A SURVEY ON NON-COMMUTATIVE DIMENSION THEORIES

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ABSTRACT. We develop an abstract theory of noncommutative dimension theories, and we show that the common theories fit into this setting, in particular the real and stable rank, the topological dimension, the decomposition rank and the nuclear dimension. Then we show how to compute or estimate the dimension theories of certain classes of C^*-algebras, in particular subhomogeneous and type I C^*-algebras.

We prove some connections between the low dimensional cases in the different theories for type I algebras. As an application, we show that type I algebras with stable rank one have torsion-free K_0-groups.

Our techniques alow to reduce essentially every question about dimension theories to the case of separable C^*-algebras. We will present applications of this by generalizing interesting results of Winter and Lin from the separable to the general case.

The theory of C^*-algebras is often considered as non-commutative topology, which is justified by the natural duality between unital, commutative C^*-algebras and the category of compact, Hausdorff spaces.

Given this fact, one tries to transfer concepts from commutative topology to C^*-algebras, and we will focus on the theory of dimension. In fact there are different non-commutative dimension theories (often called rank), and we will give an abstract setting to study these theories.

The low-dimensional case of these theories is of most interest. One reason is that the low-dimensional cases often agree for different theories. But more importantly, low dimension (in any dimension theory) is considered as a regularity property.

Such regularity properties are very important for proving the Elliott conjecture, which predicts that separable, nuclear, simple C^*-algebras are classified by their Elliott invariant, a tuple consisting of ordered K-theory, the space of traces and a pairing between the two. There are counterexamples to this general form of the conjecture, but if one restricts to certain subclasses that are regular enough, then the conjecture has been verified.

It is interesting that many regularity properties are statements about a certain dimension theory. The obvious example is the requirement of real rank zero, which implies that the C^*-algebra has many projections. In this setting many classification results have been obtained, although real rank zero is not enough to verify the Elliott conjecture.

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This paper proceeds as follows:

In Section 2 we propose a set of axioms for a (non-commutative) dimension theory, see Definition 2.1. As we will see, these axioms allow to reduce essentially every question about dimension theories to the case of separable C^*-algebras, see Lemma 2.9.

In the following Sections 3, 4 and 5 we study the real and stable rank, the topological dimension, and the decomposition rank and nuclear dimension. We will show that these theories satisfy our axioms of a dimension theory. For most cases this follows from known results, but for the topological dimension the verification of the axioms is rather involved. For each theory, we study whether it is Morita-invariant, 2.7, and how it behaves with respect to extensions, tensoring with matrices or stabilizing, tensor products and pullbacks.

In the following Sections 6 and 7 we show how to compute the dimension theories of homogeneous, continuous-trace and subhomogeneous C^*-algebras. This is mainly a presentation of known results, sometimes generalized from the separable to the non-separable case. Since the structure of these C^*-algebras is pretty well-understood, it is possible to compute their dimension theories explicitly, or at least give good estimates, see Proposition 6.10 and 7.3.

In Section 8 we study the dimension theory of type I C^*-algebras. The main result of this section estimates the real and stable rank in terms of the topological dimension, see Theorem 8.7, a result that seems not to have appeared in the literature so far.

Then we will study type I C^*-algebras that are low-dimensional in the sense of the different dimension theories. We show that all considered dimension theories (except the stable rank) do agree for dimension zero, see Proposition 8.11. This is not longer true for dimension one, where we need to distinguish whether a C^*-algebra is residually stably finite or not, see 8.21. We also show that a type I C^*-algebra with stable rank one has torsion-free K_0-group, see 8.23.

In the last Section 9 we present a few examples of C^*-algebras and their dimension theories.

This article is based on the diploma thesis of the author, [Thi09], which was written under the supervision of Wilhelm Winter at the university of Münster in 2009.

1. Preliminaries

By a morphism between C^*-algebras we mean a $*$-homomorphism, and by an ideal of a C^*-algebra we understand a closed, two-sided ideal. We use the notation $I \lhd A$ to indicate that I is an ideal of A. If A is a C^*-algebra, then we denote by $\widetilde{A}$ its minimal unitalization. We denote the set of self-adjoint elements by A_{sa}, and the set of invertible elements by A^{-1} (if A is unital). The primitive ideal space of A will be denoted by $\mathrm{Prim}(A)$, and the spectrum by $\widehat{A}$.

If $F, G \subset A$ are two subsets of a C^*-algebra, and $\varepsilon > 0$, then we write $F \subset_\varepsilon G$ if for every $x \in F$ there exists some $y \in G$ such that $\|x - y\| < \varepsilon$.

We denote by $\mathbb{K}$ the C^*-algebra of compact operators on an infinite-dimensional, separable Hilbert space.

1.1. An inductive system is a collection of C^*-algebras $(A_i)_{i\in I}$, indexed over some directed set I, together with morphisms $\varphi_{j,i}\colon A_i \to A_j$ for each indices $i \leq j$ and such that $\varphi_{k,j} \circ \varphi_{j,i} = \varphi_{k,i}$ for each indices $i \leq j \leq k$. The inductive limit of such an inductive system is constructed as follows: First consider the algebraic inductive limit, $L := (\coprod_{i\in I} A_i)_\sim$, where the equivalence relation $\sim$ identifies any two elements $a_i \in A_i$ and $a_j \in A_j$ if for some index $k \geq i, j$ we have $\varphi_{k,i}(a_i) = \varphi_{k,j}(a_j)$. For a_i set $\|a_i\| := \lim_{j\geq i} \|\varphi_{j,i}(a_i)\|$, which defines a seminorm on L. Then the completion of $L/\{a_i \mid \|a_i\| = 0\}$ is a C^*-algebras, usually denoted by $\varinjlim A_i$. We denote the canonical morphisms into the inductive limit by $\varphi_{\infty,i}\colon A_i \to \varinjlim A_i$.

1.2. We denote by $\mathcal{C}^*_{ab}$ the category whose objects are commutative C^*-algebras, and whose morphisms are $*$-homomorphisms. As pointed out in [Bla06, II.2.2.7, p.61], this category is dually equivalent to the category $\mathcal{SP}_*$ whose objects are pointed, compact Hausdorff spaces and whose morphisms are pointed continuous maps.

For a locally compact, Hausdorff space X, let X^+ be the compact, Hausdorff space obtained by attaching one additional point x_∞ to X, i.e., $X^+ = X \sqcup \{x_\infty\}$ (disjoint union) if X is compact, and otherwise if X is not compact, then X^+ is the one-point compactification of X which we also denote by αX. In both cases, the basepoint of X^+ is the attached point x_∞. For a pointed space $(X, x_\infty) \in \mathcal{SP}_*$ define:

(1) $$C_0(X, x_\infty) := \{f\colon X \to \mathbb{C} \mid f(x_\infty) = 0\}.$$

Then, the dual equivalence between $\mathcal{C}^*_{ab}$ and $\mathcal{SP}_*$ is given by the following (contravariant) functors:

(2) $$A \mapsto (\mathrm{Prim}(A)^+, x_\infty), \qquad (X, x_\infty) \mapsto C_0(X, x_\infty).$$

Thus, to study the dimension theory of commutative C^*-algebras is equivalent to study the dimension theories of pointed, compact Hausdorff spaces.

1.3. For a space X, we denote by $\mathrm{Cov}_{\mathrm{fin}}(X)$ the collection of finite, open covers of X. For two covers $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}_{\mathrm{fin}}(X)$, we write $\mathcal{U} \leq \mathcal{V}$ if the cover $\mathcal{U}$ refines the cover $\mathcal{V}$, i.e., for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subset V$. The **order** of a covering, denoted by $\mathrm{ord}(\mathcal{U})$, is the least integer k such that some point $x \in X$ is contained in k different elements of $\mathcal{U}$. We refer the reader to chapter 2 of Nagami's book [Nag70] for definitions and further explanations.

The **covering dimension** of X, denoted by $\dim(X)$, is the smallest integer $n \geq 0$ (or ∞) such that:

(1) $$\forall\, \mathcal{U} \in \mathrm{Cov}_{\mathrm{fin}}(X) \;\; \exists\, \mathcal{V} \in \mathrm{Cov}_{\mathrm{fin}}(X) \; : \; \mathcal{V} \leq \mathcal{U}, \mathrm{ord}(\mathcal{V}) \leq n+1$$

It was pointed out by Morita, [Mor75], that in general this definition of covering dimension should be modified to consider only *normal*, finite open covers. However, for normal spaces (e.g. compact spaces) every finite, open cover is normal, so that we may use the original definition.

The **local covering dimension** of X, denoted by $\mathrm{locdim}(X)$, is the smallest integer $n \geq 0$ (or ∞) such that every point $x \in X$ is contained in a closed neighborhood F such that $\dim(F) \leq n$. We refer the reader to [Dow55] and [Pea75, Chapter 5] for more information about the local covering dimension.

It was noted by Brown and Pedersen, [BP09, Section 2.2 (ii)], that $\operatorname{locdim}(X) = \dim(\alpha X)$ for a locally compact, Hausdorff space X. We propose that for a pointed space $(X, x_\infty) \in \mathcal{SP}_*$, the natural dimension to consider is $\dim(X) = \operatorname{locdim}(X \setminus \{x_\infty\})$. Then, for a commutative C^*-algebra A, the natural dimension is $\operatorname{locdim}(\operatorname{Prim}(A))$.

If $G \subset X$ is an open subset of a locally compact space, then $\operatorname{locdim}(G) \leq \operatorname{locdim}(X)$, see [Dow55, 4.1]. It was also shown by Dowker that this does not hold for the usual covering dimension (of non-normal spaces).

1.4. Let A be a C^*-algebra. A family $\mathcal{B}$ of sub-C^*-algebras of A is said to **approximate**[1] A, if for every finite subset $F \subset A$, and $\varepsilon > 0$, there exists some C^*-algebra $B \in \mathcal{B}$ such that $F \subset_\varepsilon B$.

Let $\mathcal{P}$ be some property of C^*-algebras, and assume A is approximated by sub-C^*-algebras with property $\mathcal{P}$. Then we say that A is **$\mathcal{P}$-like**[2], see [Thi11, Definition 3.2]. This is motivated by the concept of $\mathcal{P}$-likeness for commutative spaces, as defined in [MS63, Definition 1] and further developed in [MM92].

Let $\mathcal{P}$ be a non-empty class of pointed, compact spaces. Another pointed, compact space (X, x_∞) is said to be $\mathcal{P}$-like if for every $\mathcal{U} \in \operatorname{Cov}_{\text{fin}}(X)$ there exists a (pointed) map $f\colon X \to Y$ onto some $Y \in \mathcal{P}$ and $\mathcal{V} \in \operatorname{Cov}_{\text{fin}}(Y)$ such that $f^{-1}(\mathcal{V}) \leq \mathcal{U}$.

Note that we have used $\mathcal{P}$ to denote both a class of spaces and a property that spaces might enjoy. These are just different viewpoints, as we can naturally assign to a property the class of spaces with that property, and vice versa to each class of spaces the property of lying in that class.

Similarly to [Thi11, Proposition 3.4], one can show that for $(X, x_\infty) \in \mathcal{SP}_*$ and a collection $\mathcal{P} \subset \mathcal{SP}_*$ the following are equivalent:

(a) (X, x_∞) is $\mathcal{P}$-like
(b) $C_0(X, x_\infty)$ can be approximated by sub-C^*-algebras $C_0(Y, y_\infty)$ with $(Y, y_\infty) \in \mathcal{P}$

We note that the definition of covering dimension can be rephrased as follows. Let $\mathcal{P}_k$ be the collection of all k-dimensional polyhedra (polyhedra are defined by combinatoric data, and their dimension is defined by this combinatoric data). Then a compact space X satisfies $\dim(X) \leq k$ if and only if it is $\mathcal{P}_k$-like.

1.5. Recall that a unital C^*-algebra A is called **finite**, if for any $x, y \in A$ with $xy = 1$ we also have $yx = 1$. A (not necesarily unital) C^*-algebra A is called **stably finite**, if $\widetilde{A} \otimes M_k$ is finite for every $k \geq 1$. Equivalently, $\widetilde{A \otimes M_k}$ is finite for every $k \geq 1$, or equivalently $\widetilde{A \otimes \mathbb{K}}$ is finite. A C^*-algebra is called **residually stably finite** if each of its quotients is stably finite.

1.6. Recall that a group G is called torsion-free if $g = 0$ whenever $ng = 0$ for some $n \geq 1$ and $g \in G$. For a group G, let G_{tor} denote the subgroup of torsion-elements.

Let $\mathcal{G} = (G_i, \varphi_{ji})$ be an inductive system (also called directed system) of groups, and recall that its inductive limit, denoted by $\varinjlim \mathcal{G}$ or $\varinjlim G_i$, is defined as the equivalence classes of $\coprod_i G_i$ by the relation that identifies two elements $g \in G_i, h \in G_j$ whenever $\varphi_{ik}(g) = \varphi_{jk}(h)$ for some $k \geq i, k$. The maps $\varphi_{ji}\colon G_i \to G_j$ map the torsion subgroup $(G_i)_{\text{tor}}$

[1] In the literature there also appears the formulation '$\mathcal{B}$ locally approximates A'.

[2] A is also called 'locally $\mathcal{P}$', see also [Thi11, 3.1 - 3.3].

into $(G_j)_{\mathrm{tor}}$. Thus, we get a natural subsystem $\mathcal{G}_{\mathrm{tor}} = ((G_i)_{\mathrm{tor}}, \varphi_{ji})$, and a quotient system $\mathcal{G}/\mathcal{G}_{\mathrm{tor}} = (G_i/(G_i)_{\mathrm{tor}}, [\varphi_{ji}])$.

We collect some algebraic facts about torsion-free groups.

(1) A subgroup of a torsion-free group is again torsion-free.
(2) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. If A and C are torsion-free, then so is B.
(3) Let $\mathcal{G} = (G_i, \varphi_{ji})$ be an inductive system of groups, with inductive limit $G := \varinjlim \mathcal{G}$. Then $G_{\mathrm{tor}} \cong \varinjlim \mathcal{G}_{\mathrm{tor}}$ and $G/G_{\mathrm{tor}} \cong \varinjlim(\mathcal{G}/\mathcal{G}_{\mathrm{tor}})$. Thus, if all G_i are torsion-free, then so is their inductive limit G.

Recall that K-theory is continuous in the sense that $K_*(\varinjlim A_i) = \varinjlim K_*(A_i)$ for $* = 0, 1$. Hence, if $A = \varinjlim A_i$ is a direct limit of C*-algebras A_i with $K_*(A_i)$ torsion-free, then $K_*(A)$ is torsion-free. Note that the A_i are not required to be unital, and the connecting morphisms are not required to be injective or unital.

Similarly, we get the following. Let A be a C^*-algebra that is approximated by a collection $(A_i)_{i\in I}$ of sub-C^*-algebras. If each $K_*(A_i)$ is torsion-free, then $K_*(A)$ is torsion-free, for $* = 0, 1$.

2. Dimension theories for C^*-algebras

In this section we propose a set of axioms for a non-commutative dimension theory. These axioms are generalizations of properties of the dimension of locally compact, Hausdorff spaces. Below we will see that the axioms are satisfied by many theories, in particular the real and stable rank, the topological dimension, the decomposition rank and the nuclear dimension.

