D_I is upward-directed with supremum I, as desired.

We leave the proof of the sequential version as an Exercise. □

Proposition 4.17. *Let S and T be predomains, and let $f \colon S \to T$ be a* `PreDOM`*-morphism. Then for each round ideal I in S, the set $\downarrow f(I)$ is a round ideal in T. The map $\mathfrak{RI}(f) \colon \mathfrak{RI}(S) \to \mathfrak{RI}(T)$ given by $\mathfrak{RI}(I) := \downarrow f(I)$, for $I \in \mathfrak{RI}(S)$, is a* $\mathtt{DOM}_{\ll}$*-morphism.*

Proof. Exercise. □

Definition 4.18. Given a predomain S, we call the domain $\mathfrak{RI}(S)$ from Theorem 4.16 the *round ideal completion* of S. Similarly, given a ω-predomain S, we call the ω-domain $\mathfrak{RI}_\omega(S)$ the *round ω-ideal completion* of S.

We define the functor $\mathfrak{RI} \colon \mathtt{PreDOM} \to \mathtt{DOM}_{\ll}$ by sending a predomain S to its round ideal completion, and by sending a `PreDOM`-morphism $f \colon S \to T$ to the $\mathtt{DOM}_{\ll}$-morphism $\mathfrak{RI}(f) \colon \mathfrak{RI}(S) \to \mathfrak{RI}(T)$ constructed Proposition 4.17.

Similarly, we obtain a functor $\mathfrak{RI}_\omega \colon \mathtt{PreDOM}_\omega \to \mathtt{DOM}_{\ll,\omega}$, that sends a ω-predomain S to its round ω-ideal completion.

The round (ω-)ideal completion has the following universal property:

Proposition 4.19. *Let S be a predomain. Then the map $\alpha_S \colon S \to \mathfrak{RI}(S)$, given by $\alpha_S(s) := \downarrow s$ for $s \in S$, is a morphism in* `PreDOM` *(that is, continuous and open). The map α_S is an isomorphism if and only if S is a domain.*

Moreover, given a domain Q and a `PreDOM`*-morphism $f \colon S \to Q$, there exists a unique* $\mathtt{DOM}_{\ll}$*-morphism $\hat{f} \colon \mathfrak{RI}(S) \to Q$ such that $f = \hat{f} \circ \alpha_S$. This means that the following diagram can be* *completed to be commutative:*

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & \mathfrak{RI}(S) \\ & \searrow^{f} & \downarrow{\scriptstyle \hat{f}} \\ & & Q \end{array}$$

Analogously, given a ω-predomain S, we obtain a universal $\mathtt{PreDOM}_\omega$*-morphism* $\alpha_S\colon S \to \mathfrak{RI}_\omega(S)$ *with the corresponding universal property with respect to ω-domains.*

Proof. We leave it as an exercise to show that α_S is a morphism in $\mathtt{PreDOM}$, and that α_S is an isomorphism if and only if S is a domain.

Let Q be a domain, and let $f\colon S \to Q$ be a $\mathtt{PreDOM}$-morphism. Consider the map $\mathfrak{RI}(f)\colon \mathfrak{RI}(S) \to \mathfrak{RI}(Q)$ from Proposition 4.17. We obtain a commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\alpha_S} & \mathfrak{RI}(S) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{RI}(f)} \\ Q & \xrightarrow[\cong]{\alpha_Q} & \mathfrak{RI}(Q) \end{array}$$

We set $\hat{f} := \alpha_Q^{-1} \circ \mathfrak{RI}(f)$. We leave it as an exercise to show that $\hat{f}$ has the desired properties. □

The next results shows that the functor $\mathfrak{RI}\colon \mathtt{PreDOM} \to \mathtt{DOM}_\ll$ from Definition 4.18 is left-adjoint to the inclusion $\iota\colon \mathtt{DOM}_\ll \to \mathtt{PreDOM}$ from Paragraph 4.11.

In category theory, a full subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is said to be a *reflective subcateogry* if the inclusion functor $\iota\colon \mathcal{C} \to \mathcal{D}$ has a left adjoint $\gamma\colon \mathcal{D} \to \mathcal{C}$, called the *reflector* or *reflection functor*. This means that for every object X in $\mathcal{D}$ and every object Y in $\mathcal{C}$, there is a natural bijection

$$\mathcal{D}(X, \iota(Y)) \cong \mathcal{C}(\gamma(X), Y).$$

Theorem 4.20. *The round ideal completion functor* $\mathfrak{RI}\colon \mathtt{PreDOM} \to \mathtt{DOM}_\ll$ *is left adjoint to inclusion of* $\mathtt{DOM}_\ll$ *as a full subcategory of* $\mathtt{PreDOM}$*, that is, given a predomain S and a domain T, there is a natural bijection*

$$\mathtt{PreDOM}(S, T) \cong \mathtt{DOM}_\ll(\mathfrak{RI}(S), T).$$

Let $\alpha_S\colon S \to \mathfrak{RI}(S)$ be the universal $\mathtt{PreDOM}$*-morphism from Proposition 4.19. Then the bijection of the above morphism sets is given by assigning to a* $\mathtt{DOM}_\ll$*-morphism $f\colon \mathfrak{RI}(S) \to T$ the* $\mathtt{PreDOM}$*-morphism $f \circ \alpha_S$.*

Analogously, the round ω-ideal completion functor $\mathfrak{RI}_\omega\colon \mathtt{PreDOM}_\omega \to \mathtt{DOM}_{\ll,\omega}$ *is left adjoint to inclusion of* $\mathtt{DOM}_{\ll,\omega}$ *as a full subcategory of* $\mathtt{PreDOM}_\omega$*.*

4.21. Let $\mathcal{S} = ((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ be an inductive system in $\mathtt{PreDOM}$. Let $S := \text{alg--}\varinjlim S_i$ be the algebraic inductive limit. We define a relation $\prec$ on S as follows: Given $a \in S_i$ and $b \in S_j$, we set $[a] \prec [b]$ if and only if there exists $k \geq i, j$ such that $\varphi_{i,k}(a) \prec \varphi_{j,k}(b)$ in S_k.

Lemma 4.22. *The category* $\mathtt{PreDOM}$ *has inductive limits.*

More precisely, let $\mathcal{S} = ((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ be an inductive system in $\mathtt{PreDOM}$*, let $S := \text{alg--}\varinjlim S_i$ be the algebraic inductive limit, and let $\prec$ be the relation on S defined in Paragraph 4.21. Then $(S, \prec)$ is a predomain, and for each i the map $\varphi_{i,\infty}\colon S_i \to S$ is a* $\mathtt{PreDOM}$*-morphism. Then $(S, \prec)$ together with the maps $(\varphi_{i,\infty})_i$ is the inductive limit of $\mathcal{S}$ in* $\mathtt{PreDOM}$*.*

Analogously, the category $\mathtt{PreDOM}_\omega$ *has inductive limits.*

Proof. Exercise. □

Let $\mathcal{C}$ be a full, reflective subcategory of a category $\mathcal{D}$, with reflector $\gamma\colon \mathcal{D} \to \mathcal{C}$. It is a basic result in category theory that functors that have a right adjoint preserve inductive limits. Thus, γ preserves inductive limits. Moreover, if $\mathcal{D}$ has inductive limits, then so does $\mathcal{C}$. Let us recall some details. Given an inductive system $\mathcal{S} = ((X_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ in $\mathcal{C}$, we consider the inductive limit $(Y, \psi_{i,\infty})$ in $\mathcal{D}$. Set $X := \gamma(Y)$, which is an object in $\mathcal{C}$. For each i, we apply γ to the $\mathcal{D}$-morphism $\varphi_{i_\infty} \colon X_i \to Y$ to obtain the $\mathcal{C}$-morphism

$$\varphi_{i,\infty} := \gamma(\psi_{i,\infty}) \colon X_i = \gamma(X_i) \to \gamma(Y) = X.$$

Then $(X, \varphi_{i,\infty})$ is the inductive limit of the system $\mathcal{S}$ in $\mathcal{C}$.

Applying this to the full, reflective subcategory $\mathtt{DOM}_\ll$ of $\mathtt{PreDOM}$, we obtain the following result:

Proposition 4.23. *The category* $\mathtt{DOM}_\ll$ *has inductive limits.*

More precisely, let $\mathcal{S} = ((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ *be an inductive system in* $\mathtt{DOM}_\ll$. *Let* $S := \mathtt{PreDOM}\text{-}\varinjlim S_i$ *together with* $(\varphi_{i,\infty})_i$ *be the inductive limit of* $\mathcal{S}$ *in* $\mathtt{PreDOM}$. *Let* $\mathtt{DOM}_\ll\text{-}\varinjlim S_i := \mathfrak{RI}(S)$ *be the round ideal completion of* S, *and let* $\alpha_S \colon S \to \mathtt{DOM}_\ll\text{-}\varinjlim S_i$ *be the universal* $\mathtt{PreDOM}$-*morphism from Proposition 4.19. Then* $\mathtt{DOM}_\ll\text{-}\varinjlim S_i$ *together with the maps* $(\alpha_S \circ \varphi_{i,\infty})_i$ *is the inductive limit of* $\mathcal{S}$ *in* $\mathtt{DOM}_\ll$.

Analogously, the category $\mathtt{DOM}_{\ll,\omega}$ *has inductive limits.*

4.3. **The category** W **of pre-complete abstract Cuntz semigroups.** In subsection 4.2, we considered predomains and we showed that the round ω-ideal completion defines a functor $\mathfrak{RI}_\omega \colon \mathtt{PreDOM}_\omega \to \mathtt{DOM}_{\ll,\omega}$. In this subsection, we enrich the situation by considering compatible additions on our objects. We define a category W of pre-complete Cu-semigroups, and we show that the round ω-ideal completion defines a functor $\gamma \colon \mathrm{W} \to \mathrm{Cu}$.

Definition 4.24. Let S be an abelian monoid, and let $\prec$ be a binary relation on S. We call $\prec$ *additive* if the following two conditions hold:

(1) We have $0 \prec s$, for every $s \in S$.
(2) If $s' \prec s$ and $t' \prec t$ for $s', s, t', t \in S$, then $s' + t' \prec s + t$.

A W-*semigroup* is an abelian monoid S together with a binary relation $\prec$ on S such that $\prec$ is additive, such that $(S, \prec)$ is a ω-predmonain, and such that the following continuity condition holds:

(1) If $u \prec s + t$, for $u, s, t \in S$, then there exist $s', t' \in S$ with $u \prec s' + t'$, $s' \prec s$ and $t' \prec t$.

Given two W-semigroups S and T, a W-*morphism* from S to T is a map $f \colon S \to T$ that preserves the monoid structure, and that is a $\mathtt{PreDOM}_\omega$-morphism (that is, open and continuous).

4.25. Let S be a W-semigroup. Let us define an addition on $\mathfrak{RI}_\omega(S)$. Given $I, J \in \mathfrak{RI}_\omega(S)$, we set

$$I + J := \{x \in S : x \prec a + b \text{ for some } a \in I, b \in J\}.$$

We claim that $I + J$ is a round ω-ideal. (Exercise.)

In [APT14], an alternative construction of the round ω-ideal completion $\mathfrak{RI}_\omega(S)$ is given, using equivalence classes of increasing $\prec$-increasing sequences in place of round ω-ideals.

The next result has appeared as [Kei16, Proposition 3.1], see also [APT14, Proposition 3.1.6].

Theorem 4.26. *Let* S *be a* W-*semigroup. Then the round* ω-*ideal completion* $\mathfrak{RI}_\omega(S)$ *from Theorem 4.16 is a* Cu-*semigroup.*

Proof. By Theorem 4.16, $\mathfrak{RI}_\omega(S)$ is a ω-domain. It remains to verify (O3) and (O4).

To show (O3), let $I',', J', J \in \mathfrak{RI}_\omega(S)$ satisfy $I' \ll_\omega I$ and $J' \ll_\omega J$. Use Theorem 4.16 to choose $x \in I$ and $y \in J$ such that $I' \subseteq \downarrow x$ and $J' \subseteq \downarrow y$. We claim that $I' + J' \subseteq \downarrow(x + y)$. To show the claim, let $s \in I' + J'$. By definition of $I' + J'$, we can choose $a \in I'$ and $b \in J'$ with $s \prec a + b$. Then $a \prec x$ and $b \prec y$. Using that $\prec$ is additive at the second step, we obtain that

$$s \prec a + b \prec x + y,$$

as desired. We have $x + y \in I + J$. Using Theorem 4.16 again, it follows that $I' + J' \ll_\omega I + J$, as desired.

