OBERWOLFACH EXTENDED ABSTRACT: THE GENERATOR RANK FOR C^*-ALGEBRAS

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Abstract. This is the extended abstract for my talk 'The generator rank for C^*-algebras' at the Conference 'C^*-Algebras, Dynamics, and Classification', Mathematisches Forschungsinstitut Oberwolfach (MFO), 28. October – 3. November 2012, organized by Joachim Cuntz (Münster), George A. Elliott (Toronto), Andrew Toms (West Lafayette) and Wilhelm Winter (Münster).

The generator problem asks which C^*-algebras are singly generated, i.e., generated as a C^*-algebra by one of its elements. More generally, for a given C^*-algebra A one wants to determine the minimal number of generators, i.e., the minimal k such that A contains k elements that are not contained in any proper sub-C^*-algebra.

It is often convenient to consider self-adjoint generators, which only leads to a minor variation of the original generator problem, since two self-adjoint elements a, b generate the same sub-C^*-algebra as the element $a + ib$. Given a C^*-algebra A, let us denote by $\mathrm{gen}(A)$ the minimal number of *self-adjoint* generators for A, and set $\mathrm{gen}(A) = \infty$ if A is not finitely generated, see [2]. By definition, A is singly generated if and only if it is generated by two self-adjoint elements, that is, if and only if $\mathrm{gen}(A) \leq 2$.

For more details on the minimal number of self-adjoint generators we refer the reader to [2] and [5]. We just note that for a compact, metric space X, it is easy to see that $\mathrm{gen}(C(X)) \leq k$ if and only if X can be embedded into $\mathbb{R}^k$.

The problem with computing the invariant $\mathrm{gen}(\_)$ is that it does not behave well with respect to inductive limits, i.e., in general we do not have $\mathrm{gen}(A) \leq \liminf_n \mathrm{gen}(A_n)$ if $A = \varinjlim A_n$ is an inductive limit. This is unfortunate since many C^*-algebras are given as inductive limits, e.g., AF-algebras or approximately homogeneous algebras (AH-algebras).

To see an example where the minimal number of generators increases when passing to an inductive limit, let $X \subset \mathbb{R}^2$ be the topologists sine-curve. Then X can be embedded into $\mathbb{R}^2$ but not into $\mathbb{R}^1$, and therefore $\mathrm{gen}(C(X)) = 2$. However, X is an inverse limit of spaces X_n that are each homeomorphic to the interval, i.e., $X_n \cong [0, 1]$. Therefore $C(X) \cong \varinjlim_n C(X_n)$, with $\mathrm{gen}(C(X)) = 2$, while $\mathrm{gen}(C(X_n)) = 1$ for all n.

To get a better behaved theory, instead of counting the minimal number of self-adjoint generators, we will count the minimal number of "stable" self-adjoint generators. This is the underlying idea of our definition of the generator rank of a C^*-algebra. Given a C^*-algebra A, and $k \geq 1$, we let A^k_{sa} denote the space of self-adjoint k-tuples in A, and we let $\mathrm{Gen}_k(A)_{\mathrm{sa}} \subset A^k_{\mathrm{sa}}$ be the subset of tuples that generate A.

1. **Definition** ([3, Definition 2.2]). Let A be a unital C^*-algebra. The *generator rank* of A, denoted by $\mathrm{gr}(A)$, is the smallest integer $k \geq 0$ such that $\mathrm{Gen}_{k+1}(A)_{\mathrm{sa}}$ is dense in A^{k+1}_{sa}. If no such k exists, we set $\mathrm{gr}(A) = \infty$.

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Given a non-unital C^*-algebra A, let $\widetilde{A}$ denote its minimal unitization, and set $\mathrm{gr}(A) := \mathrm{gr}(\widetilde{A})$.

Thus, while "$\mathrm{gen}(A) \leq k$" records that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is not empty, "$\mathrm{gr}(A) \leq k - 1$" records that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is dense. This indicates why the generator rank is usually much larger than the minimal number of self-adjoint generators. The payoff, however, is that the generator rank is much easier to compute.

The definition of the generator rank is analogous to that of the real rank as introduced by Brown and Pedersen, [1]. This explains the index shift of the definition, and with this index shift one obtains the general estimate $\mathrm{rr}(A) \leq \mathrm{gr}(A)$, see [3, Proposition 2.5].

The most interesting value of the generator rank for A is one, which means exactly that the (single) generators are dense in A. One can show that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is a G_δ-subset of A_{sa}^k for each k (although not necessarily dense). It follows that $\mathrm{gr}(A) \leq 1$ if and only if the generators form a dense G_δ-subset of A, which means that the generic element of A is a generator.

The generator rank has many of the permanence properties that are also satisfied by other noncommutative dimension theories, see [4]. In particular, it does not increase when passing to ideals, quotients or inductive limits. Thus, the generator rank is indeed better behaved than the theory of counting the minimal number of self-adjoint generators. However, while it easy to see that for unital C^*-algebras A, B we have $\mathrm{gen}(A \oplus B) = \max\{\mathrm{gen}(A), \mathrm{gen}(B)\}$, the analog question for the generator rank seems surprisingly hard:

1. **Question.** Given two separable C^*-algebras A and B, do we have $\mathrm{gr}(A \oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$?

It is easy to see that every finite-dimensional C^*-algebra has generator rank one. We get the following consequence:

1. **Corollary** ([3, Corollary 3.3])**.** *Let A be a separable AF-algebra. Then A has generator rank at most one. In particular, A is singly generated.*

We also compute the generator rank of commutative and homogeneous C^*-algebras. If X is a compact, metric space, then:

$$\begin{aligned} \mathrm{gr}(C(X)) &= \dim(X \times X), \\ \mathrm{gr}(C(X, M_n)) &= \left\lceil \frac{\dim(X)+1}{2n-2} \right\rceil, \qquad \text{for } n \geq 2. \end{aligned}$$

This allows us to show that a unital, separable AH-algebra has generator rank one if it is either simple with slow dimension growth, or when it tensorially absorbs a UHF-algebra, see [3, Corollary 4.30]. The following natural questions remain open:

2. **Question.** Let A be unital, separable C^*-algebra that tensorially absorbs the Jiang-Su algebra. Does it follow that A has generator rank at most one?

3. **Question.** Let A be unital, separable, real rank zero, stable rank one, nuclear C^*-algebra. Does it follow that A has generator rank at most one?

Note that every II_1-factor M acting on a separable Hilbert space contains a weakly dense sub-C^*-algebra A that is unital, separable and has real rank zero and stable rank one. Thus, a positive answer to Question 5 without the assumption of nuclearity would imply that every II_1-factor M is singly generated (as a von Neumann algebra). It is known that this would imply that every separably acting von Neumann algebra M is singly generated, which is a long-standing open question first asked by Kadison in 1967.

References

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