Definition 2.1. Let $\mathcal{C}$ be a class of C^*-algebras that is closed under taking ideals, quotients, finite direct sums, and minimal unitalizations. A **dimension theory** for $\mathcal{C}$ is an assignment $d\colon \mathcal{C} \to \overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ such that $d(A) = d(A')$ if $A \cong A'$, and moreover the following axioms are satisfied:

(D1) $d(I) \leq d(A)$ whenever $I \lhd A$ is an ideal in A,
(D2) $d(A/I) \leq d(A)$ whenever $I \lhd A$,
(D3) $d(A \oplus B) = \max\{d(A), d(B)\}$,
(D4) $d(A) = d(\widetilde{A})$.
(D5) If A is approximated by sub-C^*-algebras[3] $B \subset A$ with $d(B) \leq n$, then $d(A) \leq n$.
(D6) For every separable sub-C^*-algebra $C \subset A$, there exists a separable C^*-algebra $D \in \mathcal{C}$ such that $C \subset D \subset A$ and $d(D) \leq d(A)$.

Proposition 2.2. *Let $d\colon \mathcal{C} \to \overline{\mathbb{N}}$ be a dimension theory, and let $(A_i, \varphi_{i,j})$ be an inductive system (see 1.1) with $A_i \in \mathcal{C}$ and such that the limit $A := \varinjlim A_i$ also lies in $\mathcal{C}$. Then $d(A) \leq \liminf_i d(A_i)$.*

Proof. For each i, the sub-C^*-algebra $\mathrm{im}(\varphi_{\infty,i}) \subset A$ is a quotient of A_i, and therefore $d(\mathrm{im}(\varphi_{\infty,i})) \leq d(A_i)$ by (D2). If $J \subset I$ is cofinal, then A is approximated by the collection

[3] We implicitly assume that the approximating sub-C^*-algebras B lie in the class $\mathcal{C}$ so that $d(B)$ is defined. Note that we do not assume that $\mathcal{C}$ is closed under approximation by sub-C^*-algebra, we do not even assume $\mathcal{C}$ is closed under inductive limits.

of sub-C^*-algebras $(\mathrm{im}(\varphi_{\infty,i}))_{i\in J}$. Then by (D5), $d(A)$ is bounded by $\sup_{i\in J} d(A_i)$. Since this holds for each cofinal subset $J \subset I$, we obtain:

$$d(A) \leq \inf\{\sup_{i\in J} d(A_i) \mid J \subset I \text{ cofinal}\} = \liminf_i d(A_i),$$

as desired. □

Remark 2.3. The axioms in Definition 2.1 are inspired by well-known facts of the local covering dimension of commutative spaces, see 1.3.

Let us see how the axioms translate to the commutative setting. Axiom (D1) and (D2) generalize the fact that the local covering dimension does not increase when passing to an open (resp. closed) subspace (we think of ideals and quotients of a C^*-algebra as the generalization of open and closed subsets), and axiom (D3) generalizes the fact that $\mathrm{locdim}(X \sqcup Y) = \max\{\mathrm{locdim}(X), \mathrm{locdim}(Y)\}$. Axiom (D4) generalizes that $\mathrm{locdim}(X) = \mathrm{locdim}(\alpha X)$, where αX is the one-point compactification of X.

Axiom (D5) generalizes the fact that a (compact) space is n-dimensional if it is $\mathcal{P}$-like for some class $\mathcal{P}$ of n-dimensional spaces, see 1.4. The immediate consequence noted in Proposition 2.2 generalizes the fact that $\dim(\varprojlim X_i) \leq \liminf_i \dim(X_i)$ for an inverse system of compact spaces X_i.

Axiom (D6) is a generalization of the following factorization theorem, due to Mardešić, see [Mar60, Lemma 4]: Given a compact space X and a map $f\colon X \to Y$ to a compact, metrizable space Y, there exists a compact, metrizable space Z and maps $g\colon X \to Z, h\colon Z \to Y$ such that g is onto, $\dim(Z) \leq \dim(X)$ and $f = h \circ g$. This generalizes (D6), since for a unital, commutative C^*-algebra $C(X)$ the following are equivalent: $C(X)$ is separable, X is metric, X is second countable, X is separable.

Axioms (D5) and (D6) are also related to the following concept which is due to Blackadar, [Bla06, Definition II.8.5.1, p.176]: A property $\mathcal{P}$ of C^*-algebras is called **separably inheritable** if:

(i) For every C^*-algebra A with property $\mathcal{P}$ and separable sub-C^*-algebra $C \subset A$, there exists a separable sub-C^*-algebra $D \subset A$ that contains C and has property $\mathcal{P}$,

(ii) For every inductive system (A_k, φ_k) with injective connecting morphisms if separable C^*-algebra A_n that have property $\mathcal{P}$, the inductive limit $\varinjlim A_k$ has property $\mathcal{P}$.

Thus, for a dimension theory d, the property '$d(A) \leq n$' is separably inheritable.

Axioms (D5) and (D6) imply that $d(A) \leq n$ if and only if A can be written as an inductive limit (with injective connecting morphisms) of separable C^*-algebras B with $d(B) \leq n$.

Proposition 2.4. *The assignment* $d\colon \mathcal{C}^*_{ab} \to \overline{\mathbb{N}}$, $d(A) := \mathrm{locdim}(\mathrm{Prim}(A))$, *is a dimension theory.*

Proof. Note that $\mathrm{Prim}(A)$ is a locally compact, Hausdorff space. Axioms (D1) and (D2) follow from [Dow55, 4.1, 3.1]. Axiom (D3) is easy to check, and (D4) holds since $\mathrm{locdim}(X) = \mathrm{locdim}(\alpha X)$, see 1.3.

Note that a compact space X satisfies $\dim(X) \leq k$ if it is $\mathcal{P}$-like for some class $\mathcal{P}$ consisting of k-dimensional spaces (meaning that their covering dimension is at most k). Then axiom (D5) follows from 1.4, and axiom (D6) follows from [Mar60, Lemma 4]. □

Lemma 2.5. *Let A be a C^*-algebra, and $B \subset A$ a full, hereditary sub-C^*-algebra, and $C \subset A$ a separable sub-C^*-algebra. Then there exists a separable sub-C^*-algebra $D \subset A$ containing C such that $D \cap B \subset D$ is full, hereditary.*

Proof. The proof is inspired by the arguments of [Bla78, Proposition 2.2], see also [Bla06, II.8.5.6, p.178]. We inductively define separable sub-C^*-algebras $D_k \subset A$. Set $D_0 := C$, and assume D_{k-1} has been constructed. Let $S_k := \{x_1^k, x_2^k, \ldots\}$ be a countable, dense subset of D_k. Since B is full in A, there exist for each $i \geq 1$ finitely many elements $a_{i,j}^k, c_{i,j}^k \in A$ and $b_{i,j}^k \in B$ such that

$$\|x_i^k - \sum_j a_{i,j}^k b_{i,j}^k c_{i,j}^k\| < 1/k.$$

Set $D_k := C^*(D_{k-1}, a_{i,j}^k, b_{i,j}^k, c_{i,j}^k, i, j \geq 1)$. Then define $D := \overline{\bigcup_k D_k}$, which is a separable sub-C^*-algebra of A containing C.

Note that $D \cap B \subset D$ is a hereditary sub-C^*-algebra, and let us check that it is also full. We need to show that the linear span of $D(D \cap B)D$ is dense in D. Let $d \in D$ and $\varepsilon > 0$ be given. Note that $\bigcup_k S_k$ is dense in D. Thus, we may find k and i such that $\|d - x_i^k\| < \varepsilon/2$. We may assume $k \geq 2/\varepsilon$. By construction, there are elements $a_{i,j}^k, c_{i,j}^k \in D_{k+1}$ and $b_{i,j}^k \in D_{k+1} \cap B \subset D \cap B$ such that $\|x_i^k - \sum_j a_{i,j}^k b_{i,j}^k c_{i,j}^k\| < 1/k$. It follows that the distance from d to the closed linear span of $D(D \cap B)D$ is at most ε. □

Proposition 2.6. *Let $d\colon \mathcal{C}^* \to \overline{\mathbb{N}}$ be a dimension theory. Then the following statements are equivalent:*

(1) For all C^*-algebras A, B: If $B \subset A$ is a full, hereditary sub-C^*-algebra, then $d(B) = d(A)$.
(2) For all C^*-algebras A, B: If A and B are Morita equivalent, then $d(A) = d(B)$.
(3) For all C^*-algebras A: $d(A) = d(A \otimes \mathbb{K})$.

Moreover, each of the statements is equivalent to the (a priori weaker) statement where the appearing C^*-algebras are additionally assumed to be separable.

If d satisfies the above conditions, and $B \subset A$ is a (not necessarily full) hereditary sub-C^*-algebra, then $d(B) \leq d(A)$.

Proof. For each of the statements $(1), (2), (3)$, let us denote the statement where the appearing C^*-algebras are assumed to be separable by $(1s), (2s), (3s)$ respectively. For example:

(3s) For all separable C^*-algebras A: $d(A) = d(A \otimes \mathbb{K})$.

Then the implications '$(1) \Rightarrow (1s)$', '$(2) \Rightarrow (2s)$', '$(3) \Rightarrow (3s)$' clearly hold.

'$(2s) \Rightarrow (3s)$' follows since A and $A \otimes \mathbb{K}$ are Morita equivalent.

'$(1s) \Rightarrow (3s)$' follows since $A \subset A \otimes \mathbb{K}$ is a full, hereditary sub-C^*-algebra.

'$(3s) \Rightarrow (1)$': Let A be a C^*-algebra, and $B \subset A$ a full, hereditary sub-C^*-algebra. We need to show $d(B) = d(A)$.

We will approximate A by separable sub-C^*-algebras $D \subset A$ such that $d(D \cap B) = d(D) \leq \min\{d(B), d(A)\}$. From this it easily follows that $d(B) = d(A)$.

So let $H \subset A$ be a finite set. We want to find D with the mentioned properties and $H \subset D$. We inductively define separable sub-C^*-algebras $\widetilde{F_k} \subset B$, $E_k, G_k \subset A$ such that:

(a) $E_k \supset G_{k-1}$ and $E_k \cap B \subset E_k$ is full

(b) $\widetilde{F_k} \supset E_k \cap B$ and $d(\widetilde{F_k}) \leq d(B)$
(c) $G_k \supset \widetilde{F_k}, E_k$ and $d(G_k) \leq d(A)$

We start with $G_0 := C^*(H) \subset A$. If G_{k-1} has been constructed, we apply Lemma 2.5 to find E_k satisfying (a). If E_k has been constructed, we apply axiom (D6) to $E_k \cap B \subset B$ to find $\widetilde{F_k}$ satisfying (b). If $\widetilde{F_k}$ has been constructed, we apply axiom (D6) to $C^*(\widetilde{F_k}, E_k) \subset A$ to satisfy (c).

Set $D := \overline{\bigcup_k E_k} = \overline{\bigcup_k G_k}$, which is a separable sub-C^*-algebra of A that contains H. Since D is approximated by $(G_k)_k$, we have $d(D) \leq d(A)$ by axiom (D5). Note that $D \cap B$ is approximated by $(\widetilde{F_k})_k$, and therefore $d(D \cap B) \leq d(B)$. Moreover, it is easy to check that $D \cap B \subset D$ is a full, hereditary sub-C^*-algebra. Since D is separable, we may apply Brown's stabilization theorem, [Bro77], and obtain $(D \cap B) \otimes \mathbb{K} \cong D \otimes \mathbb{K}$. By assumption, $d(D \cap B) = d((D \cap B) \otimes \mathbb{K}) = d(D \otimes \mathbb{K}) = d(D)$. This finishes the construction of D.

Lastly, if d satisfies condition (1), and $B \subset A$ is a (not necessarily full) hereditary sub-C^*-algebra, then B is full, hereditary in the ideal $I \lhd A$ generated by B. By (D1) and condition (1) we have $d(B) = d(I) \leq d(A)$. □

Definition 2.7. A dimension theory $d\colon \mathcal{C}^* \to \overline{\mathbb{N}}$ is called **Morita-invariant** if it satisfies the conditions of Proposition 2.6.

Proposition 2.8. *Let $d\colon \mathcal{C} \to \overline{\mathbb{N}}$ be a dimension theory. For any C^*-algebra A define*[4]*:*

$$\widetilde{d}(A) := \inf\{k \in \overline{\mathbb{N}} \mid A \text{ is approximated by sub-}C^*\text{-algebras } B \subset A \text{ with } d(B) \leq k\}. \tag{1}$$

Then $\widetilde{d}\colon \mathcal{C}^* \to \overline{\mathbb{N}}$ is a dimension theory that agrees with d on $\mathcal{C}$.

If moreover $\mathcal{C}$ was closed under stable isomorphism, and $d(A) = d(A \otimes \mathbb{K})$ for every (separable) $A \in \mathcal{C}$, then $\widetilde{d}$ is Morita-invariant.

Proof. If $A \in \mathcal{C}$, then clearly $\widetilde{d}(A) \leq d(A)$, and the converse inequality follows from axiom (D5). Axioms (D1)-(D5) for $\widetilde{d}$ are easy to check.

Let us check axiom (D6). Assume $C \subset A$ is separable, and let $n := d(A)$, which we may assume is finite. We first note the following: For a finite set $F \subset A$, and $\varepsilon > 0$ we can find a separable sub-C^*-algebra $A(F, \varepsilon) \subset A$ with $d(A(F,\varepsilon)) \leq n$ and $F \subset_\varepsilon A(F,\varepsilon)$. Indeed, by definition of $\widetilde{d}$ we can first find a sub-C^*-algebra $B \subset A$ with $d(B) \leq n$ and a finite subset $G \subset B$ such that $F \subset_\varepsilon G$. Applying (D6) to $C^*(G) \subset B$, we may find a separable sub-C^*-algebra $A(F,\varepsilon) \subset B$ with $d(A(F,\varepsilon)) \leq n$ and $C^*(H) \subset A(F,\varepsilon)$, which implies $F \subset_\varepsilon A(F,\varepsilon)$.

We will inductively define separable sub-C^*-algebras $D_k \subset A$ and countable dense subsets $S_k = \{x_1^k, x_2^k, \ldots\} \subset D_k$ as follows: We start with $D_1 := C$ and choose any countable dense subset $S_1 \subset D_1$. If D_i and S_i have been constructed for $i \leq k$, then set:

$$D_{k+1} := C^*(D_k, A(\{x_i^j\}_{i,j\leq k}, 1/k)) \subset A, \tag{2}$$

and choose any countable dense subset $S_{k+1} = \{x_1^{k+1}, x_2^{k+1}, \ldots\} \subset D_{k+1}$.

[4]By definition, the infimum over the empty set is $\infty \in \overline{\mathbb{N}}$. We implicitly assume that the approximating subalgebras lie in $\mathcal{C}$.

Set $D := \overline{\bigcup_k D_k} \subset A$, which is a separable C^*-algebra containing C. Let us check that $\widetilde{d}(D) \leq n$. Note that $\{x_i^j\}_{i,j\geq 1}$ is dense in D. Thus, if a finite subset $F \subset D$, and $\varepsilon > 0$ is given, we may find k such that $F \subset_{\varepsilon/2} \{x_i^j\}_{i,j\leq k}$, and we may assume $k > 2/\varepsilon$. By construction, D contains the sub-C^*-algebra $B := A(\{x_i^j\}_{i,j\leq k}, 1/k)$, which satisfies $d(B) \leq n$ and $\{x_i^j\}_{i,j\leq k} \subset_{1/k} B$. Then $F \subset_e B$, which completes the proof that $\widetilde{d}(D) \leq n$.