To show (O4), let $I, J, K \in \mathfrak{RI}_\omega(S)$ satisfy $I \ll_\omega J + K$. Use Theorem 4.16 to choose $x \in J + K$ such that $I \subseteq \downarrow x$. Since $x \in J + K$, we can choose $a \in J$ and $b \in K$ such that

$x \prec a+b$. Applying the continuity condition for the addition in a W-semigroup, we obtain $a', b' \in S$ such that $x \prec a'+b'$, $a' \prec a$ and $b' \prec b$. Set $J' := \downarrow a'$ and $K' := \downarrow b'$. Then J' and K' are round ω-ideals. Moreover, applying Theorem 4.16, we have $J' \ll_\omega J$ and $K' \ll_\omega K$. We have $x \in J'+K'$ and therefore $I \ll_\omega J'+K'$, as desired. □

Proposition 4.27. *Let S and T be* W*-semigroups, and let $f\colon S \to T$ be a* W*-morphism. Then the map $\mathfrak{RI}_\omega(f)\colon \mathfrak{RI}_\omega(S) \to \mathfrak{RI}_\omega(T)$ from Proposition 4.17 is a* Cu*-morphism.*

Proof. To simplify notation, set $\tilde{f} := \mathfrak{RI}_\omega(f)$. It follows from Proposition 4.17 that $\tilde{f}$ is a $\mathtt{DOM}_\ll$-morphism. It is clear that $\tilde{f}$ preserves the zero element. To show that $\tilde{f}$ is additive, let $I, J \in \mathfrak{RI}_\omega(S)$. Set

$$I \hat{+} J := \{a+b : a \in I, b \in J\}$$

so that $I+J = \downarrow(I\hat{+}J)$. By definition of $\mathfrak{RI}_\omega(f)$ and the sum in the round ω-ideal completion, we have

$$\tilde{f}(I) + \tilde{f}(J) = \downarrow\left(\downarrow f(I) \hat{+} \downarrow f(J)\right),$$

and

$$\tilde{f}(I+J) = \downarrow f\left(\downarrow(I\hat{+}J)\right).$$

To show equality of these sets, let $t \in \tilde{f}(I)+\tilde{f}(J)$. Choose $a \in I$, $b \in J$, and $x, y \in T$ with $t \prec x+y$, $x \prec f(a)$, and $y \prec f(b)$. Using that f is continuous, choose $a', b' \in S$ with $x \prec f(a')$, $a' \prec a$, $y \prec f(b')$ and $b' \prec b$. It follows that $a+b \in I\hat{+}J$, and then $a'+b' \in \downarrow(I\hat{+}J)$. We deduce $t \prec f(a'+b')$, which verifies $t \in \tilde{f}(I+J)$.

Conversely, let $t \in \tilde{f}(I+J)$. Choose $a \in I$, $b \in J$ and $s \in S$ with $t \prec f(s)$ and $s \prec a+b$. Using that addition in S is continuous, choose $a', b' \in S$ with $a' \prec a$, $b' \prec b$ and $s \prec a'+b'$. Using that f is open, we deduce $f(a') \prec f(a)$ and $f(b') \prec f(b)$ and

$$t \prec f(s) \prec f(a'+b') = f(a') + f(b'),$$

which verifies $t \in \tilde{f}(I)+\tilde{f}(J)$, as desired. □

Recall that $\mathtt{DOM}_\ll$ is a full subcategory of $\mathtt{PreDOM}$. Similarly, Cu is a full subcategory of W. It follows from Theorem 4.26 and Proposition 4.27 that the round ω-ideal completion defines a functor $\gamma\colon \mathrm{W} \to \mathrm{Cu}$.

Theorem 4.28. *The round ω-ideal completion defines a functor $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ that is left adjoint to inclusion of* Cu *as a full subcategory of* W*, that is, given a* W*-semigroup S and a* Cu*-semigroup T, there is a natural bijection*

$$\mathrm{W}(S,T) \cong \mathrm{Cu}(\gamma(S),T).$$

Let $\alpha_S\colon S \to \gamma(S)$ be the universal $\mathtt{PreDOM}_\omega$-morphism from Proposition 4.19. Then α_S is a monoid map and therefore a W*-morphism. The bijection of the above morphism sets is given by assigning to a* Cu*-morphism $f\colon \gamma(S) \to T$ the* W*-morphism $f \circ \alpha_S$.*

Definition 4.29. Let $((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ be an inductive system in W. Let S denote the algebraic inductive limit $\mathrm{alg}-\varinjlim S_i$. We define an addition on S as follows: Given $a \in S_i$ and $b \in S_j$, choose $k \geq i, j$ and set

$$[a]+[b] := [\varphi_{i,k}(a) + \varphi_{j,k}(b)].$$

Lemma 4.30. *The category* W *has inductive limits.*

More precisely, let $\mathcal{S} = ((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ be an inductive system in W*, let $S := \mathrm{alg}-\varinjlim S_i$ be the algebraic inductive limit. Equip S with the relation $\prec$ from Paragraph 4.21 and with addition as defined in Definition 4.29. Then S is a* W*-semigroup, and for each i the map $\varphi_{i,\infty}\colon S_i \to S$ is a* W*-morphism. Then S together with the maps $(\varphi_{i,\infty})_i$ is the inductive limit of $\mathcal{S}$ in* W*.*

Theorem 4.31. *The category* Cu *has inductive limits.*

More precisely, let $\mathcal{S} = ((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ *be an inductive system in* Cu*. Let* $S := \mathrm{W} - \varinjlim S_i$ *together with* $(\varphi_{i,\infty})_i$ *be the inductive limit of* $\mathcal{S}$ *in* W*. Set* $\mathrm{Cu} - \varinjlim S_i := \gamma(S)$ *be the round* ω*-ideal completion of* S*, and let* $\alpha_S \colon S \to \mathrm{Cu} - \varinjlim S_i$ *be the universal* W*-morphism from Theorem 4.28. Then* $\mathrm{Cu} - \varinjlim S_i$ *together with the maps* $(\alpha_S \circ \varphi_{i,\infty})_i$ *is the inductive limit of* $\mathcal{S}$ *in* Cu*.*

The following result has fist appeared as [APT14, Corollary 3.2.9], to where we refer for a proof.

Theorem 4.32. *The functor* $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$ *preserves inductive limits.*

More precisely, let $\mathcal{A} = ((A_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ *be an inductive system in* C^**. Applying the functor* Cu *we obtain an inductive system* $\mathcal{S} = ((\mathrm{Cu}(A_i))_{i\in I}, (\mathrm{Cu}(\varphi_{i,j}))_{i,j\in I, i\leq j})$ *in* Cu*. Let* $A := \varinjlim A_i$ *together with* **-homomorphisms* $(\varphi_{i,\infty})_i$ *be the inductive limit of* $\mathcal{A}$ *in* C^**.*

For each i*, consider the* Cu*-morphism* $\mathrm{Cu}(\varphi_{i,\infty})\colon \mathrm{Cu}(A_i) \to \mathrm{Cu}(A)$*. The collection* $(\mathrm{Cu}(\varphi_{i,\infty})_i$ *defines a morphism from* $\mathcal{S}$ *to* $\mathrm{Cu}(A)$*. By universality, we obtain a* Cu*-morphism*

$$\varinjlim \mathrm{Cu}(A_i) \to \mathrm{Cu}(A) = \mathrm{Cu}(\varinjlim A_i).$$

This map is an isomorphism.

Example 4.33. Consider the UHF-algebra M_{2^∞}. To compute its Cuntz semigroup, we consider M_{2^∞} as the inductive limit of the inductive system with C^*-algebras $A_n := M_{2^n}$ and $\varphi_{n,n+1}\colon A_n \to A_{n+1}$ embeddings of multiplicity 2. We have

$$\mathrm{Cu}(M_{2^\infty}) \cong \mathrm{Cu} - \varinjlim \mathrm{Cu}(A_n).$$

As in Paragraph 4.7, we consider the inductive system

$$\overline{\mathbb{N}} \xrightarrow{\cdot 2} \overline{\mathbb{N}} \xrightarrow{\cdot 2} \overline{\mathbb{N}} \to \ldots,$$

in Cu. To compute its inductive limit in Cu, we have to first compute the inductive limit in W, and then apply the completion functor. The algebraic inductive limit is $\mathbb{N}[\frac{1}{2}] \sqcup \{\infty\}$, with the usual addition. The relation $\prec$ is given as follows: For $s, t \in \mathbb{N}[\frac{1}{2}]$ we have $s \prec t$ if and only if $s \leq t$. We have $s \prec \infty$ for every $s \in \mathbb{N}[\frac{1}{2}]$. Note that $\infty \not\prec \infty$.

Let us compute the round ω-ideal completion of $\mathbb{N}[\frac{1}{2}] \sqcup \{\infty\}$. For every nonzero $a \in \mathbb{N}[\frac{1}{2}]$ we a round ω-ideals K_a given by

$$K_a := \{x \in \mathbb{N}[\tfrac{1}{2}] : x \leq a\}.$$

In particular, $K_0 = \{0\}$.

Moreover, for every $a \in (0, \infty]$, we have a round ω-ideals S_a given by

$$S_a := \{x \in \mathbb{N}[\tfrac{1}{2}] : x < a\}.$$

In particular, $S_\infty = \mathbb{N}[\frac{1}{2}]$. Note that $\mathbb{N}[\frac{1}{2}] \cup \{\infty\}$ is not a round ω-ideal, since it is not upward-directed since there is no element s with $\infty \prec s$.

It is straightforward to verify that all round ω-ideals are of this form, that is, we have

$$\gamma\left(\mathbb{N}[\tfrac{1}{2}] \sqcup \{\infty\}\right) = \{K_a : a \in \mathbb{N}[\tfrac{1}{2}]\} \cup \{S_a : a \in (0, \infty]\}.$$

Note that $S_a \subseteq K_b$ if and only if $a \leq b$, but $K_a \subseteq S_b$ if and only if $a < b$. We have $K_a + K_b = K_{a+b}$ and $S_a + S_b = S_{a+b}$. However, $K_a + S_b = S_{a+b}$.

5. Cuntz semigroups of ideals and quotients

In the first part of this section, we develop the basic theory of ideals and quotients of Cu-semigroups. This is based on [APT14, Section 5.1]. Given an ideal J in a C^*-algebra A, we show that $\mathrm{Cu}(J)$ is naturally identified with an ideal in $\mathrm{Cu}(A)$; see Lemma 5.2. This establishes a natural order-isomorphism between the lattice of ideals in A and in $\mathrm{Cu}(A)$; see Theorem 5.4. Thus, the Cuntz semigroup of a C^*-algebra encodes its ideal lattice.

Moreover, we show that $\mathrm{Cu}(A/J)$ is naturally isomorphic with the quotient $\mathrm{Cu}(A)/\mathrm{Cu}(J)$; see Theorem 5.10. Thus, the Cuntz semigroup of a C^*-algebra encodes the the Cuntz semigroups of all ideals and quotients of A.

In the second part, we study to what extend the prime and primitive ideal space of a C^*-algebra is encoded in its ideal lattice (and hence in its Cuntz semigroup). This part is based on [GHK+03, Section V-4, p.408ff].

Definition 5.1. Let S be a Cu-semigroup. An *ideal* in S is a submonoid $I \subseteq S$ that is downward-hereditary (that is, $a \leq b \in I$ implies $a \in I$) and closed under passing to suprema of increasing sequences. We say that S is *simple* if it contains only the trivial ideals $\{0\}$ and I.

By an ideal of a C^*-algebra we always mean a closed, two-sided ideal. We use the symbol '$\lhd$' to indicate that a subset is an ideal.

Lemma 5.2. *Let A be a C^*-algebra, let $J \lhd A$ be an ideal, and let $\iota\colon J \to A$ denote the inclusion map. Then the induced Cu-morphism $\iota_*\colon \mathrm{Cu}(J) \to \mathrm{Cu}(A)$ is an order-embedding that identifies $\mathrm{Cu}(J)$ with the ideal in $\mathrm{Cu}(A)$ given by*

$$\big\{[x] \in \mathrm{Cu}(A) : x \in (J \otimes \mathcal{K})_+\big\}.$$

Proof. We may assume that J and A are stable. To show that ι_* is an order-embedding, let $x, y \in J_+$ satisfy $\iota_*([x]) \leq \iota_*([y])$. This means precisely that x is Cuntz-subequivalent to y in A. Since every ideal is a hereditary sub-C^*-algebra, it follows from Proposition 2.18 that x is Cuntz-subequivalent to y in J, as desired.

It follows that ι_* identifies $\mathrm{Cu}(J)$ with the set of classes in $\mathrm{Cu}(A)$ coming from positive elements in J. Since ι_* is a Cu-morphism, it follows that $\mathrm{Cu}(J)$ is identified with a submonoid of $\mathrm{Cu}(A)$ that is closed under passing to suprema of increasing sequences. To show that $\mathrm{Cu}(J)$ is downward-hereditary in $\mathrm{Cu}(A)$, let $x \in A_+$ and $y \in J_+$ satisfy $x \precsim y$. We need to verify that $x \in J$. By definition, there exists a sequence $(r_n)_n$ in A such that $x = \lim_n r_n y r_n^*$. Since $y \in J$ and since J is an ideal, we deduce that $r_n y r_n^* \in J$ for every n. Using that J is closed, it follows that $x \in J$, as desired. □

We let $\mathrm{Lat}(A)$ denote the lattice of ideals in a C^*-algebra A. Similarly, $\mathrm{Lat}(S)$ denotes the lattice of ideals in a Cu-semigroup S.

5.3. Let A be a C^*-algebra. Applying Lemma 5.2, we define a natural map

$$\Phi\colon \mathrm{Lat}(A) \to \mathrm{Lat}(\mathrm{Cu}(A)),$$

by sending an ideal $J \lhd A$ to the ideal $\mathrm{Cu}(J) \lhd \mathrm{Cu}(A)$.

