OBERWOLFACH EXTENDED ABSTRACT: C*-ALGEBRAS OF STABLE RANK ONE AND THEIR CUNTZ SEMIGROUPS

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Abstract. This is the extended abstract for my talk 'C*-algebras of stable rank one and their Cuntz semigroups' at the Conference 'C*-Algebras', Mathematisches Forschungsinstitut Oberwolfach (MF), 11. August – 17. August 2019, organized by Mikael Rørdam (Copenhagen), Dima Shlyakhtenko (Los Angeles), Andreas Thom (Dresden) and Stefaan Vaes (Leuven).

I report on joint work with Ramon Antoine, Francesc Perera and Leonel Robert.

Stable rank one. Recall that a unital C*-algebra is said to have *stable rank one* if its invertible elements are norm-dense. This strong finiteness condition was introduced by Rieffel, [Rie83], to study nonstable K-theory. First examples of C*-algebras of stable rank one include II_1-factors and commutative C*-algebras $C(X)$ with X of covering dimension at most one.

Stable rank one has very nice permanence properties: It passes to ideals, quotients, hereditary sub-C*-algebras, directs sums, inductive limits, matrix amplifications and stabilizations. Using this, it follows for instance that the class of stable rank one C*-algebras includes all AF- and A$\mathbb{T}$-algebras.

In [Put90], Putnam proved that all irrational rotation algebras have stable rank one. Later, is was shown by Elliott-Evans, [EE93], that these algebras are even A$\mathbb{T}$-algebras. These results were generalized in two ways: First, it was shown that large classes of simple C*-algebras are ASH-algebras; see for example [LP10]. Second, it was studied when simple ASH-algebras have stable rank one; see for example [DNNP92], [EHT09].

The picture was clarified with the discovery of the Jiang-Su algebra $\mathcal{Z}$, [JS99], which is a unital, simple, nonelementary C*-algebra that is KK-equivalent to $\mathbb{C}$. One says that a C*-algebra A is *$\mathcal{Z}$-stable* if $A \cong A \otimes \mathcal{Z}$. This important regularity property is the C*-algebraic analog of the McDuff-property for von Neumann algebras. Rørdam, [Rør04], showed that $\mathcal{Z}$-stable, simple, stably finite, C*-algebras have stable rank one. In [Tom11], Toms proved that simple ASH-algebras with slow dimension growth are $\mathcal{Z}$-stable, and thus have stable rank one.

There are also numerous simple, non-$\mathcal{Z}$-stable C*-algebras of stable rank one. Amenable examples were constructed by Villadsen, [Vil98], and Toms, [Tom08]. Nonamenable examples are given by Dykema-Haagerup-Rørdam, [DHR97]: the reduced group C*-algebra $C^*_\lambda(G_1 * G_2)$ is simple and has stable rank one whenever G_1 and G_2 are nontrivial groups that are not both isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

1. **Question.** Let G be a discrete group such that its reduced group C*-algebra $C^*_\lambda(G)$ is simple. Does $C^*_\lambda(G)$ have stable rank one?

The Cuntz semigroup. Let A be a C*-algebra, and let $A \otimes \mathbb{K}$ denote its stabilization. Recall that the *Murray-von Neumann semigroup* $V(A)$ is defined

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through equivalence classes of projections in $A\otimes\mathbb{K}$. Orthogonal addition turns $V(A)$ into a commutative monoid, which is naturally isomorphic to the set of isomorphism classes of finitely generated, projective A-modules. If A is unital, then its K_0-group is isomorphic to the Grothendieck completion of $V(A)$.

Given $a,b \in A_+$, we write $a \precsim_{\mathrm{Cu}} b$ if there is a sequence $(c_n)_n$ in A with $\lim_n \|a - c_n b c_n^*\| = 0$. Further, a and b are *Cuntz equivalent*, denoted $a \sim_{\mathrm{Cu}} b$, if $a \precsim_{\mathrm{Cu}} b$ and $b \precsim_{\mathrm{Cu}} a$. These relations were introduced by Cuntz, [Cun78], in his study of dimension functions on C*-algebras. The *Cuntz semigroup* of A is

$$\mathrm{Cu}(A) := (A\otimes\mathbb{K})_{+/\sim_{\mathrm{Cu}}}.$$

Again, orthogonal addition turns $\mathrm{Cu}(A)$ into a commutative monoid, and the relation $\precsim_{\mathrm{Cu}}$ induces an additive order on $\mathrm{Cu}(A)$. There is a picture of $\mathrm{Cu}(A)$ using countably generated, Hilbert A-modules, [CEI08]. If A has stable rank one, then $\mathrm{Cu}(A)$ is naturally isomorphic to the isomorphism classes of such modules.