Lastly, assume $\mathcal{C}$ is closed under stable isomorphism, and $d(A) = d(A \otimes \mathbb{K})$ for every separable $A \in \mathcal{C}$. We want to check condition (3) of Proposition 2.6 for $\widetilde{d}$. So let A be any separable C^*-algebra.

If $\widetilde{d}(A) = \infty$, then clearly $\widetilde{d}(A \otimes \mathbb{K}) \leq \widetilde{d}(A)$. So assume $n := \widetilde{d}(A) < \infty$, which means that A is approximated by sub-C^*-algebras $(A_i)_i$ with $d(A_i) \leq n$. Then $A \otimes \mathbb{K}$ is approximated by the sub-C^*-algebras $(A_i \otimes \mathbb{K})_i$, and $d(A_i \otimes \mathbb{K}) = d(A_i) \leq n$ by assumption. Then $\widetilde{d}(A \otimes \mathbb{K}) \leq n = \widetilde{d}(A)$.

Conversely, if $\widetilde{d}(A \otimes \mathbb{K}) = \infty$, then $\widetilde{d}(A) \leq \widetilde{d}(A \otimes \mathbb{K})$. So assume $n := \widetilde{d}(A \otimes \mathbb{K}) < \infty$, which means that $A \otimes \mathbb{K}$ is approximated by sub-C^*-algebras $(A_i)_i$ with $d(A_i) \leq n$. Consider the full, hereditary sub-C^*-algebra $A \otimes e_{1,1} \in A \otimes \mathbb{K}$, which is isomorphic to A. Then $A \otimes e_{1,1}$ is approximated by the C^*-algebras $B_i := e_{1,1}A_i e_{1,1}$. Note that B_i is a hereditary sub-C^*-algebra of A_i. Since B_i and A_i are separable, we get from Brown's theorem that B_i is stably isomorphic to an ideal of A_i, and then $d(B_i) \leq d(A_i)$ from (D1) and the assumption. Then $\widetilde{d}(A) \leq n = \widetilde{d}(A \otimes \mathbb{K})$. Together we get $\widetilde{d}(A) = \widetilde{d}(A \otimes \mathbb{K})$, as desired. □

2.9. Axioms (D5) and (D6) allow to reduce essentially every question about dimension theories to the case of separable C^*-algebras. Put differently, it is enough to prove a statement about dimension theories for separable C^*-algebras, with the benefit that separable C^*-algebras are often easier to work with.

It is of course impossible to prove such a meta-statement. We just mention that one can for instance prove the following:

(i) Let $d\colon \mathcal{C}^* \to \overline{\mathbb{N}}$ be a dimension theory, A a C^*-algebra, $I \lhd A$ an ideal, and $C \subset A$ a separable sub-C^*-algebra. Then there exists a separable sub-C^*-algebra $D \subset A$ such that $d(D \cap I) \leq d(I)$ and $d(D/(D \cap I)) \leq d(A/I)$.

We will use these techniques to extend interesting results from the separable case to general C^*-algebras, in particular a theorem of Winter, [Win04, Theorem 1.6], computing the decomposition rank of subhomogeneous C^*-algebras, see Proposition 7.3, and a result of Lin, [Lin97], characterizing type I C^*-algebras of real rank zero, see Proposition 8.11.

3. Stable and real rank

The first generalization of classical dimension theory to non-commutative spaces was the stable rank as introduced by Rieffel, [Rie83]. It is a generalization of the characterization of dimension by fragility of maps.

Later, Brown and Pedersen defined the real rank in a similar way, see [BP91]. Nevertheless the real rank often behaves much different from the stable rank, but in some situations the two theories are dual to each other, see Proposition 8.12.

3.1. Let A be a unital C^*-algebra. Define:

(1) $$\mathrm{Lg}_n(A) := \{(a_1,\dots,a_n)\in A^n \mid \sum_{i=1}^n a_i^*a_i \in A^{-1}\},$$

(2) $$\mathrm{Lg}_n(A)_{\mathrm{sa}} := \mathrm{Lg}_n(A)\cap (A_{\mathrm{sa}})^n = \{(a_1,\dots,a_n)\in (A_{\mathrm{sa}})^n \mid \sum_{i=1}^n a_i^2 \in A^{-1}\},$$

where A^{-1} denotes the set of invertible elements of A. The abbreviation 'Lg' stands for 'left generators', and the reason is that a tuple $(a_1,\dots,a_n)\in A^n$ lies in $\mathrm{Lg}_n(A)$ if and only if $\{a_1,\dots,a_n\}$ generate A as a left (not necessarily closed) ideal, i.e., $Aa_1+\dots+Aa_n=A$.

Let $\mu\colon A^n\to A$ be defined as $\mu(a_1,\dots,a_n) := \sum_i a_i^*a_i$. Then $\mathrm{Lg}_n(A)=\mu^{-1}(A^{-1})$. A unital morphism $\varphi\colon A\to B$ sends A^{-1} to B^{-1}, and therefore also $\mathrm{Lg}_n(A)$ to $\mathrm{Lg}_n(B)$.

Definition 3.2 (Rieffel, [Rie83, Definition 1.4], Brown, Pedersen, [BP91])**.** Let A be a C^*-algebra. Assume first that A is unital. The **stable rank** of A, denoted by $\mathrm{sr}(A)$, is the least integer $n\geq 1$ (or ∞) such that $\mathrm{Lg}_n(A)$ is dense in A^n. The **real rank** of A, denoted by $\mathrm{rr}(A)$, is the least integer $n\geq 0$ (or ∞) such that $\mathrm{Lg}_{n+1}(A)_{\mathrm{sa}}$ is dense in A_{sa}^{n+1}.

If A is not unital, define $\mathrm{sr}(A) := \mathrm{sr}(\widetilde{A})$ and $\mathrm{rr}(A) := \mathrm{rr}(\widetilde{A})$.

Remark 3.3. There is a subtlety of indices in the above definition. Just to make things clear, we have:

(1) $\mathrm{sr}(A)\leq n :\Leftrightarrow \mathrm{Lg}_n(\widetilde{A})\subset(\widetilde{A})^n$ is dense.
(2) $\mathrm{rr}(A)\leq n :\Leftrightarrow \mathrm{Lg}_{n+1}(\widetilde{A})\subset(\widetilde{A}_{\mathrm{sa}})^{n+1}$ is dense.

The smallest possible value of the stable rank is one. If A is unital, then $\mathrm{Lg}_1(A)$ consists precisely of the left-invertible elements of A. Thus, A has stable rank one if and only if the left-invertible elements of A are dense in A. It was shown in [Rie83, Proposition 3.1] that this is also equivalent to the condition that the invertible elements of A are dense in A.

On the other hand, the smallest possible value of the real rank is zero, which by definition happens precisely if the *self-adjoint* (left-)invertible[5] elements in A are dense in A_{sa}.

Theorem 3.4. *The real and stable rank are dimension theories in the sense of Definition 2.1 for the class of all C^*-algebras.*

Proof. Axioms (D1) and (D2) are shown in [Rie83, Theorems 4.3, 4.4] for the stable rank, and in [EH95, Théorème 1.4] for the real rank. Axiom (D3) is easily verified, and (D4) holds by definition.

The special case of (D5) for countable inductive limits is shown in [Rie83, Theorem 5.1], but the same argument works for general approximations and also for the real rank. It is noted in [Bla06, II.8.5.5, p.178] that (D6) holds. □

3.5. To obtain interesting results in non-stable K-theory, see e.g. [Rie87], and to compute the stable rank of an extension, see Proposition 3.10, it is useful to study the fine structure of $\mathrm{Lg}_n(A)$. We need some notation.

Let $\mathrm{Gl}_n(A)$ denote the group of invertible elements in $M_n(A)=A\otimes M_n$, and $\mathrm{Gl}_n(A)_0$ the connected component of $\mathrm{Gl}_n(A)$ containing the identity. Note that $\mathrm{Gl}_n(A)$ acts on A^n

[5] A self-adjoint element is left-invertible if and only if it is invertible.

by multiplication (say, from the left), and this drops to an action $\lambda\colon \mathrm{Gl}_n(A)\times\mathrm{Lg}_n(A)\to\mathrm{Lg}_n(A)$. We give A^n and $Gl_n(A)$ the norms induced from A. Then λ is jointly continuous, each orbit is open and closed in $\mathrm{Lg}_n(A)$, and for each $a\in\mathrm{Lg}_n(A)$ the map $x\mapsto\lambda(x,a)$ from $\mathrm{Gl}_n(A)$ onto the orbit of a is an open map and a Serre fibration, see [Rie87].

Definition 3.6 (Rieffel, [Rie83, Definition 4.7]). Let A be a C^*-algebra. Assume first that A is unital. The **connected stable rank** of A, denoted $\mathrm{csr}(A)$, is the least integer $n\geq 1$ (or ∞) such that $\mathrm{Gl}_m(A)_0$ acts transitively on $\mathrm{Lg}_m(A)$ for all $m\geq n$. The **general stable rank** of A, denoted $\mathrm{gsr}(A)$, is the least integer $n\geq 1$ (or ∞) such that $\mathrm{Gl}_m(A)$ acts transitively on $\mathrm{Lg}_m(A)$ for all $m\geq n$.

If A is not unital, define $\mathrm{csr}(A):=\mathrm{csr}(\widetilde{A})$ and $\mathrm{gsr}(A):=\mathrm{gsr}(\widetilde{A})$.

Proposition 3.7 (Brown, Pedersen, [BP91, Proposition 1.2], Nistor, [Nis86, Lemma 2.4], Rieffel, [Rie83, Corollary 7.2]). *Let A be a C^*-algebra. Then the following estimates hold:*

$$\mathrm{rr}(A)\leq 2\cdot\mathrm{sr}(A)-1, \tag{1}$$

$$\mathrm{gsr}(A)\leq\mathrm{csr}(A)\leq\mathrm{sr}(C([0,1])\otimes A)\leq\mathrm{sr}(A)+1. \tag{2}$$

For the next result, recall that a C^*-algebra A is homotopy dominated by another C^*-algebra B if there are morphisms $\varphi\colon A\to B$ and $\psi\colon B\to A$ such that $\psi\circ\varphi$ is homotopic to id_A. If A and B are unital, then we say A is unitally homotopy dominated if the above maps φ and ψ can be chosen unital.

Proposition 3.8 (Nistor, [Nis86, Lemma 2.8]). *Let A,B be unital C^*-algebras, and assume A is unitally homotopy dominated by B. Then $\mathrm{gsr}(A)\leq\mathrm{gsr}(B)$ and $\mathrm{csr}(A)\leq\mathrm{csr}(B)$.*

Proof. The result for the connected stable rank is [Nis86, Lemma 2.8], and we vary the argument slightly for the general stable rank.

Assume unital morphisms $\varphi\colon A\to B$ and $\psi\colon B\to A$ are given such that $\psi\circ\varphi$ is homotopic to id_A. Let $k:=\mathrm{gsr}(B)$, which we may assume to be finite. Let $a=(a_1,\ldots,a_m)\in\mathrm{Lg}_m(A)$, for some $m\geq k$. Since $\psi\circ\varphi$ is homotopic to id_A through unital morphisms, a is homotopic to $\psi\circ\varphi(a)$ in $\mathrm{Lg}_n(A)$. We may lift this path to an invertible $x\in\mathrm{Gl}_m(A)_0$ such that $x\cdot a=\psi\circ\varphi(a)$, see 3.5.

Since $\mathrm{gsr}(B)=k$, there exists $y\in\mathrm{Gl}_m(B)$ such that $\varphi(x)=y\cdot e_1$, where $e_1=(1,0,\ldots,0)$. Then $(x^{-1}\psi(y))\cdot e_1=a$, which shows that $\mathrm{Gl}_m(A)$ acts transitively on $\mathrm{Lg}_m(A)$, as desired. □

3.9. We note that the general and connected stable rank satisfy axioms (D3)-(D6), but they are not dimension theories in the sense of Definition 2.1.

To see this, consider the Toeplitz algebra $\mathcal{T}=C^*(S)$, see Example 9.1. Then $\mathrm{sr}(\mathcal{T})=\mathrm{csr}(\mathcal{T})=\mathrm{gsr}(\mathcal{T})=2$. Consider also the C^*-algebra $A=C^*(S\oplus S^*)$ from 9.2, which is stably finite, but not residually stably finite. Then $\mathrm{sr}(A)=\mathrm{csr}(A)=2$, and $\mathrm{gsr}(A)=1$, which shows that the general stable rank does not satisfy (D2).

The connected stable rank does also not satisfy (D2). Indeed, for $A=C(D^2)$ (continuous functions on the two-dimensional disk) and the quotient $C(S^1)$ we have $\mathrm{csr}(A)=1$ while $\mathrm{csr}(C(S^1))=2$.

The following result shows how the connected stable rank is used to bound the stable rank of an extension.

Proposition 3.10. *Let A be a C^*-algebra, and $I \lhd A$ an ideal. Then:*

(1) $\mathrm{sr}(A) \leq \max\{\mathrm{sr}(I), \mathrm{sr}(A/I), \mathrm{csr}(A/I)\}.$

If the ideal I is σ-unital with $\mathrm{rr}(I) = 0$, $\mathrm{sr}(A) = 1$ and $K_1(I) = 0$, then: .

(2) $\mathrm{rr}(A) = \mathrm{rr}(A/I).$

Proof. The result for the stable rank is [Rie83, Theorem 4.11]. If I is σ-unital with $\mathrm{rr}(I) = 0$, $\mathrm{sr}(A) = 1$ and $K_1(I) = 0$, then the multiplier algebra $M(I)$ has real rank zero, see [Lin93]. Note that A is a pullback of $M(I)$ and A/I. Therefore, $\mathrm{rr}(A) \leq \max\{\mathrm{rr}(M(I)), \mathrm{rr}(A/I)\} = \mathrm{rr}(A/I)$, see [NOP01, Proposition 1.6], see also Proposition 3.17 and [Osa03, Proposition 3.4]. □

3.11. The result for the stable rank is much better than for the real rank. It is actually a principle that the stable rank is often easier to compute than the real rank. Note also that we can combine the above results to get the following estimates, whenever $I \lhd A$:

(1) $\max\{\mathrm{sr}(I), \mathrm{sr}(A/I)\} \leq \mathrm{sr}(A) \leq \max\{\mathrm{sr}(I), \mathrm{sr}(A/I) + 1\}.$

The connected stable rank behaves much different with respect to extensions. Instead of $\mathrm{csr}(I), \mathrm{csr}(A/I) \leq \mathrm{csr}(A)$ we have the following, see [Nag87, Lemma 2]:

(2) $\mathrm{csr}(A) \leq \max\{\mathrm{sr}(I), \mathrm{sr}(A/I)\}.$

We note that it follows from the example in 3.9 that the analog inequality does not hold for the general stable rank.

Recall that for a real number t we let $\lceil t \rceil$ denote the least integer greater than or equal to t, and similarly $\lfloor t \rfloor$ denotes the integer part of t.