Theorem 5.4. *Let A be a C^*-algebra. Then the map $\Phi\colon \mathrm{Lat}(A) \to \mathrm{Lat}(\mathrm{Cu}(A))$ as defined in Paragraph 5.3 is an order-isomorphism. Given an ideal $I \lhd \mathrm{Cu}(A)$, we have*

$$\Phi^{-1}(I) = \big\{x \in A : [xx^*] \in I\big\}.$$

Proof. Define $\Psi\colon \mathrm{Lat}(\mathrm{Cu}(A)) \to \mathrm{Lat}(A)$ by $\Psi(I) := \{x \in A : [x^*x] \in I\}$, for $I \lhd \mathrm{Cu}(A)$. We claim that Ψ is well-defined, order-preserving, and that Ψ is the inverse of Φ.

To show that Ψ is well-defined, let $I \lhd \mathrm{Cu}(A)$. Set $J := \Psi(I)$. To show that J is closed under addition, let $x, y \in J$. Using at the last step that $xy^* + yx^* \leq xx^* + yy^*$ (since $0 \leq (x-y)(x-y)^*$), we deduce that

$$(x+y)(x+y)^* = xx^* + xy^* + yx^* + yy^* \leq 2xx^* + 2yy^*,$$

which implies that

$$[(x+y)(x+y)^*] \leq [2xx^* + 2yy^*] \leq [xx^*] + [yy^*].$$

Using that I is a downward-hereditary subomonoid, it follows that $[(x+y)(x+y)^*] \in I$, and consequently $x + y \in J$, as desired.

It is straightforward to verify that J is closed under scalar multiplication. To show that J is a (not necessarily closed) ideal, let $x \in J$ and $y \in A$. Then

$$(xy)(xy)^* = xyy^*x^* \precsim yy^*, \quad \text{and} \quad (yx)(yx)^* = yxx^*y^* \sim x^*y^*yx \precsim y^*y \sim yy^*.$$

Using that I is a downward-hereditary, we deduce that $[(xy)(xy)^*]$ and $[(xy)(xy)^*]$ belong to I, which implies that $xy, yx \in J$, as desired.

To show that J is closed, let $(x_n)_n$ be a sequence in J converging to x in A. Then the sequence $(x_nx_n^*)_n$ converges to xx^*. Let $\varepsilon > 0$. Choose n such that $\|x_nx_n^* - xx^*\| < \varepsilon$. Using Paragraph 2.27, we obtain that

$$(xx^* - \varepsilon)_+ \precsim x_nx_n^*.$$

Since I is downward-hereditary, and since $[x_nx_n^*] \in I$, it follows that $[(xx^* - \varepsilon)_+] \in I$. We have

$$[xx^*] = \sup_k [(xx^* - \tfrac{1}{k})_+].$$

Using that I is closed under passing to suprema of increasing sequences, we deduce that $[xx^*] \in I$, and thus $x \in J$, as desired. This show that $J \lhd A$.

It is easy to verify that Φ and Ψ are order-preserving.

To verify that $\Phi \circ \Psi = \mathrm{id}$, let $I \lhd \mathrm{Cu}(A)$ and set $J := \Psi(I)$. Given $x \in A_+$, we have $[x] \in \Phi(J)$ if and only if $x \in J_+$, if and only if $[xx^*] \in I$. Using that $[x] = [xx^*]$, we obtain the desired equality.

To verify that $\Psi \circ \Phi = \mathrm{id}$, let $J \lhd A$ and set $I := \Phi(J)$. To show that $J \subseteq \Psi(I)$, let $x \in J$. Then $xx^* \in J$, which implies $[xx^*] \in I$, and consequently $x \in \Psi(I)$, as desired. To show that $\Psi(I) \subseteq J$, let $x \in \Psi(I)$. Then $[xx^*] \in I = \mathrm{Cu}(J)$. Thus, there exists $y \in J_+$ with $xx^* \sim y$. In particular, $xx^* \precsim y$. Using that y belongs to the ideal J, we obtain as above that $xx^* \in J$. This implies that $x \in J$, as desired. □

Corollary 5.5. *A C^*-algebra is simple if and only if its Cuntz semigroup is.*

Definition 5.6. Let S be a Cu-semigroup, and let $I \lhd S$ be an ideal. Given $a, b \in S$, we set $a \leq_I b$ if there exists $x \in I$ with $a \leq b + x$. We set $a \sim_I b$ if $a \leq_I b$ and $b \leq_I a$.

5.7. Let S be a Cu-semigroup, and let $I \lhd S$ be an ideal. Then $\sim_I$ is an equivalence relation on S. Given $a \in S$, we denote its class in S/I by a_I. We define an addition on S/I by setting $a_I + b_I := (a + b)_I$, for $a, b \in S$. We define an order on S/I by setting $a_I \leq b_I$ if and only if $a \leq_I b$, for $a, b \in S$.

Lemma 5.8 ([APT14, Lemma 5.1.2, p.37f])**.** *Let S be a* Cu*-semigroup, and let $I \lhd S$ be an ideal. Then the order and addition on S/I as defined in Paragraph 5.7 give S/I the structure of a* Cu*-semigroup. Moreover, the map $\pi_I \colon S \to S/I$ given by $\pi_I(a) := a_I$, for $a \in S$, is a surjective* Cu*-morphism.*

Lemma 5.9. *Let A be a C^*-algebra, let $J \lhd A$ be an ideal, and let $\pi \colon A \to A/J$ be the quotient *-homomorphism. Then for every $y \in (A/J)_+$ there exists $x \in A_+$ with $\pi(x) = y$.*

Proof. Let $y \in (A/J)_+$. Choose $y_0 \in A/J$ with $y = y_0y_0^*$. Since π is surjective, we can choose $x_0 \in A$ with $\pi(x_0) = y_0$. Then $x := x_0x_0^*$ is positive and satisfies $\pi(x) = y$, as desired. □

Theorem 5.10. *Let A be a C^*-algebra, let $J \lhd A$ be an ideal, and let $\pi \colon A \to A/J$ be the quotient *-homomorphism. Consider the induced* Cu*-morphism $\pi_* \colon \mathrm{Cu}(A) \to \mathrm{Cu}(A/J)$.*

Given $a, b \in \mathrm{Cu}(A)$, we have $\pi_*(a) \leq \pi_*(b)$ if and only if $a \leq_{\mathrm{Cu}(J)} b$. Moreover, π_* is surjective. Thus, π_* induces an order-isomorphism $\widehat{\pi_*} \colon \mathrm{Cu}(A)/\mathrm{Cu}(J) \to \mathrm{Cu}(A/J)$.

Proof. We may assume that J, A and A/J are stable. To show that π_* is surjective, let $b \in \mathrm{Cu}(A/J)$. Choose $y \in (A/J)_+$ with $b = [y]$. Applying Lemma 5.9, choose $x \in A_+$ with $\pi(x) = y$. Then $a := [x] \in \mathrm{Cu}(A)$ satisfies $\pi_*(a) = b$, as desired.

Next, let $a, b \in \mathrm{Cu}(A)$ satisfying $a \leq_{\mathrm{Cu}(J)} b$. Choose $c \in \mathrm{Cu}(J)$ with $a \leq b + c$. It is easy to see that $\pi_*(c) = 0$. Using that π_* is a Cu-morphism and therefore order-preserving and additive, we deduce that

$$\pi_*(a) \leq \pi_*(b + c) = \pi_*(b) + \pi_*(c) = \pi_*(b),$$

as desired.

Conversely, let $a, b \in \mathrm{Cu}(A)$ satisfy $\pi_*(a) \leq \pi_*(b)$. Choose $x, y \in A_+$ with $a = [x]$ and $b = [y]$. Then $\pi(x) \precsim \pi(y)$ in $(A/J)_+$. Let $n \geq 1$. Using Rørdam's Lemma, we can choose $s_n \in A/J$ such that

$$\big(\pi(x) - \tfrac{1}{n}\big)_+ = s_n \pi(y) s_n^*.$$

Choose $r_n \in A$ with $\pi(r_n) = s_n$. Set $d_n := r_n y r_n^* - (x - \frac{1}{n})_+$. Using that π is a *-homomorphism, we have $\pi((x - \frac{1}{n})_+) = (\pi(x) - \frac{1}{n})_+$. We obtain that

$$\pi(d_n) = \pi(r_n y r_n^* - (x - \tfrac{1}{n})_+) = s_n \pi(y) s_n^* - \big(\pi(x) - \tfrac{1}{n}\big)_+ = 0.$$

Thus, the self-adjoint element d_n belongs to J.

Claim: Given $d \in J$ self-adjoint, there is $e \in J_+$ such that $d + e \geq 0$. Indeed, using functional calculus in J, we can find $d_-, d_+ \in J_+$ such that $d = d_+ - d_-$, and such that $d_+ d_- = 0$. We have $d + d_- = d_+ \geq 0$, which shows that $e = d_-$ has the desired properties.

Applying the claim to d_n, we obtain $e_n \in J_+$ such that $d_n + e_n \geq 0$. Then

$$\big(x - \tfrac{1}{n}\big)_+ \leq r_n y r_n^* + e_n$$

in A.

We have constructed $e_n \in J_+$ for each $n \geq 1$. Set $c := \sum_n [e_n]$, which belongs to $\mathrm{Cu}(J)$. Then, for each n, we have

$$[(x - \tfrac{1}{n})_+] \leq [r_n y r_n^*] + [e_n] \leq [y] + \sum_n [e_n] = b + c.$$

Using this at the last step, we deduce that

$$a = \sup_n [(x - \tfrac{1}{n})_+] \leq b + c,$$

as desired. □

Remark 5.11. Let A be a C^*-algebra. It follows from Theorem 5.4 that $\mathrm{Cu}(A)$ encodes the ideal lattice of A. Moreover, it follows from Lemma 5.2 and Theorem 5.10 that $\mathrm{Cu}(A)$ encodes the Cuntz semigroups of all ideals and quotients of A. This is in stark contrast to K-theory.

By Theorem 5.4, the Cuntz semigroup encodes the ideal lattice of a C^*-algebra. Next, we study to what extend we can recover the primitive and prime ideal space.

Definition 5.12. Let L be a complete lattice, and let $p \in L$. Then p is called *prime* if for all $a, b \in L$ with $a \wedge b \leq p$ we have $a \leq p$ or $b \leq p$. We let $\mathrm{Spec}(L)$ denote the set of prime elements in L different from the largest element 1.

Given $a \in L$, the *hull* of a is

$$\nabla_L(a) := \big\{p \in \mathrm{Spec}(L) : p \geq a\big\} = \mathrm{Spec}(L) \cap \uparrow a.$$

Further, we set

$$\triangle_L(a) := \mathrm{Spec}(L) \setminus \nabla_L(a) = \mathrm{Spec}(L) \setminus \uparrow a.$$

Lemma 5.13. *Let L be a complete lattice, and let $M \subseteq L$. Then*

$$\bigcup_{a \in M} \triangle_L(a) = \triangle_L(\sup M), \quad \textit{and} \quad \bigcap_{a \in M} \nabla_L(a) = \nabla_L(\sup M).$$

If M is finite, then

$$\bigcap_{a \in M} \triangle_L(a) = \triangle_L(\inf M), \quad \textit{and} \quad \bigcup_{a \in M} \nabla_L(a) = \nabla_L(\inf M).$$

Moreover, $\nabla_L(0) = \triangle_L(1) = \mathrm{Spec}(L)$ and $\nabla_L(1) = \triangle_L(0) = \emptyset$.

Proof. Exercise. □

5.14. Let L be a complete lattice. It follows from Lemma 5.13 that the sets $\triangle_L(a)$, for $a \in L$, form a topology on $\mathrm{Spec}(L)$, called the the *hull-kernel topology.* Note that the closed sets are precisely $\nabla_L(a)$, for $a \in L$. The closure of a subset $M \subseteq \mathrm{Spec}(L)$ is given as

$$\overline{M} = \big\{p \in \mathrm{Spec}(L) : p \geq \inf M\big\}.$$

Definition 5.15. Let X be a topological space. A nonempty subset $M \subseteq X$ is called *irreducible* if for all closed subsets $C, D \subseteq X$ with $M \subseteq C \cup D$ we have $M \subseteq C$ or $M \subseteq D$. The space X is called *sober* if for every closed irreducible subset $C \subseteq X$ there exists a unique $x \in X$ such that $C = \overline{\{x\}}$.

Remark 5.16. A nonempty, closed subset $M \subseteq X$ is irreducible if and only if $X \setminus M$ is prime in the complete lattice $\mathcal{O}(X)$ of open sets.

Every sober space is T_0, and every Hausdorff space is sober.

Exercise 5.17. Let X be a topological space, and let $x \in X$. Show that $X \setminus \overline{\{x\}}$ is irreducible.

Proposition 5.18 ([GHK$^+$03, p.411])**.** *Let X be a topological space. Let $\xi_X : X \to \mathrm{Spec}(\mathcal{O}(X))$ be given by*

$$\xi_X(x) := X \setminus \overline{\{x\}},$$

for $x \in X$. Then ξ_X is a homeomorphism if and only if X is sober.

Remarks 5.19. (1) Proposition 5.18 shows that sober spaces can be recovered from their lattice of open sets. In particular, sober spaces X and Y are homeomorphic (denoted $X \cong Y$) if and only if the lattices $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are order-isomorphic.

(2) Let L be a complete lattice. Then $\mathrm{Spec}(L)$ is sober (see [GHK$^+$03, Proposition V-4.4, p.409]), and hence $\mathrm{Spec}(L) \cong \mathrm{Spec}(\mathcal{O}(\mathrm{Spec}(L)))$.