The following table contains some examples of the considered invariants. It becomes apparent that $\mathrm{Cu}(A)$ contains more information than $V(A)$:

A	$V(A)$	$K_0(A)$	$\mathrm{Cu}(A)$
$\mathbb{C}$ or $M_n(\mathbb{C})$	$\mathbb{N}$	$\mathbb{Z}$	$\overline{\mathbb{N}} = \mathbb{N}\cup\{\infty\}$
II_1-factor	$[0,\infty)$	$\mathbb{R}$	$[0,\infty)\sqcup(0,\infty]$
$C([0,1])$	$\mathbb{N}$	$\mathbb{Z}$	$\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$
$\mathcal{Z}$ or $C^*_\lambda(\mathbb{F}_\infty)$	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{N}\sqcup(0,\infty]$

In [CEI08], Coward-Elliott-Ivanescu introduced the category **Cu** of abstract Cuntz semigroups, and they showed that the Cuntz semigroup defines a functor from C*-algebras to **Cu**. A systematic study of **Cu** with applications to the structure theory of C*-algebras was conducted in [APT18] and [APT17]. In particular, it was shown that **Cu** admits a natural tensor product construction that gives it the structure of a closed, symmetric, monoidal category.

1. **Theorem** ([APT19]). *The category* **Cu** *is complete and cocomplete, and the Cuntz semigroup functor preserves inductive limits and direct sums. Further, a scaled version of the Cuntz semigroup preserves products and ultraproducts.*

The Cuntz semigroup is an invariant that contains a lot of information about the C*-algebra, including its lattice of ideals and its simplex of (quasi)traces. Thus, the results in [APT19] allow us to access the ideal lattice and the quasitraces of products and ultraproducts of C*-algebras.

The rank problem. Let A be a simple, stably finite, exact C*-algebra. Then $\mathrm{Cu}(A)$ decomposes as $\mathrm{Cu}(A) = V(A)^\times \sqcup \mathrm{Cu}(A)_{\mathrm{soft}}$, where $V(A)^\times$ contains the classes of nonzero projections, and where $\mathrm{Cu}(A)_{\mathrm{soft}}$ consists of classes of elements with connected spectrum. Every trace $\tau\colon A\to\mathbb{C}$ induces a dimension function:

$$d_\tau\colon \mathrm{Cu}(A)\to[0,\infty],\quad d_\tau([a]) = \lim_{n\to\infty}\tau(a^{1/n}).$$

One calls $d_\tau([a])$ the 'rank' of a with respect to τ. The rank map is

$$\alpha\colon \mathrm{Cu}(A)_{\mathrm{soft}}\to \mathrm{LAff}(T(A))_{++},\quad \alpha([a])(\tau) = d_\tau(a).$$

One says that A has *strict comparison* if α is an order-embedding, and there are examples when this fails. The *rank problem* for A is to determine the range of α. There are no examples known when α is not surjective.

2. **Question.** Given a separable, unital, simple, exact C*-algebra, and given $f \in \mathrm{LAff}(T(A))_{++}$, is there $a\in(A\otimes\mathbb{K})_+$ with $d_\tau([a]) = f(\tau)$ for all $\tau\in T(A)$?

If A is $\mathcal{Z}$-stable, then α is an isomorphism. If A is amenable, then the Toms-Winter conjecture predicts that A is $\mathcal{Z}$-stable if and only if A has strict comparison. Thus, α is predicted to be an isomorphism whenever it is an order-embedding. This establishes a connection between the Toms-Winter conjecture and the rank problem.

2. **Theorem** ([Thi17]). *If A has stable rank one, then α is surjective.*

1. **Corollary.** *If A has stable rank one and strict comparison, then*

$$\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}(T(A))_{++} \cong \mathrm{Cu}(A \otimes \mathcal{Z}).$$

If, additionally, A has locally finite nuclear dimension (for example A is an ASH-algebra), then it follows that A is $\mathcal{Z}$-stable.

Riesz interpolation. In [APRT18] we unveil new structure in the Cuntz semigroup of C*-algebras of stable rank one. The main result is:

3. **Theorem** ([APRT18]). *If A has stable rank one, then $\mathrm{Cu}(A)$ has Riesz interpolation: If $x_j \leq z_k$ for $j, k = 1, 2$, then there is y with $x_1, x_2 \leq y \leq z_1, z_2$.*

2. **Corollary.** *If A is separable and of stable rank one, then $\mathrm{Cu}(A)$ is semilattice.*

These results allow us to apply semilattice theory to study C*-algebras of stable rank one. Using this method, we confirm a conjecture of Blackadar-Handelman and we solve the Global Glimm Halving Problem in this context.

Given a unital C*-algebra, we use $\mathrm{DF}(A)$ to denote the compact convex set of not necessarily continuous dimension functions on A. For a compact, Hausdorff space X, the set $\mathrm{DF}(C(X))$ can be identified with the finitely additive probability measures on X, which is a Choquet simplex. In [BH82], Blackadar-Handelman conjectured that $\mathrm{DF}(A)$ is always a Choquet simplex. We prove:

4. **Theorem.** *If A has stable rank one, then $\mathrm{DF}(A)$ is a Choquet simplex.*

The Global Glimm Halving Problem, [KR02], seeks to characterize when a C*-algebra A has no finite-dimensional irreducible representations. A sufficient condition is that A admits a $*$-homomorphism $M_k(C_0((0,1])) \to A$ with full image for each $k \in \mathbb{N}$, and the problem is to show that this criterion is also sufficient. Under the assumption of stable rank one, we show an even stronger result:

5. **Theorem.** *Let A have stable rank one, and let $k \in \mathbb{N}$. Then A has no irreducible representation of dimension $< k$ if and only if there exists a $*$-homomorphism $M_k(C_0((0,1])) \to A$ with full image.*

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