Proposition 3.12. *Let A be a C^*-algebra, and $n \geq 1$. Then:*

(1) $$\mathrm{sr}(A \otimes M_n) = \left\lceil \frac{\mathrm{sr}(A) - 1}{n} \right\rceil + 1,$$

(2) $$\mathrm{gsr}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{gsr}(A) - 1}{n} \right\rceil + 1,$$

(3) $$\mathrm{csr}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{csr}(A) - 1}{n} \right\rceil + 1.$$

Proof. The formula for the stable rank is shown in [Rie83, Theorem 6.1]. The estimate for the connected stable rank is proved in [Rie87, Theorem 4.7] and also [Nis86, Proposition 2.10], and the proof can easily be adopted to obtain the result for the general stable rank. □

Remark 3.13. The analog of the above result for the real rank would be the statement

(1) $$\mathrm{rr}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{rr}(A)}{2n - 1} \right\rceil.$$

We note that it is not known if this statement holds in general, although it has been verified for special classes of C^*-algebras: The estimate was shown to hold for commutative C^*-algebras by Beggs and Evans, [BE91, Corollary 3.2], see also Lemma 6.9. Moreover, if

$\mathrm{rr}(A)=0$, then $\mathrm{rr}(A\otimes M_n)=0$, which shows the estimate in this particular case, see [BP91, Theorem 2.10].

See 1.5 for the definition of stable finiteness.

Proposition 3.14. *Let A be a C^*-algebra. Then:*

$$\mathrm{sr}(A\otimes\mathbb{K})=\begin{cases}1 & \text{if } \mathrm{sr}(A)=1\\ 2 & \text{if } \mathrm{sr}(A)\geq 2\end{cases} \tag{1}$$

$$\mathrm{gsr}(A\otimes\mathbb{K})=\begin{cases}1 & \text{if } A \text{ is stably finite}\\ 2 & \text{if } A \text{ is not stably finite}\end{cases} \tag{2}$$

$$\mathrm{csr}(A\otimes\mathbb{K})=\begin{cases}1 & \text{if } A \text{ is stably finite and } K_1(A)=0\\ 2 & \text{if } A \text{ is not stably finite or } K_1(A)\neq 0\end{cases} \tag{3}$$

$$\mathrm{rr}(A\otimes\mathbb{K})=\begin{cases}0 & \text{if } \mathrm{rr}(A)=0\\ 1 & \text{if } \mathrm{rr}(A)\geq 1\end{cases} \tag{4}$$

Proof. For the result on the stable rank see [Rie83, Theorem 6.4].

It follows $\mathrm{gsr}(A\otimes\mathbb{K})\leq\mathrm{csr}(A\otimes\mathbb{K})\leq\mathrm{sr}(C([0,1])\otimes A\otimes\mathbb{K})\leq 2$, by Proposition 3.7, see also [Nis86, Corollary 2.5]. Then $\mathrm{gsr}(A\otimes\mathbb{K})=1$ if and only if every left invertible element of $\widetilde{A\otimes\mathbb{K}}$ is invertible, which means precisely that A is stably finite. Similarly, $\mathrm{gsr}(A\otimes\mathbb{K})=1$ if and only if every left invertible element of $\widetilde{A\otimes\mathbb{K}}$ is invertible and in the component of the identity, which means precisely that A is stably finite and $K_1(A)=0$.

For the case of real rank, we have $\mathrm{rr}(A)=0$ if and only if $\mathrm{rr}(A\otimes\mathbb{K})=0$ by [BP91, Corollary 3.3, Theorem 2.5]. We have $\mathrm{rr}(A\otimes\mathbb{K})\leq 1$ by [BE91, Proposition 3.3]. □

3.15. Not much can be said about the behavior of real and stable rank with respect to hereditary sub-C^*-algebras, but see [Bla04] for the stable rank of a full corner.

3.16. If X and Y are spaces, then the product theorem for covering dimension shows that $\dim(X\times Y)\leq\dim(X)+\dim(Y)$ under certain weak conditions, e.g. if the spaces are metric or compact (but not in general). It follows that $\mathrm{locdim}(X\times Y)\leq\mathrm{locdim}(X)+\mathrm{locdim}(Y)$ for locally compact, Hausdorff spaces. Translated to C^*-algebras, this means that $\mathrm{rr}(A\otimes B)\leq\mathrm{rr}(A)+\mathrm{rr}(B)$ if A,B are commutative (since $\mathrm{rr}(C_0(X))=\mathrm{locdim}(X)$, see Proposition 6.8).

This estimate does not hold for general C^*-algebras. In [KO01, Example 1] it is shown that there exist two unital, separable, nuclear C^*-algebras A,B with $\mathrm{rr}(A)=\mathrm{rr}(B)=0$ but $\mathrm{rr}(A\otimes B)=1$, see also [Osa99, Theorem 2.1]. An easy (non-exact, non-separable) example is constructed using $B(H)$ (the bounded operators on a separable Hilbert space H), since $\mathrm{rr}(B(H))=0$ while $\mathrm{rr}(B(H)\otimes_{\mathrm{min}}B(H))\neq 0$, see [Osa99, Corollary 1.2], see also [KO95]. We note that the real rank of $B(H)\otimes_{\mathrm{min}}B(H)$ is not known.

In these examples, at least one of the C^*-algebras is not simple. It is an open question, if this surprising effect can also occur if both C^*-algebras are simple, see [Osa03, Question 3.11(2)].

Examples where $\mathrm{rr}(A\otimes B)$ is strictly less than $\mathrm{rr}(A)+\mathrm{rr}(B)$ are easily constructed. In fact, this can already happen in the commutative case, see [Pon30]. The same effect can occur for simple C^*-algebras: Let A be Rørdam's example of a simple, unital, separable, nuclear C^*-algebra that contains a finite and an infinite projections, see [Rør03, Corollary 7.1]. It was shown in [Rør05] that the real rank of A is not zero. Let B be any other simple, unital, non-type I (separable, nuclear) C^*-algebra with non-zero real rank (e.g. the Jiang-Su algebra). Then $A \otimes B$ is simple and purely infinite, [Rør02, Theorem 4.1.10, p.69], and every such algebra has real rank zero, [Zha90].

If one of the tensor factors is commutative, then the product formula holds for the real rank: $\mathrm{rr}(C_0(X) \otimes A) \leq \mathrm{rr}(A) + \mathrm{rr}(C_0(X))$, see [NOP01, Corollary 1.2].

For the stable rank, there exists to our knowledge no counterexample to the conjecture $\mathrm{sr}(A \otimes B) \leq \mathrm{sr}(A) + \mathrm{sr}(B)$. It is known that $\mathrm{sr}(C([0,1]) \otimes A) \leq \mathrm{sr}(A) + 1$, see [Rie83, Corollary 7.2]. It follows $\mathrm{sr}(C([0,1]^k) \otimes A) \leq \mathrm{sr}(A) + k$, and using pullbacks and inductive limits one can show $\mathrm{sr}(C(X) \otimes A) \leq \mathrm{sr}(A) + \dim(X)$ for any compact space X. However, it is not known whether $\mathrm{sr}(C([0,1]^2) \otimes A) \leq \mathrm{sr}(A) + 1$ holds. This problem goes back to Rieffel, [Rie83, Question 1.8], and if it has a positive answer than one can show that the product formula $\mathrm{sr}(A \otimes B) \leq \mathrm{sr}(A) + \mathrm{sr}(B)$ holds if at least one of the tensor factors is type I. If both A and B are type I, then the product formula can be derived from the results about the stable rank of CCR algebras, [Bro07], see Theorem 8.4, and this was noted in [Sud04b, Theorem 2.6].

Proposition 3.17 (Brown, Pedersen, [BP09, Theorem 4.1], Nagisa, Osaka, Phillips, [NOP01, Proposition 1.6])**.** *Let B, C, D be unital C^*-algebras, $\varphi\colon C \to D$ a surjective morphism, and $\psi\colon B \to D$ a unital morphism. Let $A = B \oplus_D C$ be the pullback[6] along ψ and φ. Then:*

(1) $\quad \mathrm{sr}(A) \leq \max\{\mathrm{sr}(B), \mathrm{sr}(C)\},$

(2) $\quad \mathrm{rr}(A) \leq \max\{\mathrm{rr}(B), \mathrm{rr}(C)\}.$

4. Topological dimension

One could try to define a non-commutative dimension function by simply considering the dimension of the primitive ideal space of a C^*-algebra. This will however run into problems if the primitive ideal space is not Hausdorff.

Brown and Pedersen, [BP09], suggested a way of dealing with this problem by restricting to (nice) Hausdorff subsets of $\mathrm{Prim}(A)$, and taking the supremum over the dimension of these Hausdorff subsets, see Definition 4.3. This will only be a good theory if there are enough such 'nice' Hausdorff subsets.

Definition 4.1 (Brown, Pedersen, [BP07, 2.2 (iv)])**.** Let X be a topological space. We define:

(1) A subset $C \subset X$ is called **locally closed** if there is a closed set $F \subset X$ and an open set $G \subset X$ such that $C = F \cap G$.
(2) X is called **almost Hausdorff** if every non-empty closed subset F contains a non-empty relatively open subset $F \cap G$ (so $F \cap G$ is locally closed in X) which is Hausdorff.

4.2. We consider locally closed subsets as 'nice' subsets. Then, being almost Hausdorff means having enough 'nice' Hausdorff subsets.

For a C^*-algebra A, the locally closed subsets of $\mathrm{Prim}(A)$ correspond to ideals of quotients of A (equivalently to quotients of ideals of A) up to canonical isomorphism, see

[6] The pullback $A = B \oplus_D C$ along ψ and φ is defined as $\{(b, c) \in B \oplus C \mid \psi(b) = \varphi(c)\}$

[BP07, 2.2(iii)]. Therefore, the primitive ideal space of every type I C^*-algebra is almost Hausdorff, since every non-zero quotient contains a non-zero continuous-trace ideal, see [Ped79, Theorem 6.2.11, p. 200], and the primitive ideal space of a continous-trace C^*-algebra is Hausdorff.

See Section 8 for an overview of the structure theory of type I C^*-algebras, in particular for the definition of a composition series.

Definition 4.3 (Brown, Pedersen, [BP07, 2.2(v)]). Let A be a C^*-algebra. If $\mathrm{Prim}(A)$ is almost Hausdorff, then the **topological dimension** of A, denoted by $\mathrm{topdim}(A)$, is:

(1) $\mathrm{topdim}(A) := \sup\{\mathrm{locdim}(S) \mid S \subset \mathrm{Prim}(A)$ locally closed, Hausdorff$\}$.

Proposition 4.4 (Brown, Pedersen, [BP07, Proposition 2.6]). *Let $(J_\alpha)_{\alpha\leq\mu}$ be a composition series for a C^*-algebra A. Then $\mathrm{Prim}(A)$ is almost Hausdorff if and only if each $\mathrm{Prim}(J_{\alpha+1}/J_\alpha)$ is almost Hausdorff, and if this is so, then:*

(1) $$\mathrm{topdim}(A) = \sup_{\alpha<\mu} \mathrm{topdim}(J_{\alpha+1}/J_\alpha).$$

The following result is implicit in the papers of Brown and Pedersen, e.g. [BP09, Theorem 5.6].

Proposition 4.5. *Let A be a C^*-algebra, and $B \subset A$ a hereditary sub-C^*-algebra. If $\mathrm{Prim}(A)$ is locally Hausdorff, then so is $\mathrm{Prim}(B)$, and then $\mathrm{topdim}(B) \leq \mathrm{topdim}(A)$. If B is even full hereditary, then $\mathrm{topdim}(B) = \mathrm{topdim}(A)$.*

Proof. In general, if $B \subset A$ is a hereditary sub-C^*-algebra, then $\mathrm{Prim}(B)$ is homeomorphic to an open subset of $\mathrm{Prim}(A)$. In fact, $\mathrm{Prim}(B)$ is canonically homeomorphic to the primitive ideal space of the ideal generated by B, and this corresponds to an open subset of $\mathrm{Prim}(A)$.

Note that being locally Hausdorff is a property that passes to locally closed subsets, and so it passes from $\mathrm{Prim}(A)$ to $\mathrm{Prim}(B)$. Further, every locally closed, Hausdorff subset $S \subset \mathrm{Prim}(B)$ is also locally closed (and Hausdorff) in $\mathrm{Prim}(A)$. It follows $\mathrm{topdim}(B) \leq \mathrm{topdim}(A)$.

If B is full, then $\mathrm{Prim}(B) \cong \mathrm{Prim}(A)$ and therefore $\mathrm{topdim}(B) = \mathrm{topdim}(A)$. □

Lemma 4.6. *Let A be a continuous-trace C^*-algebra, and $n \in \overline{\mathbb{N}}$. If A is approximated by sub-C^*-algebra with topological dimension $\leq n$, then $\mathrm{topdim}(A) \leq n$.*

Proof. Let us first reduce to the case that A has a global rank-one projection, i.e., that there exists a full, abelian projection $p \in A$, see [Bla06, IV.1.4.20, p.335]. To that end, let $x \in \mathrm{Prim}(A)$ be any point. Since A has continuous-trace, there exists a open neighborhood $U \subset \mathrm{Prim}(A)$ of x and an element $a \in A_+$ such that $\rho(a)$ is a rank-one projection for every $\rho \in U$. (This is Fell's condition, see [Bla06, IV.1.4.17, p.334], which roughly means that locally there are abelian projections). Then there exists a closed neighborhood $Y \subset \mathrm{Prim}(A)$ of x that is contained in U. Let $J \lhd A$ be the ideal corresponding to $\mathrm{Prim}(A)\setminus Y$. The image of a in the quotient A/J is a full, abelian projection. Moreover, if A is approximated by sub-C^*-algebras $B \subset A$ with $\mathrm{topdim}(B) \leq n$, then A/J is approximated by the sub-C^*-algebras B/J with $\mathrm{topdim}(B/J) \leq \mathrm{topdim}(B) \leq n$. If $\mathrm{topdim}(A/J) \leq n$, then every point of $\mathrm{Prim}(A)$ has a closed neighborhood of dimension $\leq n$, which means $\mathrm{topdim}(A) = \mathrm{locdim}(\mathrm{Prim}(A)) \leq n$.

We assume from now on that A has continuous-trace with a full, abelian projection $p \in A$. Thus, $pAp \cong C(X)$ where we set $X := \mathrm{Prim}(A)$. If A is approximated by sub-C^*-algebras $B_\lambda \subset A$, then one checks that pAp is approximated by sub-C^*-algebra $pB_\lambda p$, and these algebras are commutative so that $pB_\lambda p \cong C(X_\lambda)$. It follows from Proposition 4.5 that $\dim(X_\lambda) \leq \mathrm{topdim}(B_\lambda) \leq n$.

Thus, $C(X)$ is approximated by sub-C^*-algebras $C(X_\lambda) \subset C(X)$ with $\dim(X_\lambda) \leq n$. One checks that this implies $\dim(X) \leq n$, e.g. by noting that X is $\{X_\lambda\}$-like, see 1.4. Then $\mathrm{topdim}(A) = \dim(X) \leq n$ as desired. □

Lemma 4.7. *Let A be a type I C^*-algebra, and $n \in \overline{\mathbb{N}}$. If A is approximated by sub-C^*-algebra with topological dimension $\leq n$, then* $\mathrm{topdim}(A) \leq n$.

Proof. Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for A such that each successive quotient is of continuous-trace, and assume A is approximated by sub-C^*-algebras $A_\lambda \subset A$ with $\mathrm{topdim}(A_\lambda) \leq n$.

Then $J_{\alpha+1}/J_\alpha$ is approximated by the sub-C^*-algebras $(A_\lambda \cap J_{\alpha+1})/J_\alpha$. Since $\mathrm{topdim}((A_\lambda \cap J_{\alpha+1})/J_\alpha) \leq \mathrm{topdim}(A_\lambda) \leq n$, we obtain from the above Lemma 4.6 that $\mathrm{topdim}(J_{\alpha+1}/J_\alpha) \leq n$. By Proposition 4.4, $\mathrm{topdim}(A) = \sup_{\alpha<\mu} \mathrm{topdim}(J_{\alpha+1}/J_\alpha) \leq n$, as desired. □

Remark 4.8. It is noted in [BP07, Remark 2.5(v)] that a weaker version of Lemma 4.7 would follow from [Sud04a]. However, the statement is formulated as an axiom there, and it is not clear that the formulated axioms are consistent and give a dimension theory that agrees with the topological dimension.