Definition 5.20. Let A be a C^*-algebra, and let $J \lhd A$ be an ideal with $J \neq A$. Then J is called *primitive* if there exists an irreducible representation π of A with $J = \ker(\pi)$. Further, J is called *prime* if for all ideals $K, L \lhd A$ with $K \cap L \subseteq J$, we have $K \subseteq J$ or $L \subseteq J$. We denote the set of all primitive and prime ideals in A by $\mathrm{Prim}(A)$ and $\mathrm{Prime}(A)$, respectively.

An ideal $J \lhd A$ with $J \neq A$ is prime in the sense of Definition 5.20 if and only if it is prime in the sense of Definition 5.12 (as an element in the complete lattice $\mathrm{Lat}(A)$). Thus, The following result is an immediate consequence of the definitions:

Lemma 5.21. *Let A be a C^*-algebra. Then* $\mathrm{Prime}(A) = \mathrm{Spec}(\mathrm{Lat}(A))$.

Remarks 5.22. (1) Let A be a C^*-algebra. In general, every primitive ideal is prime, that is, $\mathrm{Prim}(A) \subseteq \mathrm{Prime}(A)$. If A is separable, then the converse holds and we have $\mathrm{Prim}(A) = \mathrm{Prime}(A)$. However, it was shown by Weaver, [Wea03], that there exist nonseparable C^*-algebras with $\mathrm{Prim}(A) \subsetneqq \mathrm{Prime}(A)$.

(2) Let A be a C^*-algebra. The hull-kernel topology on $\mathrm{Prime}(A)$ (and its subset $\mathrm{Prime}(A)$) is defined as in Paragraph 5.14. In particular, given $M \subseteq \mathrm{Prime}(A)$, the closure of M in $\mathrm{Prime}(A)$ is defined as

$$\overline{M} := \Big\{J \in \mathrm{Prime}(A) : J \supseteq \bigcap M\Big\}.$$

Similarly, the the closure of $M \subseteq \mathrm{Prim}(A)$ in $\mathrm{Prim}(A)$ is $\overline{M} = \{J \in \mathrm{Prim}(A) : J \supseteq \bigcap M\}$.

We have

$$\mathcal{O}(\mathrm{Prim}(A)) \cong \mathcal{O}(\mathrm{Prime}(A)) \cong \mathrm{Lat}(A).$$

In particular, the prime ideal space $\mathrm{Prime}(A)$ is sober. Moreover, the primitive ideal space $\mathrm{Prim}(A)$ is sober if and only if every primitive ideal in A is prime.

Proposition 5.23. *Let A be a C^*-algebra. Then there is a natural homeomorphism*

$$\mathrm{Prime}(A) \cong \mathrm{Spec}(\mathrm{Lat}(\mathrm{Cu}(A))).$$

If A is separable, then $\mathrm{Prim}(A) \cong \mathrm{Spec}(\mathrm{Lat}(\mathrm{Cu}(A)))$.

Proof. By Theorem 5.4, the lattices $\mathrm{Lat}(A)$ and $\mathrm{Lat}(\mathrm{Cu}(A))$ are order-isomorphic. We obtain that

$$\mathrm{Prime}(A) = \mathrm{Spec}(\mathrm{Lat}(A)) \cong \mathrm{Spec}(\mathrm{Lat}(\mathrm{Cu}(A))).$$

If A is separable, then $\mathrm{Prim}(A) = \mathrm{Prime}(A)$. □

Corollary 5.24. *Let A and B be commutative C^*-algebras. Then $A \cong B$ if and only if $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$.*

Proof. The forward implication is clear. To show that backward implication, assume that $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$. Using this at the second step, using the Gelfand-Naimark theorem for A and B at the first and last step, and using Proposition 5.23 at the second and fourth step, we obtain that

$$A \cong C_0(\mathrm{Prime}(A)) \cong C_0(\mathrm{Spec}(\mathrm{Lat}(\mathrm{Cu}(A)))) \cong C_0(\mathrm{Spec}(\mathrm{Lat}(\mathrm{Cu}(B)))) \cong C_0(\mathrm{Prime}(B)) \cong B,$$

as desired. □

Let A and B be separable, nuclear C^*-algebras. It was shown by Kirchberg that $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ if and only if $A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$. Using this, we obtain the following result. (A C^*-algebra A is called $\mathcal{O}_2$-stable if $A \cong A \otimes \mathcal{O}_2$.)

Corollary 5.25 (Kirchberg)**.** *Let A and B be nuclear, stable, $\mathcal{O}_2$-stable C^*-algebras. Then $A \cong B$ if and only if $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$.*

6. Traces and Functionals on Cuntz semigroups

In this section, we will first show that there is a natural bijection between lower semicontinuous quasitraces on a C^*-algebra and functionals on its Cuntz semigroup; see Theorem 6.17.

Definition 6.1. Let A be a C^*-algebra. A *trace* on A is a function $\tau\colon A_+ \to [0,\infty]$ that is additive, that is homogeneous ($\tau(0) = 0$ and $\tau(ra) = r\tau(a)$ for $a \in A_+$ and $r \in (0,\infty)$) and that satisfies the trace property: $\tau(x^*x) = \tau(xx^*)$ for $x \in A$.

We say that a trace τ is *lower semicontinuous* if for every $t \in (0,\infty)$ the set $\tau^{-1}((t,\infty])$ is open, or equivalently, if for every converging sequence $(a_n)_n$ in A_+ we have $\liminf_n \tau(x) \geq \tau(\lim_n x_n)$.

We let $T(A)$ denote the set of all lower semicontinuous traces on A.

6.2. Let $\tau \in T(A)$. Set

$$K_\varphi := \big\{x \in A : \varphi(x^*x) = 0\big\}, \quad \text{and} \quad F_\varphi := \big\{x \in A : \varphi(x^*x) < \infty\big\}.$$

Then K_τ is a closed, two-sided ideal in A, and τ induces a lower semicontinuous trace on the quotient A/K_τ. Further, F_τ is a two-sided ideal in A that is usually not closed. We say that τ is *densely defined* if F_φ is dense in A.

Let $B \subseteq A$ be a sub-C^*-algebra with $B \subseteq F_\tau$. Then the restriction of τ to B_+ extends (uniquely) to a bounded, tracial functional $B \to \mathbb{C}$.

We first study lower semicontinuous traces on commutative C^*-algebras.

6.3. Let X be a locally compact, Hausdorff space, and let $\mathcal{B}(X)$ denote the family of Borel subsets of X. Recall that a *positive Borel measure* on X is a map $\mu\colon \mathcal{B}(X) \to [0,\infty]$ that is strict ($\mu(\emptyset) = 0$) and countably additive (given pairwise disjoint sets $E_1, E_2, \ldots \in \mathcal{B}(X)$, we have $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$).

Given $E \in \mathcal{B}(X)$, we say that μ is *inner regular at E* if

$$\mu(E) = \sup\big\{\mu(K) : K \subseteq E, K \text{ compact}\big\},$$

and we say that μ is *outer regular at E* if

$$\mu(E) = \inf\big\{\mu(U) : E \subseteq U, U \text{ open}\big\}.$$

Recall that a *Radon measure* on X is a positive Borel measure $\mu\colon \mathcal{B}(X) \to [0,\infty]$ that is finite on compact subsets and that is inner regular at all Borel sets. A Radon measure is automatically outer regular on compact subsets. For details on Radon measures, we refer to Arveson's notes, [Arv96].

Given a positive Borel measure μ on X, we define $\tau_\mu\colon C_0(X)_+ \to [0,\infty]$ by

$$\tau_\mu(f) := \int_X f d\mu = \int_0^\infty \mu\big(\{x \in X : f(x) \geq t\}\big) dt, \tag{6.1}$$

for $f \in C_0(X)_+$.

Recall that $C_c(X)$ denotes the (non-closed) ideal of compactly supported functions in $C_0(X)$. If μ is a Radon measure on X, then the restriction of τ_μ to $C_c(X)_+$ takes finite values. One version of the Riesz theorem states that this assignment is a one-to-one correspondence between Radon measures on X and positive linear functionals $C_c(X)_+ \to [0,\infty)$.

A Radon measure is finite (that is, $\mu(X) < \infty$) if and only if $\tau_\mu\colon C_0(X)_+ \to [0,\infty]$ takes finite values. Then τ_μ extends to a unique positive bounded functional $C_0(X) \to \mathbb{C}$. Therefore, another version of the Riesz theorem states that finite Radon measures on X correspond to positive bounded functionals $C_0(X) \to \mathbb{C}$.

For our purposes, we need to refrain from the assumption that μ is finite on compact sets. We therefore define: An *extended Radon measure* on X is a positive Borel measure $\mu\colon \mathcal{B}(X) \to [0,\infty]$ that is inner regular on all Borel sets, and that is outer regular on compact sets. We let $M_+(X)$ denote the set of all extended Radon measures on X.

Proposition 6.4 (Riesz theorem for extended Radon measures)**.** *Let X be a locally compact, Hausdorff space. Given an extended Radon measure μ, let $\tau_\mu\colon C_0(X)_+ \to [0,\infty]$ be defined as in* (6.1)*. Then τ_μ is a lower semicontinuous trace on $C_0(X)$.*

Moreover, the assignment $M_+(X) \to T(C_0(X))$, $\mu \mapsto \tau_\mu$, is a bijection.

Proof. Let $\mu \in M_+(X)$. It is straightforward to check that τ_μ is additive, homogeneous and order-preserving.

Claim 1: Let $f \in C_0(X)_+$. Then $\tau_\mu(f) = \sup_{\varepsilon>0} \tau_\mu((f-\varepsilon)_+)$.

Indeed, since $(f-\varepsilon)_+ \leq f$ for all $\varepsilon > 0$, and since τ_μ is order-preserving, we obtain that $\tau_\mu(f) \geq \sup_{\varepsilon>0} \tau_\mu((f-\varepsilon)_+)$. The converse inequality follows using that μ is inner regular. This proves the claim.

To show that τ_μ is lower semicontinuous, let $t \in [0,\infty)$, and set $U_t := \{f \in C_0(X)_+ : \tau_\mu(f) > t\}$. To show that U_t is open, let $f \in U_t$. Using claim 1, choose $\varepsilon > 0$ such that $\tau_\mu((f-\varepsilon)_+) > t$. Given $g \in C_0(X)_+$ with $\|g-f\| < \varepsilon$, we have $g \geq (f-\varepsilon)_+$ and consequently

$$\tau_\mu(g) \geq \tau_\mu((f-\varepsilon)_+) > t.$$

This shows that the open ball of radius ε around f belongs to U_t. It follows that U_t is open, as desired.

Thus, the map $\Phi\colon M_+(X) \to T(C_0(X))$, $\mu \mapsto \tau_\mu$, is well-defined.

To show that Φ is surjective, let $\tau\colon C_0(X)_+ \to [0,\infty]$ be a trace. We define a Borel measure μ on X in two stages: First, given a compact subset $K \subseteq X$, let $\mathbb{1}_K$ denote the characteristic function of K, and set

$$\mu(K) := \inf\big\{\tau(f) : f \in C_0(X)_+, f \geq \mathbb{1}_K\big\}.$$

Second, given a Borel subset $E \subseteq X$, set

$$\mu(E) := \sup\big\{\mu(K) : K \subseteq E, K \text{ compact }\big\}.$$

We omit the details to show that μ is a extended Radon measure.

We also leave it to the reader to show that Φ is injective. □

6.5. Let X be a locally compact, Hausdorff space, let $\mu \in M_+(X)$, and let $\tau \in T(C_0(X))$ be the associated lower semicontinuous trace. Set

$$F_\mu := \bigcup\{U : U \in \mathcal{O}(X), \mu(U) < \infty\}.$$

Then F_μ is an open subset of X, and the restriction of μ to F_μ is 'normal' Radon measure. (Indeed, if $K \subseteq F_\mu$ is compact, then K can be covered by finitely many subsets of finite measure, and consequently $\mu(K) < \infty$.) Given $x \in X$, we have $x \in F_\mu$ if and only if $\mu(\{x\}) < \infty$. (This follows using that $\{x\}$ is compact, and that μ is outer regular on compact sets.)

Recall the definition of F_τ from Paragraph 6.2. Then the open subset of X corresponding to F_τ is exactly F_μ, that is:

$$F_\tau = \big\{ f \in C_0(X)_+ : f(x) = 0 \text{ for } x \notin F_\mu \big\}.$$

Remark 6.6. Let X be a locally compact, Hausdorff space, let $\mu \in M_+(X)$, and let $\tau \in T(C_0(X))$ be the associated lower semicontinuous trace. Given $f, g \in C_0(X)_+$, recall that $f \precsim g$ in $C_0(X)$ if and only if $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$. Thus, Cuntz (sub)equivalence of functions in $C_0(X)_+$ is determined by the support.

Hence, the map $C_0(X)_+ \to [0,\infty]$, $f \mapsto \mu(\operatorname{supp}(f))$, respects Cuntz subequivalence and therefore induces a well-defined, order-preserving map $d_\tau\colon C_0(X)_+/\sim \to [0,\infty]$ by $d_\tau([f]) := \mu(\operatorname{supp}(f))$ for $f \in C_0(X)_+$.

For each $t \geq 0$, we have

$$\{x \in X : f(x) \geq t\} = \operatorname{supp}((f-t)_+).$$

Using (6.1), we can recover τ from d_τ as follows:

$$\tau(f) = \int_0^\infty \mu\big(\{x \in X : f(x) \geq t\}\big)dt = \int_0^\infty \mu\big(\operatorname{supp}((f-t)_+)\}\big)dt = \int_0^\infty d_\tau([(f-t)_+])dt.$$

This is the underlying idea for the correspondence between traces on C^*-algebras and functionals on its Cuntz semigroup; see Theorem 6.17.