Lemma 4.9. *Let A be a continuous-trace C^*-algebra, and $C \subset A$ a separable sub-C^*-algebra. Then there exists a separable, continuous-trace sub-C^*-algebra $D \subset A$ that contains C, and such that the inclusion $C \subset D$ is proper, and* $\mathrm{topdim}(D) \leq \mathrm{topdim}(A)$.

Proof. Let us first reduce to the case that A is σ-unital, and the inclusion $C \subset A$ is proper. To this end, consider the hereditary sub-C^*-algebra $CAC \subset A$ generated by C. Since C is separable, it contains a strictly positive element which is also strictly positive in CAC. Moreover, having continuous-trace passes to hereditary sub-C^*-algebras, see [Ped79, Proposition 6.2.10, p.199]. By Proposition 4.5, $\mathrm{topdim}(CAC) \leq \mathrm{topdim}(A)$.

We assume from now on that A is σ-unital and the inclusion $C \subset A$ is proper. Set $X := \mathrm{Prim}(A)$. By Brown's stabilization theorem, [Bro77], there is an isomorphism $\Phi\colon A \otimes \mathbb{K} \to C_0(X) \otimes \mathbb{K}$. Let $e_{i,j} \in \mathbb{K}$ be the canonical matrix units, and consider the following C^*-algebra:

$$E := C^*(\bigcup_{i,j} e_{1,i}\Phi(C)e_{j,1}) \subset C_0(X) \otimes e_{1,1}.$$

Note that E is separable and commutative. Thus, there exists a separable sub-C^*-algebra $C_0(Y) \subset C_0(X)$ such that $E = C_0(Y) \otimes e_{1,1}$. We chose E such that $\Phi(C \otimes \mathbb{K}) \subset C_0(Y) \otimes \mathbb{K}$.

The inclusion $C_0(Y) \subset C_0(X)$ is induced by a pointed continuous map $f\colon X^+ \to Y^+$, where X^+ is the space X with one additional point ∞ adjoint[7]. From commutative dimension theory, see [Nag70, Corollary 27.5, p.159] or [Mar60, Lemma 4], we get the following:

[7] Thus, if X is non-compact, then X^+ is the one-point compactification of X. Otherwise, if X is compact, then $X^+ = \sqcup\{\infty\}$ is the disjoint union of X with one additional point. In both cases, the added point ∞ is the base point.

There exists a compact, metrizable space Z with $\dim(Z) \leq \dim(X)$ and continuous (surjective) maps $g\colon X \to Z$ and $h\colon Z \to Y$ such that $f = h \circ g$. Set $Z_0 := Z \setminus \{g(\infty)\}$, and note that g^* induces an embedding $C_0(Z_0) \subset C_0(X)$. Moreover, $C_0(Z_0)$ is separable.

Consider $D' := C_0(Z_0) \otimes \mathbb{K} \subset C_0(X) \otimes \mathbb{K}$. We have that D' is a separable, continuous-trace C^*-algebra, and $\Phi(C \otimes \mathbb{K}) \subset D'$, and $\operatorname{topdim}(D') = \dim(Z) \leq \dim(X) = \operatorname{topdim}(A)$. We think of C as included in $C \otimes \mathbb{K}$ via $C \cong C \otimes e_{11}$. Set $D := e_{11}(\Phi^{-1}(D'))e_{11}$, which is a hereditary sub-C^*-algebra of $\Phi^{-1}(D') \cong D'$, and therefore is a separable, continuous-trace C^*-algebra with $\operatorname{topdim}(D) \leq \operatorname{topdim}(D' =\leq \operatorname{topdim}(A)$. By construction, $C \otimes e_{11} \subset D$, and this inclusion is proper since $D \subset A \otimes e_{11}$ and the inclusion $C \otimes e_{11} \subset A \otimes e_{11}$ is proper. □

Lemma 4.10. *Let A be a C^*-algebra, and $J \lhd A$ an ideal, and $C \subset A$ a sub-C^*-algebra. Assume $K \subset J$ is a sub-C^*-algebra that contains $C \cap J$ and such that the inclusion $C \cap J \subset K$ is proper. Then K is an ideal in the sub-C^*-algebra $C^*(K, C) \subset A$ generated by K and C. Moreover, naturally $C^*(K, C)/K \cong C/(C \cap J)$.*

Proof. Set $B := A/J$ and denote the quotient morphisms by $\pi\colon A \to B$. Set $D := \pi(C) \subset B$. Clearly, $C^*(K, C)$ contains both K and C, and it is easy to see that the restriction of π to $C^*(K, C)$ maps onto D. The situation is shown in the following commutative diagram, where the top and bottom row are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & A & \xrightarrow{\pi} & B & \longrightarrow & 0 \\
 & & \cup & & \cup & & \cup & & \\
 & & K & \longrightarrow & C^*(K,C) & \longrightarrow & D & & \\
 & & \cup & & \cup & & \| & & \\
0 & \longrightarrow & C \cap J & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0
\end{array}$$

Let us show that K is an ideal in $C^*(K, C)$. Since $C^*(K, C)$ is generated by elements of K and C, it is enough to show that xy and yx lie in K whenever $x \in K$ and $y \in K$ or $y \in C$. For $y \in K$ that is clear, so assume $y \in C$.

Since $C \cap J \subset K$ is proper, for any $\varepsilon > 0$ there exists $z \in C \cap J$ such that $\|zyz - y\| < \varepsilon$. Then $\|xy - xzyz\|, \|yx - zyzx\| < \varepsilon\|x\|$ and $xzyz, zyzx \in K$. Since $\varepsilon > 0$ was arbitrary, it follows $xy, yx \in K$.

This shows that the middle row in the above diagram is also exact. □

Proposition 4.11. *Let A be a C^*-algebra, and $J \lhd A$ an ideal of type I, and $C \subset A$ a separable sub-C^*-algebra. Then there exists a separable sub-C^*-algebra $D \subset A$ such that $C \subset D$ and $\operatorname{topdim}(D \cap J) \leq \operatorname{topdim}(J)$.*

Proof. The ideal $J = J_\mu$ has a composition series $(J_\alpha)_{\alpha \leq \mu}$ with successive quotients of continuous-trace. We prove the proposition by transfinite induction over μ.

Step 1: The proposition holds for $\mu = 0$. This follows since J is assumed to have a composition series with length 0 and so $J = \{0\}$ and we can simply set $D := C$.

Step 2: If the proposition holds for a finite ordinal μ, then it also holds for $\mu + 1$.

So assume $J = J_{\mu+1}$ has a composition series $(J_\alpha)_{\alpha \leq \mu+1}$. Consider the quotient J/J_1 which is an ideal in A/J_1, and note that J/J_1 has the canonical composition series $(J_\alpha/J_1)_{1 \leq \alpha \leq \mu}$.

By assumption, the proposition holds for μ, and so there is a separable sub-C^*-algebra $E \subset A/J_1$ such that $C/J_1 \subset E$ and $\operatorname{topdim}(E \cap (J_{\mu+1}/J_1)) \leq \operatorname{topdim}(J_{\mu+1}/J_1)$. Find a separable sub-$C^*$-algebra $D_0 \subset A$ such that $D_0/J_1 = E$ and $C \subset D_0$. The situation is shown in the following commutative diagram, whose rows are short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 & \longrightarrow & A & \longrightarrow & A/J_1 & \longrightarrow & 0 \\
 & & \cup & & \cup & & \cup & & \\
0 & \longrightarrow & D_0 \cap J_1 & \longrightarrow & D_0 & \longrightarrow & E & \longrightarrow & 0 \\
 & & \cup & & \cup & & \cup & & \\
0 & \longrightarrow & C \cap J_1 & \longrightarrow & C & \longrightarrow & C/(C \cap J_1) & \longrightarrow & 0
\end{array}$$

We apply Lemma 4.9 to the inclusion $D_0 \cap J_1 \subset J_1$ to find a separable sub-C^*-algebra $K \subset J_1$ containing $D_0 \cap J_1$ and such that the inclusion $D_0 \cap J_1 \subset K$ is proper, and $\operatorname{topdim}(K) \leq \operatorname{topdim}(J_1)$. Set $D := C^*(K, D_0) \subset A$, which is a separable C^*-algebra with $C \subset D$. By Lemma 4.10, D is an extension of E by K, and therefore:

$$\begin{aligned}
\operatorname{topdim}(D \cap J_{\mu+1}) &= \max\{\operatorname{topdim}(D \cap J_1), \operatorname{topdim}((D \cap J_{\mu+1})/(D \cap J_1))\} \\
&= \max\{\operatorname{topdim}(K), \operatorname{topdim}(E \cap (J_{\mu+1}/J_1))\} \\
&\leq \max\{\operatorname{topdim}(J_1), \operatorname{topdim}(J_{\mu+1}/J_1)\} \\
&= \operatorname{topdim}(J_{\mu+1}).
\end{aligned}$$

Step 3: Assume λ is a limit ordinal, and n is finite. If the statement holds for all $\alpha < \lambda$, then it holds for $\lambda + n$.

If λ has cofinality at most ω, then there are ordinals $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda$ such that $\lambda = \sup_k \lambda_k$. If λ has cofinality larger than ω, then set $\lambda_k := 0$.

We inductively define ordinals $\alpha_k \leq \mu$ and sub-C^*-algebras $D_k, E_k \subset A$ with the following properties:

(i) $D_k \subset E_k$ and $E_k \subset D_{k+1}$
(ii) $\alpha_1 \leq \alpha_2 \leq \ldots$ and $\lambda_k \leq \alpha_k$
(iii) $\operatorname{topdim}(D_k \cap J_{\alpha_k}) \leq \operatorname{topdim}(J_{\alpha_k})$ and $\operatorname{topdim}((E_k \cap J_{\lambda+n})/J_\lambda) \leq \operatorname{topdim}(J_{\lambda+n}/J_\lambda)$

We set $\alpha_0 = 0$ and $D_0 := C$.

Given D_k, let us construct E_k. Consider $D_k/J_\lambda \subset A/J_\lambda$ and the ideal $J_{\lambda+n}/J_\lambda \lhd A/J_\lambda$ which has a composition series of length n. Since the statement holds for n, there exists a separable sub-C^*-algebra $\widetilde{E_k} \subset A/J_\lambda$ such that $D_k/J_\lambda \subset \widetilde{E_k}$ and $\operatorname{topdim}(\widetilde{E_k} \cap J_{\lambda+n}) \leq \operatorname{topdim}(J_{\lambda+n}/J_\lambda)$. Let $E_k \subset A$ be any separable C^*-algebra such that $D_k \subset E_k$ and $E_k/J_\lambda = \widetilde{E_k}$.

Given E_k, let us construct α_{k+1} and D_{k+1}. Define ordinals:

$$\beta_{k+1} := \inf\{\alpha \leq \lambda \mid E_k \cap J_\alpha = E_k \cap J_\lambda\}$$

$$\alpha_{k+1} := \begin{cases} \sup\{\beta_{k+1}, \lambda_{k+1}\} & \text{,if } \beta_k < \lambda \\ \sup\{\alpha_k, \lambda_{k+1}\} & \text{,if } \beta_k = \lambda \end{cases}.$$

In any case we have $\alpha_k \leq \alpha_{k+1} < \lambda$, and by assumption the proposition holds for α_{k+1}. Thus, we can find a separable sub-C^*-algebra $D_{k+1} \subset A$ such that $E_k \subset D_{k+1}$ and $\operatorname{topdim}(D_{k+1} \cap J_{\alpha_{k+1}}) \leq \operatorname{topdim}(J_{\alpha_{k+1}})$.

Now set $D := \overline{\bigcup_k D_k} = \overline{\bigcup_k E_k}$. This is a separable sub-C^*-algebra of A with $C \subset D$. Moreover, $(D \cap J_{\lambda+n})/J_\lambda = \overline{\bigcup_k (E_k \cap J_{\lambda+n})/J_\lambda}$ and so:

$$\begin{aligned} \operatorname{topdim}((D \cap J_{\lambda+n})/J_\lambda) &\leq \liminf \operatorname{topdim}((E_k \cap J_{\lambda+n})/J_\lambda) \\ &\leq \operatorname{topdim}(J_{\lambda+n}/J_\lambda) \end{aligned}$$

. Set $\alpha := \sup_k \alpha_k$ and note that by construction $D \cap J_\lambda = D \cap J_\alpha$. One checks that $D \cap J_\alpha = \overline{\bigcup_k (D_k \cap J_{\alpha_k})}$. Therefore:

$$\begin{aligned} \operatorname{topdim}(D \cap J_\lambda) &\leq \liminf_k \operatorname{topdim}(D_k \cap J_{\alpha_k}) \\ &\leq \sup_k \operatorname{topdim}(J_{\alpha_k}) \\ &\leq \operatorname{topdim}(J_\lambda) \end{aligned}$$

Together we get:

$$\begin{aligned} \operatorname{topdim}(D \cap J_{\lambda+n}) &= \max\{\operatorname{topdim}(D \cap J_\lambda), \operatorname{topdim}((D \cap J_{\lambda+n})/J_\lambda)\} \\ &\leq \max\{\operatorname{topdim}(J_\lambda), \operatorname{topdim}(J_{\lambda+n}/J_\lambda)\} \\ &= \operatorname{topdim}(J_{\lambda+n}). \end{aligned}$$

This completes the proof. □

Theorem 4.12. *The topological dimension is a dimension theory in the sense of Definition 2.1 for the class of type I C^*-algebras.*

Proof. Axioms (D1)-(D4) follow from Proposition 4.4. Axiom (D5) is Lemma 4.7, and (D6) is a corollary of Proposition 4.11. □

4.13. Let us denote the class of type I C^*-algebras by $\mathcal{GCR}$. By the above Theorem 4.12, $\operatorname{topdim}\colon \mathcal{GCR} \to \overline{\mathbb{N}}$ is a dimension theory. Let us denote by $\operatorname{topdim}^\sim : \mathcal{C}^* \to \overline{\mathbb{N}}$ the extension by approximation, as defined in Proposition 2.8. This defines a Morita-invariant dimension theory, since $\operatorname{topdim}(A) = \operatorname{topdim}(A \otimes \mathbb{K})$ for any type I C^*-algebra A.

For any C^*-algebra A, if $\operatorname{topdim}^\sim(A) < \infty$, then A is in particular approximated by type I sub-C^*-algebras. This implies that A is nuclear, satisfies the universal coefficient theorem (UCT), see [Dad03], and is not properly infinite.

4.14. The topological dimension behaves well with respect to tensor products. If A, B are type I C^*-algebras, then $\operatorname{Prim}(A \otimes B) \cong \operatorname{Prim}(A) \times \operatorname{Prim}(B)$, see [Bla06, IV.3.4.25, p.390], and therefore:

(1) $$\operatorname{topdim}(A \otimes B) \leq \operatorname{topdim}(A) + \operatorname{topdim}(B).$$

This property is preserved when passing to the extension as defined in Proposition 2.8. Let A, B be any C^*-algebras, then:

(1) $$\operatorname{topdim}^\sim(A \otimes B) \leq \operatorname{topdim}^\sim(A) + \operatorname{topdim}^\sim(B).$$

Note that we need not specify the tensor product, since $\operatorname{topdim}^\sim(A) < \infty$ implies that A is nuclear.

4.15. For type I C^*-algebras, a characterization of $\operatorname{topdim}^{\sim}(A) = 0$ is given in Proposition 8.11. For general C^*-algebras, see 5.11 for a characterization of $\operatorname{topdim}^{\sim}(A) = 0$.

5. Decomposition rank and nuclear dimension

The decomposition rank and nuclear dimension are defined in terms of completely positive approximations that fulfill a certain decomposability condition. The decomposition rank was introduced by Kirchberg and Winter in [KW04], and the nuclear dimension was introduced by Winter and Zacharias in [WZ10].

Definition 5.1 (Winter, [Win03, Definition 3.1])**.** Let A be a C^*-algebra, $\varphi\colon F \to A$ a completely positive (c.p.) map from a finite-dimensional[8] C^*-algebra F, and $n \in \mathbb{N}$. Then the **strict order** of φ is not bigger than n, denoted by $\operatorname{ord}(\varphi) \leq n$, if for any set $\{e_0, \dots, e_{n+1}\}$ of pairwise orthogonal, minimal projections in F there are some indices $i, j \in \{0, \dots, n+1\}$ such that $\varphi(e_i)$ and $\varphi(e_j)$ are orthogonal[9].

The case of strict order zero is of special interest, see [WZ09]. There are nice characterizations of this case, see [Win03, Proposition 4.1.1(a)].

Definition 5.2 (Kirchberg, Winter, [KW04, Definition 2.2])**.** Let A be a C^*-algebra, and $\varphi\colon \oplus_{i=1}^{s} M_{r_i} \to A$ a c.p. map, and $n \in \mathbb{N}$. Then φ is n**-decomposable** if there exists a decomposition $\{1, 2, \dots, s\} = \coprod_{j=0}^{n} I_j$ such that for each j the restriction of φ to $\oplus_{i \in I_j}$ has strict order zero.

We say φ is decomposable if it is n-decomposable for some n.

5.3. Let A be C^*-algebra. Given a finite set $E \subset A$, and $\varepsilon > 0$, a **c.p. approximation** for E within ε is a tuple (F, ψ, φ) where F is a finite-dimensional C^*-algebra, $\psi\colon A \to F$ is a completely positive contractive (c.p.c.) map, $\varphi\colon F \to A$ is a c.p. map, and such that $\|\varphi \circ \psi(a) - a\| < \varepsilon$ for each $a \in E$. A c.p. approximation (F, ψ, φ) is called c.p.c. approximation if φ is also contractive.

Recall that a C^*-algebra is nuclear if it has the completely positive approximation property (CPAP):

(1) For every finite $E \subset A$, and $\varepsilon > 0$, there exists a c.p.c. approximation for E within ε.

Recently, it was shown that nuclear C^*-algebras satisfy a much stronger approximation property, see [HKW11]:

(2) For every finite $E \subset A$, and $\varepsilon > 0$, there exists a c.p.c. approximation (F, ψ, φ) for E within ε such that φ is decomposable.

Note that each of the maps φ is n-decomposable for some n, but the number n may depend on φ. If we put an upper bound on n, then we end up with the definition of decomposition rank (for c.p.c. approximations) and nuclear dimension (for c.p. approximations). We note that the above mentioned stronger version of the CPAP was discovered after the decomposition rank and nuclear dimension were established.

[8] In this context, 'finite-dimensional' is not referring to a dimension theory, but to the dimension as a vector space.

[9] Two elements a, b of a C^*-algebra are said to be orthogonal if $ab = a^*b = ab^* = a^*b^* = 0$

Definition 5.4 (Kirchberg, Winter, [KW04, Definition 3.1], Winter, Zacharias, [WZ10, Definition 2.1])**.** Let A be a C^*-algebra. The **nuclear dimension** of A, denoted by $\dim_{\mathrm{nuc}}(A)$, is the smallest integer $n \geq 0$ (or ∞) such that the following holds:

(1) For every finite subset $E \subset A$, and $\varepsilon > 0$, there is a c.p. approximation (F, ψ, φ) for E within ε such that φ is n-decomposable.

The **decomposition rank** of A, denoted by $\mathrm{dr}(A)$, is defined similarly, using c.p.c. approximations, i.e., $\mathrm{dr}(A) \leq n$ if the following holds:

(2) For every finite subset $E \subset A$, and $\varepsilon > 0$, there is a *c.p.c.* approximation (F, ψ, φ) for E within ε such that φ is n-decomposable.

5.5. It is clear from the definition that $\dim_{\mathrm{nuc}}(A) \leq \mathrm{dr}(A)$ for every C^*-algebra A. A C^*-algebra with finite decomposition rank is strongly quasidiagonal, see [KW04, Theorem 5.3]. Since there are non-quasidiagonal C^*-algebras with finite nuclear dimension (e.g. the Toeplitz algebra, or Kirchberg algebras), the nuclear dimension and decomposition rank can differ in general.

It is natural to ask if there are conditions for a C^*-algebra A that imply $\mathrm{dr}(A) = \dim_{\mathrm{nuc}}(A)$, and possible such conditions are being strongly quasidiagonal or the related property of having a faithful tracial state, see [WZ10, Question 9.1].

Remark 5.6. The original definition of the decomposition rank in [KW04, Definition 3.1] is only for separable C^*-algebras. As noted in [WZ10, Remark 2.2(v)], this is unnecessary and most results of [KW04] also hold for non-separable C^*-algebras.

Theorem 5.7. *The decomposition rank and the nuclear dimension are Morita-invariant dimension theories in the sense of Definition 2.1 for the class of all C^*-algebras.*

Proof. For the nuclear dimension, axioms (D1), (D2), (D3), (D6) and (D4) follow from Propositions 2.5, 2.3, 2.6 and Remark 2.11 in [WZ10]. Axiom (D5) is easily verified.

For the decomposition rank, axiom (D5) is also easily verified, and axiom (D6) follows from [WZ10, Proposition 2.6] adapted for c.p.c. approximations instead of c.p. approximations. The other axioms (D1)-(D4) follow from Proposition 3.8, 3.11 and Remark 3.2 of [KW04] for separable C^*-algebras. Using axioms (D5) and (D6) this can be extended to all C^*-algebras. □

5.8. The Toeplitz extension $0 \to \mathbb{K} \to \mathcal{T} \to C(S^1) \to 0$, see Example 9.1, shows that in general the decomposition rank of an extension can not be estimated by that of the ideal and quotient. Indeed, $\mathrm{dr}(\mathcal{T}) = \infty$ since $\mathcal{T}$ is not quasidiagonal, while $\mathrm{dr}(\mathbb{K}) = 0$ and $\mathrm{dr}(C(S^1)) = 1$.

If A is strongly quasidiagonal (or has a faithful tracial state), then one could hope that the estimate $\mathrm{dr}(A) \leq \max\{\mathrm{dr}(J), \mathrm{dr}(A/J)\}$ holds for every ideal $J \lhd A$, but this is not known and would probably be hard to prove. But the estimate does hold under the additional assumption that the ideal $J \lhd A$ contains a quasicentral approximate unit of projections, so that in this case $\mathrm{dr}(A) = \max\{\mathrm{dr}(J), \mathrm{dr}(A/J)\}$, see [KW04, Proposition 6.1].

For the nuclear dimension we have the estimate $\dim_{\mathrm{nuc}}(A) \leq \dim_{\mathrm{nuc}}(J) + \dim_{\mathrm{nuc}}(A/J) + 1$ whenever $J \lhd A$, see [WZ10, Proposition 2.9].

5.9. For a tensor product (of nuclear C^*-algebra) we have $\mathrm{dr}(A\otimes B) \leq (\mathrm{dr}(A)+1)\cdot(\mathrm{dr}(B)+1)-1$ and $\dim_{\mathrm{nuc}}(A\otimes B) \leq (\dim_{\mathrm{nuc}}(A)+1)\cdot(\dim_{\mathrm{nuc}}(B)+1)-1$, see [KW04, 3.2] and [WZ10, Proposition 2.3].

5.10. It is unknown whether the decomposition rank and nuclear dimension behave well with respect to pullbacks, i.e., whether the analog of 3.17 holds.

If it is true, then it is probably hard to prove, as it is already difficult for the case of (pullbacks) of subhomogeneous C^*-algebras, see [Win04].

Proposition 5.11. *Let A be a C^*-algebra. Then the following are equivalent:*

(i) $\mathrm{dr}(A) = 0$
(ii) $\dim_{nuc}(A) = 0$
(iii) $\mathrm{topdim}\tilde{}(A) = 0$
(iv) *A is an LF-algebra (i.e. approximated by finite-dimensional sub-C^*-algebras)*

Proof. By [KW04, Example 4.1] and [WZ10, Remarks 2.2 (iii)], a separable C^*-algebra A satisfies $\mathrm{dr}(A) = 0$ if and only if $\dim_{\mathrm{nuc}}(A) = 0$, and this also happens precisely if A is an AF-algebra. It follows that (1), (2) and (4) are equivalent also in the non-separable case.

By [Lin97], a separable type I C^*-algebra A satisfies $\mathrm{topdim}(A)0 =$ if and only if it is an AF-algebra. Thus, for a a general type I C^*-algebra, topological dimension zero is equivalent to being LF. Every LF-algebra A satisfies $\mathrm{topdim}\tilde{}\,(A) = 0$. Conversely, if $\mathrm{topdim}\tilde{}(A) = 0$ then A is approximated by type I sub-C^*-algebra A_λ with $\mathrm{topdim}(A_\lambda) = 0$, and so in turn A is LF, which shows that (3) and (4) are equivalent in general. □

Remark 5.12. Every AF-algebra is an LF-algebra, and the converse holds for separable C^*-algebras. However, it was recently shown by Farah and Katsura, [FK10, Theorem 1.5], that there exist (non-separable) LF-algebras that are not AF.

6. HOMOGENEOUS AND CONTINUOUS-TRACE C^*-ALGEBRAS

Homogeneous and continuous-trace C^*-algebras are well-understood and in their structure pretty close to commutative C^*-algebras. We will see that this makes it possible to compute their dimension theories quite explicitly. Let us begin with the basic notions.

Definition 6.1 (Fell, [Fel61, 3.2])**.** Let A be a C^*-algebra and $n \geq 1$. Then A is called n-**homogeneous** if all its irreducible representations are n-dimensional. We further say that A is **homogeneous** if it is n-homogeneous for some n.

6.2. Let us recall a general construction: Assume $\mathfrak{B} = (E \xrightarrow{p} X)$ is a locally trivial fibre bundle (over a locally compact, Hausdorff space X) whose fiber has the structure of a C^*-algebra. Let

$$\Gamma_0(\mathfrak{B}) = \{f\colon X \to E \mid p\circ f = id_X, (x \to \|f(x)\|) \in C_0(X)\} \tag{1}$$

be the sections of $\mathfrak{B}$ that vanish at infinity. Then $\Gamma_0(\mathfrak{B})$ has a natural structure of a C^*-algebra, with the algebraic operations defined fibrewise, and norm $\|f\| := \sup_{x\in X}\|f(x)\|$.

If the bundle has fibre M_n (a so-called M_n-bundle), then $A := \Gamma_0(\mathfrak{B})$ is n-homogeneous and $\mathrm{Prim}(A) \cong X$. Thus, every M_n-bundle defines an n-homogeneous C^*-algebra. The converse does also hold:

Proposition 6.3 (Fell, [Fel61, Theorem 3.2])**.** *Let A be a C^*-algebra, $n \in \mathbb{N}$. Then the following are equivalent:*

(1) A is n-homogeneous,
(2) $A \cong \Gamma_0(\mathfrak{B})$ for a locally trivial M_n-bundle $\mathfrak{B}$.

6.4. If $A \cong \Gamma_0(\mathfrak{B})$, then the primitive ideal space of A is (canonically) homeomorphic to the base space of the bundle, a Hausdorff space. Thus, the primitive ideal space of a homogeneous C^*-algebra is Hausdorff. Let us consider another natural class of C^*-algebras with Hausdorff primitive ideal space, the so-called continuous-trace algebras. We need to briefly recall some concepts:

Recall that for each Hilbert-space H there is a trace-map $\mathrm{tr}\colon B(H) \to \mathbb{C}$ (which might take value ∞). It has the property that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for any $a, b \in B(H)$, in particular $\mathrm{tr}(uau^*) = \mathrm{tr}(a)$ for a unitary $u \in A$. For a representation $\pi\colon A \to B(H)$ we can consider the map $\mathrm{tr}_\pi := \mathrm{tr} \circ \pi : A \to \mathbb{C}$. It is well defined up to unitary equivalence, i.e., for each $[\pi] \in \widehat{A}$ we get a well-defined map $\mathrm{tr}_{[\pi]}$, see [Bla06, IV.1.4.8, p.333].

We say a that a positive element $a \in A_+$ has continuous-trace if the map $\hat{a}\colon \widehat{A} \to \mathbb{C}$ defined as $[\pi] \mapsto \mathrm{tr}_{[\pi]}(a) = \mathrm{tr}(\pi(a))$ is finite, bounded and continuous. We let $\mathfrak{m}_+(A)$ denote all these elements. They form the positive cone of an ideal which is denote by $\mathfrak{m}(A)$, see [Bla06, IV.1.4.11, p.333].

Definition 6.5 (see [Bla06, IV.1.4.12, p.333])**.** A C^*-algebra A is of **continuous-trace** if the ideal $\mathfrak{m}(A)$, generated by the positive elements with continuous-trace, is dense in A.

6.6. Homogeneous C^*-algebras (and their finite direct sums) are of continuous-trace. Indeed, for a homogeneous C^*-algebra A we have $\mathfrak{m}_+(A) = A_+$ and $\mathfrak{m}(A) = A$. However, not every continuous-trace algebra is homogeneous, in fact not even $\mathfrak{m}_+(A) = A_+$ does imply that. For example $C_0(X, \mathbb{K})$ is of continuous-trace with $\mathfrak{m}_+(A) = A_+$, but A is not homogeneous. The reason is simply that all irreducible representations are infinite-dimensional. But this algebra is the section algebra of a bundle (with fibre $\mathbb{K}$), and some authors call these algebras $\aleph_0$-homogeneous.

Another example is

$$A = \{ f \in C([0,1], M_2) \mid f(0), f(1) \in \left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset M_2 \}, \tag{1}$$

which is more obviously not homogeneous, but still of continuous-trace. This example should not be confused with the so-called dimension-drop algebras, which are not of continuous-trace.

Proposition 6.7 (see [Bla06, IV.1.4.18, p.335])**.** *Let A be a C^*-algebra. Then the following are equivalent:*

(1) A is of continuous-trace,
(2) $\widehat{A}$ is Hausdorff, and A satisfies Fell's condition, i.e., for every $\pi \in \widehat{A}$ there is a neighborhood $U \subset \widehat{A}$ of π and some $x \in A$ such that $\rho(x)$ is a rank-one projection for each $\rho \in U$.

The easiest homogeneous C^*-algebras are the commutative C^*-algebras. Thus, to get some feeling let us first recall how non-commutative dimension theories see the commutative world.