Let A be a C^*-algebra, and let $\lambda\colon \operatorname{Cu}(A) \to [0,\infty]$ be a nice map. We define Generalizing this idea to the noncommutative setting, we want to associate to a map $\lambda\colon \operatorname{Cu}(A) \to [0,\infty]$ the map $\tau_\lambda\colon A_+ \to [0,\infty]$ given by

$$\tau_\lambda(a) := \int_0^\infty \lambda([(a-t)_+])dt,$$

for $a \in A_+$. It turns out that this works well if under reasonable assumptions on λ (see Definition 6.12). However, it is not clear that the associated map τ_λ is additive. This makes it necessary to consider potentially nonlinear maps $A_+ \to [0,\infty]$. The following definition makes this precise.

Definition 6.7. Let A be a C^*-algebra. A 1-*quasitrace* on A is a function $\tau\colon A_+ \to [0,\infty]$ that is homogeneous, that is additive on commutating elements (that is, $\tau(a+b) = \tau(a) + \tau(b)$ for $a, b \in A_+$ that commute), and that satisfies the trace property.

For $n \geq 2$, an n-quasitrace on A is a 1-quasitrace $\tau\colon A_+ \to [0,\infty]$ that extends to a 1-quasitrace on $A \otimes M_n$. We use $\operatorname{QT}(A)$ to denote the set of lower-semicontinuous 2-quasitraces on A.

Remarks 6.8. (1) If τ is a 2-quasitrace, then it is an n-quasitrace for all $n \geq 2$; see [BH82, Proposition II.4.1]. Moreover, the extension to $A \otimes M_n$ is unique. (It is $\tau \otimes \operatorname{tr}_n$, where $\operatorname{tr}_n\colon M_n \to [0,\infty)$ denotes the normalized trace on M_n.) It follows that every 2-quasitrace extends to a 1-quasitrace $A \otimes \mathcal{K}_+ \to [0,\infty]$. On the other hand, it was noted by Haagerup that Kirchberg proved that there exists a unital C^*-algebra with a 1-quasitrace that is not a 2-quasitrace; see [Haa14].

It turns out that the interesting objects are the 2-quasitraces. For that reason, and to simplify notition, by a *quasitrace* we always mean a 2-quasitrace. Thus, $\operatorname{QT}(A)$ is the set of lower semicontinuous quasitraces on A.

(2) Blackadar writes: 'One of the oldest, most famous, and most important open general structure questions for C^*-algebras is whether every quasitrace is a trace'; see [Bla06, II.6.8.16, p.124]. It was shown by Haagerup that every quasitrace on a unital, exact C^*-algebra is a

trace; see [Haa14, Theorem 5.11]. In [Kir97], Kirchberg shows that every quasitrace on a not necessarily unital, simple, exact C^*-algebra is a trace.

(3) A 1-quasitrace is not assumed to be order-preserving. However, this property follows automatically; see [BH82, Corollary II.2.5] and Corollary 6.18.

Notation 6.9. For each $n \geq 1$, let $f_n\colon \mathbb{R} \to [0,1]$ be the function such that $f_n(t) = 0$ for $t \leq \frac{1}{n}$, such that $f_n(t) = 1$ for $t \geq \frac{2}{n}$, and such that f_n is linear on the interval $[\frac{1}{n}, \frac{2}{n}]$.

Let A be a C^*-algebra, and let $\tau \in \mathrm{QT}(A)$. We abuse notation and use τ to also denote the unique extension to a lower semicontinuous quasitrace $(A \otimes \mathcal{K})_+ \to [0,\infty]$. Given $a \in (A \otimes \mathcal{K})_+$, we set $\widehat{d}_\tau(a) := \sup_n \tau(f_n(a))$.

Lemma 6.10. *Let A be a C^*-algebra, and let $\tau \in \mathrm{QT}(A)$. Then, for every $a \in (A \otimes \mathcal{K})_+$, we have $\widehat{d}_\tau(a) = \lim_n \tau(f_n(a))$, and*

$$\widehat{d}_\tau(a) = \sup_{\varepsilon > 0} \widehat{d}_\tau((a-\varepsilon)_+).$$

Moreover, if $a, b \in A_+$ satisfy $a \precsim b$, then $\widehat{d}_\tau(a) \leq \widehat{d}_\tau(b)$.

Proof. We may assume that A is stable.

Let $B \subseteq A$ be a commutative sub-C^*-algebra. Let X be the spectrum of B. We identify B with $C_0(X)$. The restriction of τ to B is a trace, which by Proposition 6.4 corresponds to integration with respect to an extended Radon measure μ on X.

Claim 1: Let $b \in B_+$, considered as a function $b\colon X \to [0,\infty)$. Then $\widehat{d}_\tau(b) = \mu(\mathrm{supp}(b))$.

Set $K_n := \{x : b(x) \geq \frac{2}{n}\}$, for $n \in \mathbb{N}_{>0}$. Then $\mathrm{supp}(b) = \bigcup_n K_n$. Using that μ is inner regular, we deduce that

$$\mu(\mathrm{supp}(b)) = \sup_n \mu(K_n).$$

Let $n \in \mathbb{N}_{>0}$. Note that $f_n(b)$ belongs to B. Given $x \in X$, we have $b(x) \geq \frac{2}{n}$ if and only if $f_n(b) = 1$. We deduce that

$$\tau(f_n(b)) = \int_X f_n(b) d\mu \geq \mu(\{x : f_n(b)(x) = 1\}) = \mu(K_n).$$

On the other hand, we have $f_n(b) \leq \mathbb{1}_{\mathrm{supp}(b)}$ and therefore

$$\tau(f_n(b)) = \int_X f_n(b) d\mu \leq \int_X \mathbb{1}_{\mathrm{supp}(b)} d\mu = \mu(\mathrm{supp}(b)).$$

It follows that

$$\mu(\mathrm{supp}(b)) = \sup_n \mu(K_n) \leq \sup_n \tau(f_n(b)) = \widehat{d}_\tau(b) = \sup_n \tau(f_n(b)) \leq \mu(\mathrm{supp}(b)),$$

which proves the claim.

A similar argument shows that $\widehat{d}_\tau(a) = \lim_n \tau(f_n(a))$ and $d_\tau(a) = \sup_{\varepsilon>0} d_\tau((a-\varepsilon)_+)$, for $a \in A_+$.

From claim 1, we immediately deduce the following claim:

Claim 2: If $a, b \in A_+$ commute and satisfy $a \leq b$, then $\widehat{d}_\tau(a) \leq \widehat{d}_\tau(b)$. Moreover, given $a \in A_+$ and $r \in (0,\infty)$, we have $\widehat{d}_\tau(a) = \widehat{d}_\tau(ra)$.

Claim 3: Let $a, b \in A_+$ satisfy $a \sim_{\mathrm{u}} b$. Then $\widehat{d}_\tau(a) = \widehat{d}_\tau(b)$.

Indeed, choose $u \in \mathcal{U}(\widetilde{A})$ such that $uau^* = b$. Since conjugating with u is an automorphism on A, and since automorphisms preserve functional calculus, we deduce that $uf_n(a)u^* = f_n(b)$ for every n. Using the trace property of τ at the second step, we obtain that

$$\widehat{d}_\tau(a) = \sup_n \tau(f_n(a)) = \sup_n \tau(uf_n(a)u^*) = \sup_n \tau(f_n(b)) = \widehat{d}_\tau(b).$$

Claim 4: Let $a, b \in A_+$ satisfy $a \precsim b$. Then $\widehat{d}_\tau(a) \leq \widehat{d}_\tau(b)$.

To prove the claim, let $\varepsilon > 0$. Using that Cuntz comparison in A is unitarily implemented, choose $\delta > 0$ and $x \in A_+$ such that $(a-\varepsilon)_+ \sim_{\mathrm{u}} x \subseteq (b-\delta)_+$.

Choose $k \in \mathbb{N}_{>0}$ with $\frac{2}{k} < \delta$. Then $f_k(b)$ acts as a unit for $(b-\delta)_+$ and consequently also for x. It follows that x and $f_k(b)$ commute and that $x \leq \|x\| f_k(b)$. Using claim 2, we deduce that $\widehat{d_\tau}(x) \leq \widehat{d_\tau}(f_k(b))$.

Choose $r > 0$ such that $f_k(b) \leq rb$. (For instance, $r = \frac{n}{2}$.) By an analogous argument, we deduce that $\widehat{d_\tau}(f_k(b)) \leq \widehat{d_\tau}(b)$. Using claim 3 at the first step, we obtain that

$$\widehat{d_\tau}((a-\varepsilon)_+) = \widehat{d_\tau}(x) \leq \widehat{d_\tau}(f_k(b)) \leq \widehat{d_\tau}(b).$$

Using that $d_\tau(a) = \sup_{\varepsilon>0} d_\tau((a-\varepsilon)_+)$, we deduce that $\widehat{d_\tau}(a) \leq \widehat{d_\tau}(b)$, as desired. □

The previous result justifies the following definition.

Definition 6.11. Let A be a C^*-algebra, and let $\tau \in \mathrm{QT}(A)$. Define $d_\tau\colon \mathrm{Cu}(A) \to [0,\infty]$ by $d_\tau([a]) := \widehat{d}(a) = \lim_n \tau(f_n(a))$ for $a \in (A \otimes \mathcal{K})_+$.

Definition 6.12. Let S be a Cu-semigroup. A *functional* on S is a generalized Cu-morphism $S \to [0,\infty]$, that is, a map $\lambda\colon S \to [0,\infty]$ that preserves order, addition, the zero element, and suprema of increasing sequences. We let $F(S)$ denote the set of all functionals on S.

Lemma 6.13. *Let S be a* Cu*-semigroup, and let $f\colon S \to [0,\infty]$ be a* PoM*-morphism, that is, f preserves addition, order and the zero element. Define $\tilde{f}\colon S \to [0,\infty]$ by*

$$\tilde{f}(a) := \sup\big\{ f(a') : a' \ll_\omega a \big\},$$

for $a \in S$. Then $\tilde{f}$ is a functional on S, called the regularization *of f. Moreover, $\tilde{f}$ is the largest functional below f. Thus, f is a functional if and only if $f = \tilde{f}$.*

Proof. Exercise. □

Proposition 6.14. *Let A be a C^*-algebra, let $\tau \in \mathrm{QT}(A)$, and let $d_\tau\colon \mathrm{Cu}(A) \to [0,\infty]$ be as in Definition 6.11. Then d_τ is a functional on* $\mathrm{Cu}(A)$.

Proof. We may assume that A is stable. It follows from Lemma 6.10 that d_τ is well-defined and order-preserving. It is also clear that $d_\tau(0) = 0$. To show additivity, let $a, b \in \mathrm{Cu}(A)$. Choose orthogonal elements $x, y \in A_+$ representing a and b, respectively. For each n, we have $f_n(x+y) = f_n(x) + f_n(y)$, and the elements $f_n(x)$ and $f_n(y)$ commute. It follows that

$$\begin{aligned} d_\tau(a+b) = \lim_n \tau(f_n(x+y)) &= \lim_n \big[\tau(f_n(x)) + \tau(f_n(y))\big] \\ &= \lim_n \tau(f_n(x)) + \lim_n \tau(f_n(y)) = d_\tau(a) + d_\tau(b), \end{aligned}$$

as desired.

It follows that d_τ is a PoM-morphism. We apply Lemma 6.13 to show that d_τ is a functional. Let $a \in \mathrm{Cu}(A)$. We need to verify that $d_\tau(a) = \sup\{d_\tau(a') : a' \ll_\omega a\}$. Choose $x \in A_+$ such that $a = [x]$. Given $a' \in \mathrm{Cu}(A)$, by Lemma 3.14 we have $a' \ll_\omega [x]$ if and only if there exists $\varepsilon > 0$ with $a' \leq [(x-\varepsilon)_+]$. Using this at the first step, and using Lemma 6.10 at the last step, we deduce

$$\sup\big\{ d_\tau(a') : a' \ll_\omega a \big\} = \sup\big\{ d_\tau([(x-\varepsilon)_+]) : \varepsilon > 0 \big\} = \sup_{\varepsilon>0} \widehat{d}((x-\varepsilon)_+) = \widehat{d}(x) = d_\tau(a),$$

as desired. □

Lemma 6.15. *Let A be a C^*-algebra, and let $\tau\colon A_+ \to [0,\infty]$ be a* 1*-quasitrace (*2*-quasitrace, trace). Define $\tilde{\tau}\colon A_+ \to [0,\infty]$ by*

$$\tilde{\tau}(a) := \sup_{\varepsilon>0} \tau((a-\varepsilon)_+),$$

for $a \in A_+$. Then $\tilde{\tau}$ is a lower semicontinous 1*-quasitrace (*2*-quasitrace, trace), called the* regularization *of τ. Moreover, $\tilde{\tau}$ is the largest lower semicontinuous* 1*-quasitrace (*2*-quasitrace, trace) below τ. Thus, τ is lower semicontinuous if and only if $\tau = \tilde{\tau}$.*

Proof. Exercise. □

Proposition 6.16. *Let A be a C^*-algebra, and let $\lambda\colon \mathrm{Cu}(A) \to [0,\infty]$ be a functional. Define $\tau_\lambda\colon (A\otimes\mathcal{K})_+ \to [0,\infty]$ by*

$$\tau_\lambda(a) := \int_0^\infty \lambda\big([(a-t)_+] \big) dt,$$

for $a \in (A\otimes\mathcal{K})_+$. Then τ_λ is a lower-semicontinuous quasitrace on A.

Proof. We clearly have $\tau_\lambda(0)=0$. Let $a,b\in A_+$ commute. Choose a commutative sub-C^*-algebra $B\subseteq A$ with $a,b\in B$. Let X be the spectrum of B. We identify B with $C_0(X)$. The restriction of τ to B is induced by an extended Radon measure μ on X. The elements a and b correspond to functions $f,g\in C_(X)$. Then

$$\tau_\lambda(a) = \int_0^\infty \lambda\big([(a-t)_+] \big) dt = \int_X f d\mu.$$

Using the additivity of the integral, we deduce that

$$\tau_\lambda(a+b) = \int_X (f+g)d\mu = \int_X f d\mu + \int_X g d\mu = \tau_\lambda(a)+\tau_\lambda(b),$$

as desired.

To verify the trace property, let $x\in A$. Given $\varepsilon>0$, we have $(xx^*-\varepsilon)_+ \sim (x^*x-\varepsilon)_+$ by Corollary 2.53. It follows that that $\lambda([xx^*-t)_+]) = \lambda([x^*x-t)_+])$ for every $t\geq 0$. Using this at the second step, we deduce

$$\tau_\lambda(xx^*) = \int_0^\infty \lambda\big([(xx^*-t)_+] \big) dt = \int_0^\infty \lambda\big([(x^*x-t)_+] \big) dt = \tau_\lambda(x^*x),$$

as desired.

Let us show that τ_λ is lower semicontinuous. By Lemma 6.15, it is enough to verify that $\tau_\lambda(a) = \sup_{\varepsilon>0}\tau_\lambda((a-\varepsilon)_+)$ for every $a\in A_+$. Given $a\in A_+$, choose a commutative sub-C^*-algebra $B\subseteq A$ containing a and the elements $(a-\varepsilon)_+$ for $\varepsilon>0$. The restriction of τ_λ to B is given by integration with respect to an extended Radon measure on the spectrum of B. Then, it follows from Proposition 6.4 that $\tau_\lambda(a) = \sup_{\varepsilon>0}\tau_\lambda((a-\varepsilon)_+)$, as desired. □

Theorem 6.17. *Let A be a C^*-algebra. Then there is a natural bijection $\mathrm{QT}(A)\cong F(\mathrm{Cu}(A))$. Given $\tau\in\mathrm{QT}(A)$, the corresponding functional d_τ is given by*

$$d_\tau(a) := \sup_n \tau(f_n(a)),$$

for $a\in(A\otimes\mathcal{K})_+$. Given $\lambda\in F(\mathrm{Cu}(A))$, the corresponding lower semicontinuous quasitrace τ_λ is given by

$$\tau_\lambda(a) := \int_0^\infty \lambda\big([(a-t)_+] \big) dt,$$

for $a\in(A\otimes\mathcal{K})_+$. Thus, for $\tau\in\mathrm{QT}(A)$ and $a\in(A\otimes\mathcal{K})_+$, we have

$$\tau(a) = \int_0^\infty d_\tau\big([(a-t)_+] \big) dt.$$

Similarly, for $\lambda\in F(\mathrm{Cu}(A)$ and $a\in(A\otimes\mathcal{K})_+$, we have

$$\lambda([a]) = \sup_n \int_0^\infty \lambda\big([(f_n(a)-t)_+] \big) dt.$$

Proof. It follows from Propositions 6.14 and 6.16 that the two assignments $\mathrm{QT}(A)\to F(\mathrm{Cu}(A))$ and $F(\mathrm{Cu}(A))\to \mathrm{QT}(A)$ are well-defined. To show that the assignments are inverses of each other, it is enough to verify this for commutative sub-C^*-algebras. In that case, the result follows as observed in Remark 6.6. □

Corollary 6.18. *Let A be a C^*-algebra, and let $\tau\in\mathrm{QT}(A)$. Then τ is order-preserving: Given $a,b\in A_+$ with $a\leq b$, we have $\tau(a)\leq\tau(b)$.*

Proof. Let $a, b \in A_+$ satisfy $a \leq b$. For each $t \geq 0$, we deduce that $a - t \leq b - t \leq (b-t)_+$. Multiplying by $(a-t)_+$ to obtain the third step, we deduce that

$$(a-t)_+ \sim (a-t)_+^3 = (a-t)_+(a-t)(a-t)_+ \leq (a-t)_+(b-t)_+(a-t)_+ \precsim (b-t)_+.$$

By Lemma 6.10, we obtain $d_\tau([(a-t)_+]) \leq d_\tau([(b-t)_+])$ for all $t \geq 0$. Using Theorem 6.17, it follows that

$$\tau(a) = \int_0^\infty d_\tau([(a-t)_+])dt \leq \int_0^\infty d_\tau([(b-t)_+])dt = \tau(b),$$

as desired. □

6.19. Let S be a Cu-semigroup, and let $a, b \in S$. We set $\infty a := \sup_n na$. Recall that $\downarrow\infty a = \{x \in S : x \leq \infty a\}$. One can show that $\downarrow\infty a$ is an ideal in S. It is the ideal generated by a, that is, the smallest ideal containing a. (Exercise.) Note that ∞a is the (unique) largest element of the ideal $\downarrow\infty a$. It follows that a and b generate the same ideal if and only if $\infty a = \infty b$.

Thus, if S is simple, then $\infty a = \infty b$ for all nonzero $a, b \in S$. We simply denote this largest element of S by ∞.

Definition 6.20. Let S be a Cu-semigroup, and let $a \in S$. Then a is said to be *finite* if $a \neq a+b$ for every nonzero $b \in S$. If a is not finite, then it is said to be *infinite.*

Further, S is said to be *stably finite* if an element $a \in S$ is finite whenever there exists $\tilde{a} \in S$ with $a \ll_\omega \tilde{a}$.

Lemma 6.21. *Let S be a nonzero, simple* Cu*-semigroup, and let $a \in S$. Then a is infinite if and only if $a = \infty$. Further, S is stably finite if and only if ∞ is not compact.*

Proof. Assume first that a is infinite. Choose $b \in S$ nonzero such that $a = a + b$. Then

$$a + 2b = (a+b) + b = a + b = a,$$

and inductively $a = a + nb$ for all $n \in \mathbb{N}$. Using this at the fourth step, it follows that

$$\infty = \infty b \leq a + \infty b = \sup_n (a + nb) = \sup_n a = a \leq \infty,$$

and hence $a = \infty$, as desired. Conversely, since $S \neq \{0\}$, it is clear that ∞ is infinite.

If S is stably finite, then every compact element is finite, and consequently ∞ can not be compact. Conversely, assume that S is not stably finite. Choose $a, \tilde{a} \in S$ with $a \ll_\omega \tilde{a}$ and such that a is infinite. Then $a = \infty$ and then

$$\infty = a \ll_\omega \tilde{a} \leq \infty,$$

which implies $\infty \ll_\omega \infty$, as desired. □

Recall that a C^*-algebra A is called *finite* if every projection in A is finite, that is, if $q \precsim p \leq q$ implies $p = q$ for all projections $p, q \in A$.

Proposition 6.22. *Let A be a unital, simple C^*-algebra. Then A is stably finite if and only if* $\mathrm{Cu}(A)$ *is stably finite.*

Proof. Assume first that $\mathrm{Cu}(A)$ is not stably finite, and let us show that A is not stably finite. Since A is simple, so is $\mathrm{Cu}(A)$. It follows from Lemma 6.21 that ∞ is compact. The class $[1_A]$ of the unit in A is nonzero. Then

$$\infty \ll_\omega \infty = \sup_n [1_A],$$

which allows us to choose $k \in \mathbb{N}$ with $\infty \leq k[1_A]$. Let $1_A \otimes 1_n$ denote the unit in $A \otimes M_n$. Then $k[1_A]$ is the class of $1_A \otimes 1_n$. It follows that the projection $1_A \otimes 1_n$ is Murray-von Neumann subequivalent to the projection $1_A \otimes 1_n \oplus 1_A$, which implies that $1_A \otimes 1_n$ is infinite. Thus A is not stably finite, as desired.

Conversely, assume that A is not stably finite. Choose an infinite projection $p \in A \otimes \mathcal{K}$. The class $[p]$ in $\mathrm{Cu}(A)$ is compact. Using that p is infinite, it easily follows that $[p]$ is infinite. Thus $\mathrm{Cu}(A)$ is not stably finite, as desired. □

Proposition 6.23. *Let S be a simple* Cu*-semigroup, and let $e \in S$ be a nonzero, compact element. Then S is stably finite if and only there exists a functional $\lambda \in F(S)$ with $\lambda(e) = 1$.*

Proof. First, assume that we are given $\lambda \in F(S)$ with $\lambda(e) = 1$. To show that S is stably finite, it is enough to verify that ∞ is not compact; see Lemma 6.21. To reach a contradiction, assume that ∞ is compact. Then

$$\infty \ll_\omega \infty = \sup_n ne,$$

which allows us to choose $k \in \mathbb{N}$ with $\infty \leq ke$. Then $(k+1)e \leq \infty \leq ke$, and thus

$$k+1 = \lambda((k+1)e) \leq \lambda(ke) = k,$$

which is the desired contradiction.

Conversely, assume that S is stably finite. Consider the subset $S_0 := \{ne : n \in \mathbb{N}\}$ of S. Given $m, n \in \mathbb{N}$, using that S is stably finite, we have $me \leq ne$ in S if and only if $m \leq n$. Thus, we may define an order-preserving map $f_0 \colon S_0 \to [0, \infty]$ by $f_0(ne) := n$, for $n \in \mathbb{N}$. Since $[0, \infty]$ is an injective object among positively ordered monoids, we can extend f_0 to a PoM-morphism $f \colon S \to [0, \infty]$. Let $\tilde{f} \colon S \to [0, \infty]$ be the regularization of f, as in Lemma 6.13. Since e is compact, we have

$$\tilde{f}(e) = \sup\big\{ f(a) : a \ll_\omega e \big\} = f(e) = 1.$$

Thus, $\lambda := \tilde{f}$ has the desired properties. □

Theorem 6.24 (Cuntz, [Cun78])**.** *Let A be a unital, simple C^*-algebra. Then A is stably finite if and only if A has a normalized quasitrace.*

Proof. Let $e \in \mathrm{Cu}(A)$ denote the class of the unit 1_A. Let $\tau \in \mathrm{QT}(A)$. For each $n \in \mathbb{N}$, we have $f_n(1_A) = 1_A$ and therefore

$$d_\tau(e) := \sup_n \tau(f_n(1_A)) = \tau(1_A).$$

Thus, by Theorem 6.17, a functional $\lambda \in F(\mathrm{Cu}(A))$ with $\lambda(e) = 1$ corresponds to a quasitrace $\tau \in \mathrm{QT}(A)$ with $\tau(1_A) = 1$.

By Proposition 6.22, A is stably finite if and only if $\mathrm{Cu}(A)$ is. Since e is a nonzero compact element, it follows from Proposition 6.23 that $\mathrm{Cu}(A)$ is stably finite if and only if there exists $\lambda \in F(\mathrm{Cu}(A))$ with $\lambda(e) = 1$. As mentioned above, such a functional λ corresponds to a normalized quasitrace. Combining these equivalence, we obtain the desired statement. □

Haagerup showed that every quasitrace on a unital, simple, exact C^*-algebra is a trace. Combining this with the result of Cuntz about the existence of quasitraces, we obtain the following:

Corollary 6.25 (Cuntz, Haagerup)**.** *Let A be a unital, simple, exact C^*-algebra. Then A is stably finite if and only if A has a tracial state.*

6.26. Let S be a Cu-semigroup. Recall that an element a in S is said to be *compact* if $a \ll_\omega a$. We let S_c denote the set of compact elements in S. Further, we set $S_c^\times := S_c \setminus \{0\}$. It is easy to see that S_c is a submonoid of S.

Let A be a C^*-algebra. Given a projection p in $A \otimes \mathcal{K}$, the corresponding class $[p]$ in $\mathrm{Cu}(A)$ is compact. Thus, the natural map $V(A) \to \mathrm{Cu}(A)$ takes image in $\mathrm{Cu}(A)_c$. If A is stably finite, then this map is an order-embedding; see Corollary 2.33. It is a natural question to ask whether this map is surjective. That is, given a compact element $a \in \mathrm{Cu}(A)$ we ask if there is a projection $p \in A \otimes \mathcal{K}$ with $a = [p]$.

By [BC09, Corollary 3.3], this is the case for stable finite C^*-algebras. Thus, given a stably finite C^*-algebra A, we have an natural isomorphism of positively ordered monoids $V(A) \xrightarrow{\cong} \mathrm{Cu}(A)_c$.

Definition 6.27. Let S be a Cu-semigroup, and let $a, b \in S$. Then a is said to be *stably below* b, denoted $a <_s b$, if there exists $k \in \mathbb{N}$ such that $(k+1)a \leq kb$.

Further, a is said to be *soft* if for every $a' \in S$ with $a' \ll_\omega a$ we have $a' <_s a$. We denote the set of soft elements in S by S_{soft}. We set $S_{\mathrm{soft}}^\times := S_{\mathrm{soft}} \setminus \{0\}$.