Proposition 6.8. *Let $A = C_0(X)$ be a commutative C^*-algebra. Then:*

$$(1)\qquad \mathrm{sr}(A) = \left\lfloor \frac{\mathrm{locdim}(X)}{2} \right\rfloor + 1,$$

$$(2)\qquad \mathrm{csr}(A) \leq \left\lfloor \frac{\mathrm{locdim}(X)+1}{2} \right\rfloor + 1,$$

$$(3)\qquad \mathrm{rr}(A) = \mathrm{dr}(A) = \mathit{dim}_{nuc}(A) = \mathrm{topdim}(A) = \mathrm{locdim}(X).$$

Proof. The equality $\mathrm{topdim}(A) = \mathrm{locdim}(X)$ follows since $\mathrm{Prim}(A) = X$ is Hausdorff. Note that for any dimension theory rk we have $\mathrm{rk}(A) = \mathrm{rk}(\widetilde{A}) = \mathrm{rk}(C(\alpha X))$, and $\dim(\alpha X) = \mathrm{locdim}(X)$. Thus, the formula for the stable rank follows from Rieffel, [Rie83, Proposition 1.7], and for the connected stable rank it follows from Nistor, [Nis86, Corollary 2.5]. We get $\mathrm{rr}(A) = \mathrm{locdim}(A)$ from Brown, Pedersen, [BP91, Proposition 1.1].

It follows from Kirchberg, Winter, [KW04, 3.3], and Winter, Zacharias, [WZ10, Proposition 2.4] that $\mathrm{dr}(A) = \mathrm{dim}_{\mathrm{nuc}}(A) = \mathrm{topdim}(A)$ if A is separable. The general result follows using axioms (D5) and (D6) for the different theories. For instance, let us prove $\mathrm{dr}(A) \leq \mathrm{topdim}(A)$. So let $n := \mathrm{topdim}(A)$, which we may assume is finite. By (D6) for the topological dimension, there exists an approximating family of separable sub-C^*-algebras $A_\lambda \subset A$ with $\mathrm{topdim}(A_\lambda) \leq n$. We have $\mathrm{dr}(A_\lambda) = \mathrm{topdim}(A_\lambda) \leq n$, since A_λ is separable (and commutative). By (D5) for the decomposition rank, we get $\mathrm{dr}(A) \leq n$. The other inequalities are proved the same way. □

6.9. Thus, the real rank, the decomposition rank, the nuclear dimension and the topological dimension all agree for commutative C^*-algebras. This is no longer the case for non-commutative homogeneous C^*-algebras. Let us first consider the case $A = C_0(X, M_n)$, which is the easiest n-homogeneous C^*-algebra.

We may view A as the sections (vanishing at infinity) of the trivial bundle $X \times M_n \to X$. Now things change. While still $\mathrm{dr}(A) = \mathrm{dim}_{\mathrm{nuc}}(A) = \mathrm{topdim}(A) = \mathrm{locdim}(X)$, the stable and real rank 'break down'. For the stable rank there is a general formula expressing $\mathrm{sr}(A \otimes M_n)$ in terms of $\mathrm{sr}(A)$, see Proposition 3.12. Combining it with Proposition 6.8, we get:

$$(1)\qquad \mathrm{sr}(C_0(X) \otimes M_n) = \left\lceil \left\lfloor \frac{\mathrm{locdim}(X)}{2} \right\rfloor /n \right\rceil + 1 = \left\lceil \frac{\mathrm{locdim}(X) + 2n - 1}{2n} \right\rceil.$$

For the real rank, no general formula expressing $\mathrm{rr}(A \otimes M_n)$ in terms of $\mathrm{rr}(A)$ is known. But Beggs and Evans, [BE91, Corollary 3.2], proved the following formula:

$$(2)\qquad \mathrm{rr}(C(X) \otimes M_n) = \left\lceil \frac{\dim(X)}{2n-1} \right\rceil.$$

By Proposition 6.3, a general n-homogeneous C^*-algebra A is nothing else than the section algebra of a locally trivial M_n-bundles over $X = \mathrm{Prim}(A)$. The bundle need not be globally trivial, it may have a twist. However, if the bundle has finite type (meaning that the base space can be covered by finitely many open sets over which the bundle is trivial), then A has the structure of a finite iterated pullback and one can compute the real and stable rank using Proposition 3.17. The following result contains the most general result.

Proposition 6.10. *Let A be a n-homogeneous C^*-algebra with primitive ideal space $X := \mathrm{Prim}(A)$. Then:*

(1) $$\mathrm{sr}(A) = \mathrm{sr}(C_0(X, M_n)) = \left\lceil \frac{\mathrm{locdim}(X) + 2n - 1}{2n} \right\rceil,$$

(2) $$\mathrm{rr}(A) = \mathrm{rr}(C_0(X, M_n)) = \left\lceil \frac{\mathrm{locdim}(X)}{2n - 1} \right\rceil,$$

(3) $$\mathrm{dr}(A) = \mathit{dim}_{nuc}(A) = \mathrm{topdim}(A) = \mathrm{locdim}(X).$$

Proof. The formulas for the stable and real rank are [Bro07, Proposition 3.8]. The equalities in the bottom line hold in more generality, see Proposition 7.3. □

Remark 6.11. Thus, if $\mathfrak{B}_1, \mathfrak{B}_2$ are M_n-bundles over some (locally compact, Hausdorff) space X, then the considered dimension theories cannot distinguish between $\Gamma_0(\mathfrak{B}_1)$ and $\Gamma_0(\mathfrak{B}_2)$, i.e., they cannot see the twist of the M_n-bundles.

Proposition 6.12. *Let A be a continuous-trace C^*-algebra with primitive ideal space $X := \mathrm{Prim}(A)$. Then:*

(1) $$\mathrm{dr}(A) = \mathit{dim}_{nuc}(A) = \mathrm{topdim}(A) = \mathrm{locdim}(X).$$

Proof. For separable, continuous-trace C^*-algebras, the result is [WZ10, Corollary 2.10], see also [KW04, Corollary 3.10]. For the non-separable case, note first that the inequalities $\mathrm{topdim}(A) \leq \dim_{\mathrm{nuc}}(A) \leq \mathrm{dr}(A)$ hold for general type I C^*-algebras, see Theorem 8.7. To show $\mathrm{dr}(A) \leq \mathrm{topdim}(A)$, use 4.9 to approximate A by separable, continous-trace sub-C^*-algebras A_λ with $\mathrm{topdim}(A_\lambda) \leq \mathrm{topdim}(A)$. We have $\mathrm{dr}(A_\lambda) = \mathrm{topdim}(A_\lambda) \leq \mathrm{topdim}(A)$, and so from (D5) for the decomposition rank we get $\mathrm{dr}(A) \leq \mathrm{topdim}(A)$. □

6.1. **Low dimensions and K-theory.**

Let A be a continuous-trace C^*-algebra (e.g. a homogeneous C^*-algebra). We will see that $K_1(A)$ is torsion-free whenever $\mathrm{topdim}(A) \leq 2$, and that $K_0(A)$ is torsion-free whenever $\mathrm{topdim}(A) \leq 1$. For this we relate the K-theory of A to that of $\mathrm{Prim}(A)$. This indicates that torsion is a high-dimensional behavior.

Lemma 6.13. *Let A be a separable, continuous-trace C^*-algebra. If $\mathrm{topdim}(A) \leq 2$, then:*

(1) $$A \otimes \mathbb{K} \cong C_0(\mathrm{Prim}(A)) \otimes \mathbb{K}.$$

Proof. This follows from the Dixmier-Douady theory for continuous-trace C^*-algebras, see e.g. [Bla06, Iv.1.7, p.344ff]. If $\dim(\mathrm{Prim}(A)) \leq 2$, then $H^3(\mathrm{Prim}(A)) = 0$. Hence, all separable, continuous trace algebras with spectrum $\mathrm{Prim}(A)$ are stably isomorphic, in particular stably isomorphic to $C_0(\mathrm{Prim}(A))$. □

6.14. Lemma 6.13 is well-known, and it implies that for spaces of dimension ≤ 2 there is no twisted K-theory. We get the following: If A is a separable, continuous-trace C^*-algebra with primitive ideal space X and $\mathrm{topdim}(A) = \mathrm{locdim}(X) \leq 2$, then $K_*(A) \cong K_*(C_0(X))$.

Note that $K_0(C_0(X))$ is isomorphic to $K^0(X)$ if X is compact. However, if X is non-compact, then $K_0(C_0(X))$ is isomorphic to $\widetilde{K}^0(\alpha X)$, the reduced K-theory of the minimal compactification αX. We may unify the two cases by considering the space X^+, see 1.2. Note that also $K_1(C_0(X)) \cong \widetilde{K}^0(X^+)$, so that we get the following:

$$K_*(A) \cong K_*(C_0(X)) \cong \widetilde{K}^*(X_+). \tag{2}$$

Next, we will relate the K-theory and cohomology of low-dimensional spaces.

6.15. The Chern character is a (graded) ring-homomorphism from the K-theory of a space X to its cohomology ring. We only consider Čech-cohomology, and denote it by $H^n(X;G)$. Forgetting the ring structure, we consider only the two group-homomorphisms:

$$\mathrm{ch}^0 : K^0(X) \to H^{\mathrm{ev}}(X;\mathbb{Q}) = \bigoplus_{k\geq 0} H^{2k}(X;\mathbb{Q}), \tag{1}$$

$$\mathrm{ch}^1 : K^1(X) \to H^{\mathrm{odd}}(X;\mathbb{Q}) = \bigoplus_{k\geq 0} H^{2k+1}(X;\mathbb{Q}). \tag{2}$$

Both ch^0 and ch^1 are group-isomorphisms after tensoring with $\mathbb{Q}$. This means that K-theory and cohomology of a space agree *up to torsion*.

If X is a compact space with $\dim(X) \leq 2$, then one has an integral version of the Chern character which induces the following natural group-isomorphisms:

$$\chi^0 : K^0(X) \xrightarrow{\cong} H^0(X) \oplus H^2(X), \tag{3}$$

$$\chi^1 : K^1(X) \xrightarrow{\cong} H^1(X). \tag{4}$$

We get the following result, which has applications to the K-theory of low-dimensional homogeneous or continuous-trace C^*-algebras, and in this way also for AH-algebras.

Proposition 6.16. *Let X be a compact, Hasudorff space.*
(1) If $\dim(X) \leq 2$, then $K^1(X)$ is torsion-free.
(2) If $\dim(X) \leq 1$, then $K^0(X)$ is torsion-free.

Proof. (1): Assume first that X is a (finite) CW-complex with $\dim(X) \leq 2$. Then $K^1(X) \cong H^1(X)$ which in turn is isomorphic to $\mathrm{Hom}(H_1(X),\mathbb{Z})$ by the UCT. Since $H_1(X)$ is finitely-generated, $\mathrm{Hom}(H_1(X),\mathbb{Z}) \cong H_1(X)/H_1(X)_{\mathrm{tor}} \cong \mathbb{Z}^k$ for some k, where $H_1(X)_{\mathrm{tor}}$ is the torsion subgroup of $H_1(X)$.

We extend this to general compact, Hausdorff spaces in two steps. First, by Freudenthal's expansion theorem, every metric, compact space X can be written as an inverse limit, $X \cong \varprojlim X_i$, of finite CW-complexes X_i with $\dim(X_i) \leq \dim(X)$, see [Nag70, Corollary 27.4, p.158]. Then $K^1(X_i)$ is torsion-free for each i. Since K-theory is a continuous functor, we get $K_0(X) \cong \varinjlim_i K^1(X_i)$. Then $K_0(X)$ is torsion-free, since torsion-freeness is preserved by inductive limits, see 1.6.

Secondly, by Mardešić's expansion theorem, a general compact, Hausdorff space X can be written as an inverse limit, $X \cong \varprojlim X_i$, of metric, compact spaces X_i with $\dim(X_i) \leq \dim(X)$, see [Nag70, Theorem 27.8, p.160]. Arguing as above we get that $K_0(X)$ is torsion-free.

We note that this two-step argument is necessary, since there exists a compact, Hausdorff space X that can not be written as an inverse limit of finite CW-complexes X_i with $\dim(X_i) \leq \dim(X)$.

(2): Assume first that X is a finite CW-complex with $\dim(X) \leq 1$. Then $K^0(X) \cong H^0(X) \cong \mathbb{Z}^k$ where k denotes the (finite) number of connected components of X. Arguing as in (1) we can extend this result to the general case. □

Corollary 6.17. *Let A be a continuous-trace C^*-algebra.*

(1) If $\operatorname{topdim}(A) \leq 2$, then $K_1(A)$ is torsion-free.
(2) If $\operatorname{topdim}(A) \leq 1$, then $K_0(A)$ is torsion-free.

Proof. Let us first prove the statements for the separable case. Set $X := \operatorname{Prim}(A)$. By 6.14, we have $K_*(A) \cong \widetilde{K}^*(X_+)$, and moreover $\dim(X_+) = \operatorname{topdim}(A)$. Note that $\widetilde{K}^*(X_+)$ is torsion-free if $K^*(X_+)$ is. Thus, the statements about the K-theory for A follow from Proposition 6.16.

For a general continuous-trace C^*-algebra A, we use Lemma 4.9 to find a nested subcollection of separable, continuous-trace sub-C^*-algebras $A_\lambda \subset A$ with $\operatorname{topdim}(A_\lambda) \leq \operatorname{topdim}(A)$ that exhaust A. If $\operatorname{topdim}(A) \leq 2$, then each $K_1(A_\lambda)$ is torsion-free. Since K-theory is a continuous functor, and torsion-freeness is preserved by inductive limits, this implies $K_1(A)$ is torsion-free, see 1.6.

The same argument works for $\operatorname{topdim}(A) \leq 1$ and the K_0-group. □

7. Subhomogeneous C^*-algebras

Subhomogeneous C^*-algebras are in some sense one step further away from commutative C^*-algebras than homogeneous C^*-algebras. Their primitive ideal space need not be Hausdorff, but as we will see they have a nice description as iterated pullbacks of homogeneous C^*-algebras.

The dimension theories of subhomogeneous C^*-algebras can be computed quite explicitly. We will in particular see that subhomogeneous C^*-algebras of topological dimension at most one have stable rank one. From a more general result in the next section, we get that the K_0-group is torsion-free.

Definition 7.1 (see [Bla06, IV.1.4.1, p.330]). Let A be a C^*-algebra and $n \in \mathbb{N}$. Then A is called **n-subhomogeneous** if all its irreducible representations are at most n-dimensional. Further, a C^*-algebra is called **subhomogeneous** if it is n-subhomogeneous for some n.

Proposition 7.2 (see [Bla06, IV.1.4.3, p.331]). *Let A be a C^*-algebra. Then the following are equivalent:*

(1) A is subhomogeneous,
(2) A is a sub-C^*-algebra of a (sub)homogeneous C^*-algebra,
(3) A is a sub-C^*-algebra of $C_0(X) \otimes M_n$ (for some X and n).

For a subhomogeneous C^*-algebra A we consider the homogeneous parts $\operatorname{Prim}_k(A) \subset \operatorname{Prim}(A)$ consisting of all primitive ideals correpsonding to irreducible representations of dimension k. Since each $\operatorname{Prim}_k(A)$ is locally closed in $\operatorname{Prim}(A)$, there is a C^*-algebras A_k corresponding to $\operatorname{Prim}_k(A)$, and this algebra is n-homogeneous. We will see that the dimension theories cannot distinguish A from $\oplus_k A_k$, i.e., they do not see the 'faults' in a subhomogeneous C^*-algebra.

To prove the following theorems in the separable case, the theory of recursive subhomogeneous C^*-algebras is very useful. It was developed by Phillips, [Phi07], and it implies that separable, subhomogeneous C^*-algebras with finite topological dimension are iterated pullbacks of separable, homogeneous C^*-algebras.