Proposition 6.28 ([APT14, Proposition 5.3.16, p.57])**.** *Let S be a simple, stably finite* Cu*-semigroup satisfying (O5). Then every element in S is compact or soft. Moreover, only the zero element is both compact and soft. Thus, S is the disjoint union $S = S_c \sqcup S_{\mathrm{soft}}^\times$.*

Proof. Let $a \in S$ be soft and compact. Since a is compact, we have $a \ll_\omega a$. Since a is soft, we can choose $k \in \mathbb{N}$ such that $(k+1)a \leq ka$. If a is nonzero, then this implies that a is infinite, which contradicts the assumption that S is stably finite. Thus a has to be zero.

Next, let $a \in S$ be an element that is nonzero and noncompact. To show that a is soft, let $x' \in S$ with $x' \ll_\omega a$. Choose $x \in S$ with $x' \ll_\omega x \ll_\omega a$. Apply (O5) to choose $d \in S$ such that

$$x' + d \leq a \leq x + d.$$

If d were zero, then $a = x$ and thus $a = x \ll_\omega a$, which contradicts that a is not compact. Thus, d is not zero. Then

$$x' \ll_\omega x \leq \infty = \sup_n nd,$$

which allows us to choose $k \in \mathbb{N}$ with $x' \leq kd$. Then

$$(k+1)x' = kx' + x' \leq kx' + kd \leq ka.$$

This show that $x' <_s a$, as desired. □

Notation 6.29. Let S be a Cu-semigroup, and let $a \in S$. We define $\hat{a} \colon F(S) \to \overline{\mathbb{P}}$ by $\hat{a}(\lambda) := \lambda(a)$ for $\lambda \in F(S)$.

We let $\mathrm{LAff}(F(S))$ denote the collection of maps $F(S) \to \overline{\mathbb{P}}$ that are lower semicontinuous, additive, order-preserving and homogeneous (that is, $\hat{a}(t\lambda) = t\hat{a}(\lambda)$ for $t \in (0,\infty), \lambda \in F(S)$.) We equip $\mathrm{LAff}(F(S))$ with pointwise order and addition, giving it the structure of a positively ordered monoid.

It is easy to check that $\hat{a}$ belongs to $\mathrm{LAff}(F(S))$. We define $\kappa_S \colon S \to \mathrm{LAff}(F(S))$ by $\kappa_S(a) := \hat{a}$, for $a \in S$.

6.30. It is easy to see that κ_S is additive and order-preserving. Thus, given $a, b \in S$ with $a \leq b$, then $\hat{a} \leq \hat{b}$. If the converse implication holds, then we say that the order (of a and b) is determined by functionals.

Theorem 6.31 ([APT14, Theorem 5.2.13, p.49])**.** *Let S be a* Cu*-semigroup, and let $a, b \in S$ with $\hat{a} <_s \hat{b}$. (That is, there exists $k \in \mathbb{N}$ such that $(k+1)\lambda(a) \leq k\lambda(b)$ for every $\lambda \in F(S)$). Then every $a' \in S$ with $a' \ll_\omega a$ satisfies $a' <_s b$.*

Lemma 6.32. *Let S be a* Cu*-semigroup, and let $a, b \in S$ with $\hat{a} \leq \hat{b}$. Assume that a is soft. Then every $a' \in S$ with $a' \ll_\omega a$ satisfies $a' <_s b$.*

Proof. Let $a' \in S$ satisfy $a' \ll_\omega a$. Choose $x \in S$ with $a' \ll_\omega x \ll_\omega a$. Since a is soft, we obtain that $x <_s a$. This implies that $x <_s b$. Then $\hat{x} <_s \hat{b}$. By Theorem 6.31, we obtain $x' <_s b$ for every $x' \in S$ with $x' \ll_\omega x$. In particular, since $a' \ll_\omega x$, we have $a' <_s b$, as desired. □

Definition 6.33. Let S be a Cu-semigroup. Then S is said to be *almost unperforated* if for all $a, b \in S$ with $a <_s b$ we have $a \leq b$. Further, S is said to be *almost divisible* if for all $k \in \mathbb{N}$, and for all $a', a \in S$ with $a' \ll_\omega a$ there exists $x \in S$ such that $kx \leq a$ and $a \leq (k+1)x$.

Theorem 6.34 ([APT14, Theorem 5.3.12, p.56])**.** *Let S be an almost unperforated* Cu*-semigroup, and let $a, b \in S$. Then $a \leq b$ whenever $\hat{a} \leq \hat{b}$ and a is soft.*

Proof. Assume that a is soft, and that $\hat{a} \leq \hat{b}$. Use (O2) to choose a $\ll_\omega$-increasing sequence $(a_n)_n$ in S with $a = \sup_n a_n$. Let $n \in \mathbb{N}$. Then $a_n \ll_\omega a$. By Lemma 6.32, we have $a_n <_s b$. Since S is almost unperforated, we obtain that $a_n \leq b$. Then $a = \sup_n a_n \leq b$, as desired. □

Corollary 6.35. *Let S be an almost unperforated* Cu*-semigroup, and let $a, b \in S_{\mathrm{soft}}$. Then $a \leq b$ if and only if $\hat{a} \leq \hat{b}$. Thus, the restriction of κ_S to S_{soft} is an order-embedding $S_{\mathrm{soft}} \to \mathrm{LAff}(F(S))$.*

Remark 6.36. Let S be an almost unperforated Cu-semigroup. It follows from Theorem 6.34 and Corollary 6.35 that there is a close connection between the order in S and comparison by functionals on S. However, even in an almost unperforated Cu-semigroup, we do in general not have that $\hat{a} \leq \hat{b}$ implies $a \leq b$ for all elements a and b.

Theorem 6.37. *Let S be a simple, stably finite, almost unperforated, almost divisible* Cu*-semigroup satisfying (O5). Then $S \cong S_c \sqcup \mathrm{LAff}(F(S))^{\times}$.*

6.38. Given a prime number p, recall the definition of the UHF-algebra M_{p^∞}; see Example 4.6. We computed the Cuntz semigroup of M_{p^∞} in Example 4.33:

$$\mathrm{Cu}(M_{p^\infty}) \cong \mathbb{N}[\tfrac{1}{p}] \sqcup (0, \infty].$$

The elements in $\mathbb{N}[\frac{1}{p}]$ are compact, while the elements in $(0, \infty]$ are soft.

The Jiang-Su algebra $\mathcal{Z}$ is a unital, separable, simple, nuclear, stably finite C^*-algebra that has a unique normalized trace, and that is KK-equivalent to the complex numbers (hence $K_0(\mathcal{Z}) \cong \mathbb{Z}$ and $K_1(\mathcal{Z}) = 0$). It was originally constructed by Jiang and Su, [JS99], as the inductive limit of so-called dimension-drop algebras. Given natural numbers p, q, the dimension-drop algebra of type (p, q) is defined as

$$\mathcal{Z}_{p,q} := \big\{ f\colon [0,1] \to M_p \otimes M_q : f \text{ continuous}, f(0) \in 1 \otimes M_q, f(1) \in M_p \otimes 1 \big\}.$$

Later, it was shown by Rørdam and Winter, [RW10], that $\mathcal{Z}$ can also be obtained as the inductive limit $\mathcal{Z} \cong \varinjlim_n A_n$, where each A_n is the same (generalized) dimension-drop algebra of type $(2^\infty, 3^\infty)$, defined as

$$\mathcal{Z}_{2^\infty,3^\infty} := \big\{ f\colon [0,1] \to M_{2^\infty} \otimes M_{3^\infty} : f \text{ continuous}, f(0) \in 1 \otimes M_{3^\infty}, f(1) \in M_{2^\infty} \otimes 1 \big\}.$$

Unfortunately, a more concrete description of $\mathcal{Z}$ is not known.

Using that $\mathcal{Z}$ has stable rank one, it follows that $V(\mathcal{Z}) \cong \mathbb{N}$. One can then compute the Cuntz semigoup of the Jiang-Su algebra as

$$\mathrm{Cu}(\mathcal{Z}) \cong \mathbb{N} \sqcup (0, \infty].$$

It is easy to check that $\mathrm{Cu}(\mathcal{Z})$ is almost unperforated and almost divisible.

Recall that a C^*-algebra A is said to be *$\mathcal{Z}$-stable* if $A \cong A \otimes \mathcal{Z}$.

Theorem 6.39 (Rørdam, [Rør04, Theorem 4.5])**.** *Let A be a $\mathcal{Z}$-stable C^*-algebra. Then* $\mathrm{Cu}(A)$ *is almost unperforated and almost divisible.*

The following result shows that the Cuntz semigroup of a unital, simple, stably finite, $\mathcal{Z}$-stable C^*-algebras is determined by the ordered K_0-group (which determines $V(A)$), the quasitrace simplex $\mathrm{QT}(A)$ (which is the trace simplex $T(A)$ if A is additionally assumed to be exact) and the pairing between K_0 and quastitraces, given by a natural pairing map $V(A) \times \mathrm{St}(K_0(A)) \to \overline{\mathbb{P}}$, which together with the natural map $\mathrm{QT}(A) \to \mathrm{St}(K_0(A))$ induces a map $V(A) \to \mathrm{LAff}(\mathrm{QT}(A))$.

Corollary 6.40. *Let A be a unital, simple, stably finite, $\mathcal{Z}$-stable C^*-algebras. Then $\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}(\mathrm{QT}(A))$.*

7. An introduction to tensor products of Cuntz semigroups

Definition 7.1. Let S, T and P be Cu-semigroups. A map $\varphi\colon S \times T \to P$ is called a Cu-*bimorphism* if it satisfies the following conditions:

(1) φ is a generalized Cu-morphism is each variable, that is, for fixed $a \in S$, the map $\varphi(a, \_)\colon T \to P$ preserves addition, order, the zero element and suprema of increasing sequence; and similarly for the first variable.

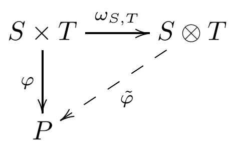
(2) φ preserves the joint way-below relation, that is, given $a', a \in S$ and $b', b \in T$ with $a' \ll_\omega a$ and $b' \ll_\omega b$, we have $\varphi(a', b') \ll_\omega \varphi(a, b)$.

We let $\mathrm{BiCu}(S \times T, P)$ denote the set of Cu-bimorphisms $S \times T \to P$.

Remark 7.2. Let A and B be stable C^*-algebras, and consider their maximal tensor product $A \otimes_{\max} B$. Given $a \in A_+$ and $b \in B_+$, the element $a \otimes b \in A \otimes_{\max} B$ is positive. Moreover, if $a', a \in A_+$ satisfy $a' \precsim a$, and if $b', b \in B_+$ satisfy $b' \precsim b$, then $a' \otimes b' \precsim a \otimes b$ in $A \otimes_{\max} B$. Thus, the map $A_+ \times B_+ \to (A \otimes B)_+$, $(a, b) \mapsto a \otimes b$, induces a well-defined map $\tau^0_{A,B} \colon \mathrm{Cu}(A) \times \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B)$.

Exercise 7.3. Show that the map $\tau^0_{A,B}$ from Remark 7.2 is a Cu-bimorphism.

Theorem 7.4 ([APT14, Theorem 6.3.3])**.** *Let S and T be* Cu*-semigroups. Then there exists a* Cu*-semigroup $S \otimes T$ and a* Cu*-bimorphism $\omega_{S,T} \colon S \times T \to S \otimes T$ satisfying the following universal property: For every* Cu*-semigroup P and every* Cu*-bimorphism $\varphi \colon S \times T \to P$ there exists a unique* Cu*-morphism $\tilde{\varphi} \colon S \otimes T \to P$ such that $\varphi = \tilde{\varphi} \circ \omega_{S,T}$. This means that the following diagram of undotted arrows can be completed to commute:*



Thus, given a Cu*-semigroup P, the assignment $f \mapsto f \circ \omega_{S,T}$ defines a bijection between the following sets of* Cu*-(bi)morphisms:*

$$\mathrm{Cu}(S \otimes T, P) \xrightarrow{\cong} \mathrm{BiCu}(S \times T, P).$$

Moreover, this assigment preserves the structure of these (bi)morphism sets as partially ordered monoids.

Remark 7.5. We say that the universal Cu-bimorphism $\omega_{S,T} \colon S \times T \to S \otimes T$ *linearizes* bimorphisms. In the language of category theory, Theorem 7.4 can also be stated by saying that $\omega_{S,T}$ *represents* the bimorphism functor $\mathrm{BiCu}(S \times T, \_)$; see [APT14, Section 6.1].

Notation 7.6. Let S and T be Cu-semigroups, let $\omega_{S,T} \colon S \times T \to S \otimes T$ be the universal Cu-bimorphism, and let $s \in S$ and $t \in T$. We write $s \otimes t$ for $\omega_{S,T}(s, t)$ and we call $s \otimes t$ an *elementary tensor*.

The following result collects the basic properties of the tensor product in Cu. Proofs and more details can be found in Corollary 6.3.6, Paragraph 6.3.7, Proposition 6.4.1 in [APT14,].

Proposition 7.7. *The tensor product in the category* Cu *has the following properties:*

1. *(1) The tensor product is associative: Given* Cu*-semigroups S, T and P, there is a natural isomorphism $S \otimes (T \otimes P) \cong (S \otimes T) \otimes P$.*
2. *(2) The tensor product is symmetric: Given* Cu*-semigroups S and T, the* Cu*-bimorphism $S \times T \to T \otimes S$, $(s, t) \mapsto t \otimes s$, induces a natural isomorphism $S \otimes T \cong T \otimes S$.*
3. *(3) The* Cu*-semigroup $\overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ is a tensorial unit: Given a* Cu*-semigroup S, we have natural isomorphisms $\overline{\mathbb{N}} \otimes S \cong S \cong S \otimes \overline{\mathbb{N}}$. The isomorphism $S \otimes \overline{\mathbb{N}} \cong S$ identifies the simple tensor $s \otimes k$ with $ks = s + s + \ldots + s$, for $s \in S$ and $k \in \mathbb{N}$. Further, it identifies $s \otimes \infty$ with $\infty s := \sup_n na$, for $a \in S$.*
4. *(4) The tensor product is functorial in both variables: Given* Cu*-semigroups S, S', T, T', and* Cu*-morphisms $\alpha \colon S \to S'$ and $\beta \colon T \to T'$, the* Cu*-bimorphism $S \times T \to S' \otimes T'$ given by $(s, t) \mapsto \alpha(s) \otimes \beta(t)$ induces a* Cu*-morphism $S \otimes T \to S' \otimes T'$, denoted by $\alpha \otimes \beta$.*
5. *(5) The tensor product is continuous in both variables: Given a* Cu*-semigroup S and an inductive system $(T_i, \varphi_{i,j})$ in* Cu*, $(S \otimes T_i, \mathrm{id}_S \otimes \varphi_{i,j})$ is an inductive system in* Cu *and*

there is an isomorphism

$$S \otimes \left(\varinjlim_i T_i \right) \cong \varinjlim_i (S \otimes T_i).$$

An analogous statement holds for the other variable.

It follows that the defined tensor product gives Cu *the structure of a symmetric, monoidal category.*

Example 7.8. Let p be a prime number. Consider the inductive system

$$\overline{\mathbb{N}} \xrightarrow{p\cdot} \overline{\mathbb{N}} \xrightarrow{p\cdot} \overline{\mathbb{N}} \xrightarrow{p\cdot} \dots$$

We have seen in Example 4.33 that the inductive limit of this system in Cu is isomorphic to the Cuntz semigroup of the UHF-algebra M_{p^∞}. To simplify notation, set $R_p := \mathrm{Cu}(M_{p^\infty})$. We identify $R_p = \mathbb{N}[\frac{1}{p}] \sqcup (0, \infty]$.

Let us compute $R_p \otimes R_p$. Using that the tensor product in Cu is compatible with inductive limits, we have

$$R_p \otimes R_p \cong \varinjlim \left(R_p \otimes \overline{\mathbb{N}} \xrightarrow{\mathrm{id} \otimes (p\cdot)} R_p \otimes \overline{\mathbb{N}} \xrightarrow{\mathrm{id} \otimes (p\cdot)} R_p \otimes \overline{\mathbb{N}} \to \dots \right).$$

Applying the natural isomorphism $R_p \otimes \overline{\mathbb{N}} \cong R_p$, the Cu-morphism $\mathrm{id} \otimes (p\cdot) \colon R_p \otimes \overline{\mathbb{N}} \to R_p \otimes \overline{\mathbb{N}}$ is turned into the Cu-morphism $p\cdot \colon R_p \to R_p$ given by multiplication by p. Thus

$$R_p \otimes R_p \cong \varinjlim \left(R_p \xrightarrow{p\cdot} R_p \xrightarrow{p\cdot} R_p \to \dots \right).$$

Note that multiplication by p is an isomorphism on $\mathbb{N}[\frac{1}{p}] \sqcup (0, \infty]$, with inverse given by multiplication by $\frac{1}{p}$. It follows that $R_p \otimes R_p \cong R_p$.

Notation 7.9. Let A and B be C^*-algebras, and let $\tau^0_{A,B} \colon \mathrm{Cu}(A) \times \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B)$ be the Cu-bimorphism from Remark 7.2, satisfying $\tau^0_{A,B}([x],[y]) = [x \otimes y]$, for $x \in A_+$ and $y \in B_+$. By Theorem 7.4, $\tau^0_{A,B}$ corresponds to a Cu-morphism $\tau_{A,B} \colon \mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B)$.

Proposition 7.10 ([APT14, Proposition 6.4.13])**.** *Let A and B be C^*-algebras. Then the universal* Cu*-morphism $\tau_{A,B} \colon \mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B)$ from Notation 7.9 is an isomorphism if A or B is an AF-algebra.*

Example 7.11. Let p be a prime number. The UHF-algebra M_{p^∞} is an AF-algebra. Moreover, it satisfies $M_{p^\infty} \otimes M_{p^\infty} \cong M_{p^\infty}$. (We say that M_{p^∞} is self-absorbing.) Thus

$$R_p \otimes R_p \cong \mathrm{Cu}(M_{p^\infty}) \otimes \mathrm{Cu}(M_{p^\infty}) \cong \mathrm{Cu}(M_{p^\infty} \otimes M_{p^\infty}) \cong \mathrm{Cu}(M_{p^\infty}) \cong R_p,$$

which gives an alternative argument for the observation from Example 7.8 that $R_p \otimes R_p$ is isomorphic to R_p.

Note that the isomorphism $\mu \colon R_p \otimes R_p \xrightarrow{\cong} R_p$ can be used to define a multiplication on R_p by setting $ab := \mu(a \otimes b)$, for $a, b \in R_p$. The class of the unit of M_{p^∞} is the compact element of value '1' in $\mathbb{N}[\frac{1}{p}]$ and it acts as a multiplicative unit element. One can check that this gives R_p the structure of a unital, commutative semiring.

7.12. Let us try to generalize Example 7.11 to define a multiplication on $\mathrm{Cu}(D)$ for more general C^*-algebras. Let D be a unital C^*-algebra, and let $\varphi \colon D \otimes_{\max} D \to D$ be a *-isomorphism. Postcomposing the Cu-bimorphism $\tau_{D,D}$ with $\mathrm{Cu}(\varphi)$, we obtain a Cu-bimorphism

$$\mu \colon \mathrm{Cu}(D) \otimes \mathrm{Cu}(D) \xrightarrow{\tau_{D,D}} \mathrm{Cu}(D \otimes_{\max} D) \xrightarrow{\mathrm{Cu}(\varphi)} \mathrm{Cu}(D).$$

Note that $\mu([x] \otimes [y]) = [\varphi(x \otimes y)]$ for $x, y \in D_+$. We want that μ defines a multiplication on $\mathrm{Cu}(D)$, turning it into a unital semiring.

It is natural to require that the class of the unit of D acts as a multiplicative unit for the semiring structure on $\mathrm{Cu}(D)$. Thus, we want that

$$\mu([1] \otimes [x]) = [x] = \mu([x] \otimes [1]),$$

for all $x \in D_+$. Applying the definition of μ, we expect that

$$\varphi(1 \otimes x) \sim x \sim \varphi(x \otimes 1),$$

for all $x \in D_+$.

A prominent class of C^*-algebras where this condition is satisfied are the so-called *strongly self-absorbing* C^*-algebras, introduced by Toms and Winter in [TW07]. We say that a unital C^*-algebra D is *strongly self-absorbing* if $D \not\cong \mathbb{C}$ and if there exists a *-isomorphism $\psi\colon D \to D \otimes_{\max} D$ such that ψ is approximately unitarily equivalent to the first factor embedding $\mathrm{id}_D \otimes 1\colon D \to D \otimes_{\max} D$, $x \mapsto x \otimes 1$. (This means that there is a sequence of unitaries $(u_n)_n$ in $D \otimes_{\max} D$ such that $\|u_n \psi(x) u_n^* - x \otimes 1\| \to 0$ for $n \to \infty$, for all $x \in D$.) It follows that ψ is also unitarily equivalent to the second factor embedding $1 \otimes \mathrm{id}_D\colon D \to D \otimes_{\max} D$, $x \mapsto 1 \otimes x$. Since approximate unitary equivalence is stronger than Cuntz equivalence, we obtain

$$1 \otimes x \sim \psi(x) \sim x \otimes 1,$$

for every $x \in D_+$. Applying ψ^{-1}, we deduce

$$\psi^{-1}(1 \otimes x) \sim x \sim \psi^{-1}(x \otimes 1),$$

for all $x \in D_+$, which shows that ψ^{-1} has the desired properties.

Let D be a strongly self-absorbing C^*-algebra. The *-isomorphism $\psi\colon D \to D \otimes_{\max} D$ is unique up to approximate unitary equivalence. It follows that there is a canonical multiplication on $\mathrm{Cu}(A)$ that satisfies

$$[x][y] = [\psi^{-1}(x \otimes y)],$$

for $x, y \in A_+$. One can show that this gives $\mathrm{Cu}(D)$ the structure of a unital, commutative semiring.

The known examples of strongly self-absorbing C^*-algebras are:

(1) The Jiang-Su algebra $\mathcal{Z}$.
(2) The UHF-algebras M_p of infinite type. (Infinite type means that the supernatural number p satisfies $p = p^2 \neq 1$, which ensures $M_p \not\cong \mathbb{C}$ and $M_p \otimes M_p \cong M_p$.)
(3) The Cuntz algebra $\mathcal{O}_\infty$. (We think of $\mathcal{O}_\infty$ as the purely infinite analogue of $\mathcal{Z}$.)
(4) The tensor products $\mathcal{O}_\infty \otimes M_p$, for any UHF-algebra M_p of infinite type.
(5) The Cuntz algebra $\mathcal{O}_2$.

Let D be a strongly self-absorbing C^*-algebra. If a C^*-algebra A satisfies $A \cong D \otimes A$, then we say that A is *D-stable*. If A is D-stable, then there is a *-isomorphism $\psi\colon A \to D \otimes A$ that is approximately untiarily equivalent to the second factor embedding. Moreover, ψ is unique up to approximately untiarily equivalence. This allows us to define a canonical action of the semiring $\mathrm{Cu}(D)$ on the Cu-semigroup $\mathrm{Cu}(A)$ by setting

$$[d][x] := [\psi^{-1}(d \otimes x)],$$

for $d \in D_+$ and $x \in A_+$. See [APT14, Section 7.1] for more details.

The following result follows by combining Proposition 7.1.4 with Paragraph 7.6.1 (which shows that the Cuntz semigroup of all known strongly self-absorbing is 'solid') and Proposition 7.1.6 in [APT14].

Proposition 7.13. *Let D be any of the known examples of strongly self-absorbing C^*-algebras, and let A be a D-stable C^*-algebra. Then* $\mathrm{Cu}(A) \cong \mathrm{Cu}(D) \otimes \mathrm{Cu}(A)$. *(We say that* $\mathrm{Cu}(A)$ *is* $\mathrm{Cu}(D)$-stable.*)*

The converse is not true in general. However, it is important to determine when a given C^*-algebra is D-stable. As a first step, we characterize when a Cu-semigroup is Cu(D)-stable.

Given a purely infinite, strongly self-absorbing C^*-algebra D (in the above list, these are $\mathcal{O}_\infty$, $\mathcal{O}_\infty \otimes M_p$, and $\mathcal{O}_2$), we have $\mathrm{Cu}(D) \cong \{0, \infty\}$. The following result characterizes when a Cu-semigroup is $\{0, \infty\}$-stable.

Theorem 7.14 ([APT14, Theorem 7.2.2, Proposition 7.2.8]). *Let S be a* Cu*-semigroup. Then $S \cong \{0, \infty\} \otimes S$ if and only if every $a \in S$ satisfies $a = 2a$. (Such a then also satisfies $a = \infty a$, and is called properly infinite.)*

A C^*-algebra A is purely infinite if and only if $\mathrm{Cu}(A) \cong \{0, \infty\} \otimes \mathrm{Cu}(A)$.

The following result characterizes when a Cu-semigroup is R_p-stable.

Theorem 7.15 ([APT14, Theorem 7.4.10]). *Let S be a* Cu*-semigroup, and let p be a prime number. Then $S \cong R_p \otimes S$ if and only if S is p-unperforated (that is, we have $pa \leq pb$ if and only if $a \leq b$, for $a, b \in S$) and p-divisible (that is, for every $a \in S$ there exists $b \in S$ with $pb = a$).*

The following result characterizes when a Cu-semigroup is Z-stable.

Theorem 7.16 ([APT14, Theorem 7.3.8]). *Let S be a* Cu*-semigroup. Then $S \cong Z \otimes S$ if and only if S is almost unperforated and almost divisible.*

Remark 7.17. Given a simple, unital, separable, nuclear C^*-algebra A, the Toms-Winter conjecture predicts that A is $\mathcal{Z}$-stable if and only if Cu(A) is almost unperforated.

Thus, we can consider Theorem 7.16 as a Cu-semigroup-version of the Toms-Winter conjecture. Note that $A \cong \mathcal{Z} \otimes A$ implies that $\mathrm{Cu}(A) \cong Z \otimes \mathrm{Cu}(A)$, which in turn implies that Cu(A) is almost unperforated (and almost divisible). This even holds for arbitrary C^*-algebras and was first observed by Rørdam, [Rør04].

Conversely, assume that Cu(A) is almost unperforated. We may decompose the conjecture of Toms and Winter into two parts: First, assuming that Cu(A) is almost unperforated, does it follow that Cu(A) is almost divisible? Second, assuming that Cu(A) is almost unperforated and almost divisible (and hence Cu(A) is Z-stable by Theorem 7.16) does it follow that A is $\mathcal{Z}$-stable?

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