Proposition 7.3. *Let A be a subhomogeneous C^*-algebra. Then:*

$$\text{(1)} \qquad \mathrm{sr}(A) = \max_k \mathrm{sr}(A_k) = \max_k \left\lceil \frac{\mathrm{locdim}(\mathrm{Prim}_k(A)) + 2k - 1}{2k} \right\rceil,$$

$$\text{(2)} \qquad \mathrm{rr}(A) = \max_k \mathrm{rr}(A_k) = \max_k \left\lceil \frac{\mathrm{locdim}(\mathrm{Prim}_k(A))}{2k-1} \right\rceil,$$

$$\text{(3)} \qquad \mathrm{dr}(A) = \mathit{dim}_{\mathit{nuc}}(A) = \mathrm{topdim}(A).$$

Proof. The results for the stable and real rank are [Bro07, Lemma 3.4, Theorem 3.9].

It was shown by Winter, [Win04, Theorem 1.6], that $\mathrm{dr}(A) = \mathrm{topdim}(A)$ if A is a separable, subhomogeneous C^*-algebra. For the non-separable case, note first that the inequalities $\mathrm{topdim}(A) \leq \dim_{\mathrm{nuc}}(A) \leq \mathrm{dr}(A)$ hold for general type I C^*-algebras, see Theorem 8.7. To show $\mathrm{dr}(A) \leq \mathrm{topdim}(A)$, use (D6) for the topological dimension to approximate A by separable sub-C^*-algebras A_λ (that are necessarily subhomogeneous) with $\mathrm{topdim}(A_\lambda) \leq \mathrm{topdim}(A)$. Then $\mathrm{dr}(A_\lambda) = \mathrm{topdim}(A_\lambda) \leq \mathrm{topdim}(A)$, and by (D5) for the decomposition rank we get $\mathrm{dr}(A) \leq \mathrm{topdim}(A)$. □

8. TYPE I C^*-ALGEBRAS

In this section we take a look at dimension theories of general type I C^*-algebras. Since these algebras can be fairly complicated, the statements cannot be as explicit as for subhomogeneous C^*-algebras. Still, for CCR algebras there is an almost complete picture, see Theorem 8.4 and Corollary 8.5. Using composition series we can draw some nice conclusions for general type I C^*-algebras, see Theorem 8.7.

For information about type I C^*-algebras and their rich structure we refer the reader to Chapter IV.1 of Blackadar's book, [Bla06], and Chapter 6 of Pedersen's book, [Ped79].

Definition 8.1 (Kaplansky, [Kap51])**.** Let A be a C^*-algebra. Then A is called a **CCR algebra** if for each of its irreducible representations $\pi\colon A \to B(H)$ we have $\pi(A) = K(H)$ (the image of A under every irreducible representation lands in the compact operators).

8.2. Recall that a **composition series** for a C^*-algebra A is a collection of ideals of A, $(J_\alpha)_{\alpha\leq\mu}$, indexed over all ordinal numbers (less than some ordinal μ), such that $A = J_\mu$ and:

(i) if $\alpha \leq \beta$, then $J_\alpha \subset J_\beta$,
(ii) if α is a limit ordinal, then $J_\alpha = \overline{\bigcup_{\gamma<\alpha} J_\gamma} = \varinjlim_{\gamma<\alpha} J_\gamma$

The C^*-algebras $J_{\alpha+1}/J_\alpha$ are called the successive quotients of the composition series.

A CCR C^*-algebras is also called **liminal**. A C^*-algebra is called **postliminal** if it has a composition series with liminal successive quotients. As it turns out, this is equivalent to having a composition series whose successive quotients have continuous trace.

Originally, a postliminal C^*-algebra was called a GCR algebra, see [Kap51]. However, some authors define a GCR algebra to be a C^*-algebra A such that for each irreducible representation $\pi\colon A \to B(H)$ we have $\pi(A) \supset K(H)$. Fortunately, these conditions and other natural conditions are all equivalent, which was first proved by Glimm for the separable case, [Gli61], and later generalized to the non-separable case by Sakai, [Sak66] and [Sak67].

Proposition 8.3 (Glimm's theorem). *Let A be a C^*-algebra. Then the following are equivalent:*

(1) A is postliminal (i.e. has a composition series with liminal successive quotients),
(2) A is internally type I[10]*,*
(3) A is bidual type I[11]*,*
(4) A is a GCR algebra (i.e. $\pi(A) \supset K(H)$ for each irreducible representation π of A).

*If A satisfies these conditions then it is called a **type I** C^*-algebra.*

To obtain estimates about the dimension theories of general type I C^*-algebras, see Theorem 8.7, we will use the following result for CCR algebras. For a C^*-algebra A we denote by A_k the successive quotient of A that corresponds to the irreducible representations of dimension k.

Theorem 8.4 (Brown, [Bro07, Theorem 3.10]). *Let A be a CCR C^*-algebra, and $\operatorname{topdim}(A) < \infty$. Then:*

(1) If $\operatorname{topdim}(A) \leq 1$, then $\operatorname{sr}(A) = 1$.
(2) If $\operatorname{topdim}(A) > 1$, then $\operatorname{sr}(A) = \sup_{k\geq 1} \max\{\left\lceil \frac{\operatorname{topdim}(A_k)+2k-1}{2k} \right\rceil, 2\}$.
(3) If $\operatorname{topdim}(A) = 0$, then $\operatorname{rr}(A) = 0$.
(4) If $\operatorname{topdim}(A) > 0$, then $\operatorname{rr}(A) = \sup_{k\geq 1} \max\{\left\lceil \frac{\operatorname{topdim}(A_k)}{2k-1} \right\rceil, 1\}$.

We may draw the following conclusion:

Corollary 8.5. *Let A be a CCR C^*-algebra. Then:*

(1) $$\operatorname{sr}(A) \leq \left\lfloor \frac{\operatorname{topdim}(A)}{2} \right\rfloor + 1,$$

(2) $$\operatorname{csr}(A) \leq \left\lfloor \frac{\operatorname{topdim}(A)+1}{2} \right\rfloor + 1,$$

(3) $$\operatorname{rr}(A) \leq \operatorname{topdim}(A).$$

Proof. If $\operatorname{topdim}(A) = \infty$, then the statements hold. So we may assume $\operatorname{topdim}(A) < \infty$, whence Theorem 8.4 applies.

[10] A is internally type I if every quotient of A contains a nonzero abelian element x. An element $x \in A$ is called abelian if the hereditary sub-C^*-algebras $\overline{x^*Ax}$ it generates is commutative.

[11] A is bidual type I if for every representation $\pi\colon A \to B(H)$ the enveloping von Neumann algebra $\pi(A)''$ is type I, or equivalently if the universal enveloping von Neumann algebra of A is type I.

(1): If $\mathrm{topdim}(A) \leq 1$, then $\mathrm{sr}(A) = 1 \leq \lfloor \mathrm{topdim}(A)/2 \rfloor + 1$. If $d := \mathrm{topdim}(A) \geq 2$, then set $d_k := \mathrm{topdim}(A_k) \leq d$ and compute:

$$\begin{aligned}\mathrm{sr}(A) &= \sup_k \max\{\left\lceil \frac{d_k + 2k - 1}{2k} \right\rceil, 2\} \\ &\leq \sup_k \max\{\left\lceil \frac{d + 2k - 1}{2k} \right\rceil, 2\} \\ &\leq \max\{\left\lceil \frac{d+1}{2} \right\rceil, 2\} \\ &\leq \left\lfloor \frac{d}{2} \right\rfloor + 1.\end{aligned}$$

(2): The statement for the connected stable rank follows from Proposition 3.7.

(3): Again, we use Theorem 8.4. If $\mathrm{topdim}(A) = 0$, then $\mathrm{rr}(A) = 0 \leq \mathrm{topdim}(A)$. If $d := \mathrm{topdim}(A) \geq 1$, then set $d_k := \mathrm{topdim}(A_k) \leq d$ and compute:

$$\begin{aligned}\mathrm{rr}(A) &= \sup_k \max\{\left\lceil \frac{d_k}{2k-1} \right\rceil, 1\} \\ &\leq \sup_k \max\{\left\lceil \frac{d}{2k-1} \right\rceil, 1\} \\ &\leq \max\{\lceil d \rceil, 1\} \\ &\leq d.\end{aligned}$$

□

We can now give general comparison results for type I C^*-algebras. It seems that these results do not appear in the literature so far. But first some remark is in order, that might also be of general interest.

Remark 8.6. What makes type I C^*-algebra so accessible is the presence of composition series with successive quotients that are easier to handle (e.g. continuous-trace or CCR). They allow us to prove statements by transfinite induction, for which one has to consider the case of a successor and limit ordinal. Let us see that for statements about dimension theories one only needs to consider successor ordinals.

Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for a C^*-algebra. If α is a limit ordinal, then $J_\alpha = \overline{\bigcup_{\gamma<\alpha} J_\gamma} = \varinjlim_{\gamma<\alpha} J_\gamma$. For any dimension theory rk we have

$$\begin{aligned}\mathrm{rk}(J_\alpha) &\leq_{(D5)} \liminf_{\gamma<\alpha} \mathrm{rk}(J_\gamma) \\ &\leq \quad \sup_{\gamma<\alpha} \mathrm{rk}(J_\gamma) \\ &\leq_{(D1)} \sup_{\gamma<\alpha} \mathrm{rk}(J_\alpha),\end{aligned}$$

and thus $\mathrm{rk}(J_\alpha) = \sup_{\gamma<\alpha} \mathrm{rk}(J_\gamma)$.

Thus, any reasonable estimate about dimension theories that holds for $\gamma < \alpha$ will also hold for α. It follows that we only need to consider a successor ordinal α, in which case $A = J_\alpha$ is an extension of $B = J_\alpha / J_{\alpha-1}$ by $I = J_{\alpha-1}$. By assumption the result is true for I and has to be proved for A (by using that B is CCR or continuous-trace). This idea is used to prove the next theorem:

Theorem 8.7. *Let A be a type I C^*-algebra. Then:*

$$\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)+1}{2} \right\rfloor + 1, \tag{1}$$

$$\mathrm{csr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)}{2} \right\rfloor + 2, \tag{2}$$

$$\mathrm{rr}(A) \leq \mathrm{topdim}(A) + 2, \tag{3}$$

$$\mathrm{topdim}(A) \leq \mathit{dim}_{nuc}(A) \leq \mathrm{dr}(A). \tag{4}$$

Proof. (1): Let $(J_\alpha)_{\alpha\leq\mu}$ be a composition series for A such that the successive quotients are CCR algebras. Set $d := \mathrm{topdim}(A)$. By Remark 8.6 it is enough to consider short exact sequences of C^*-algebras $0 \to I \to A \to B \to 0$ where the quotient B is a CCR algebra. By assuming that the inequality is valid for I we need to show that it holds for A. We have $\mathrm{topdim}(I) \leq d$ and $\mathrm{topdim}(B) \leq d$, and then:

$$\begin{aligned} \mathrm{sr}(A) &\leq \max\{\mathrm{sr}(I), \mathrm{sr}(B), \mathrm{csr}(B)\} \\ &\leq \max\{\left\lfloor \frac{d+1}{2} \right\rfloor + 1, \left\lfloor \frac{d}{2} \right\rfloor + 1, \left\lfloor \frac{d+1}{2} \right\rfloor + 1\} \\ &= \left\lfloor \frac{d+1}{2} \right\rfloor + 1, \end{aligned}$$

where the first inequality follows from Proposition 3.10, and the second follows using the assumption on B and Corollary 8.5.

(2), (3): The inequalities for the connected stable rank and real rank follow from (1) and Proposition 3.7.

(4): The inequality $\mathrm{dim}_{\mathrm{nuc}}(A) \leq \mathrm{dr}(A)$ holds for every C^*-algebra A. To show $\mathrm{topdim}(A) \leq \mathrm{dim}_{\mathrm{nuc}}(A)$, consider a composition series $(J_\alpha)_{\alpha\leq\mu}$ with successive quotients of continuous-trace. For each $\alpha < \mu$, we have $\mathrm{dim}_{\mathrm{nuc}}(J_{\alpha+1}/J_\alpha) \leq \mathrm{dim}_{\mathrm{nuc}}(A)$ by axioms (D1) and (D2) for the nuclear dimension. By Proposition 6.12, $\mathrm{topdim}(J_{\alpha+1}/J_\alpha) = \mathrm{dim}_{\mathrm{nuc}}(J_{\alpha+1}/J_\alpha)$, and then from Proposition 4.4 we get $\mathrm{topdim}(A) = \sup_{\alpha<\mu} \mathrm{topdim}(J_{\alpha+1}/J_\alpha) \leq \mathrm{dim}_{\mathrm{nuc}}(A)$. □

Remark 8.8. We get the following:

(1) $\mathrm{sr}(A) = \left\lfloor \frac{\mathrm{topdim}(A)}{2} \right\rfloor + 1,$ if A is commutative.

(2) $\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)}{2} \right\rfloor + 1,$ if A is CCR.

(3) $\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)+1}{2} \right\rfloor + 1,$ if A is type I.

This also shows that the inequality for the stable rank in Corollary 8.5 can not be improved (the same is true for the estimates of real rank and connected stable rank). The Toeplitz algebra, see Example 9.1, shows that the estimate of Theorem 8.7 for the stable rank can not be improved either.

8.9. For a C^*-algebra A let us use the notation $\mathbb{I}^k A := C([0,1]^k) \otimes A$. For $k \geq 1$, we can prove $\mathrm{sr}(\mathbb{I}^k A) \leq \left\lfloor \frac{\mathrm{topdim}(\mathbb{I}^k A)}{2} \right\rfloor + 1$ along the lines of Theorem 8.7 using $\mathrm{csr}(\mathbb{I}^k A) = \mathrm{csr}(A)$, see [Nis86, Lemma 2.8, Corollary 2.9]. Then $\mathrm{rr}(\mathbb{I}^k A) \leq \mathrm{topdim}(A) + 1 + k$ for $k \geq 1$.

It is unclear whether this also holds for $k = 0$, i.e., whether Theorem 8.7 (3) can be improved to the inequality $\mathrm{rr}(A) \leq \mathrm{topdim}(A) + 1$ for type I C^*-algebras.

In fact, the author does not know of a type I C^*-algebra A where the even stronger inequality $\mathrm{rr}(A) \leq \mathrm{topdim}(A)$ does not hold.

8.10. Let A be a type I C^*-algebra, and set $k := \mathrm{dr}(A)$. If $k \in \{0, 1\}$, then $\mathrm{rr}(A) \leq \mathrm{dr}(A)$, see Proposition 8.11 and Corollary 8.19. It is unclear whether this also holds if $k \geq 2$.

It is possible that the inequality is even true for all (separable) C^*-algebras, see [Win03, 4.1] and [KW04, 4.11].

8.1. **Low dimensions and K-theory.**

It was shown by Lin, [Lin97], that a separable, type I C^*-algebras has real rank zero if and only if it is an AF-algebra. For general C^*-algebras, the case of dimension zero agrees for the decomposition rank, the nuclear dimension and the extended topological dimension, and moreover it is equivalent to being an LF-algebra, see 5.11. It follows that there is a natural concept of 'dimension zero' for type I C^*-algebras, see Proposition 8.11.

We will study the relations of dimension one for the different dimension theories of a type I C^*-algebra A. It follows from [BP09, Proposition 5.2] that $\mathrm{sr}(A) = 1$ implies $\mathrm{topdim}(A) \leq 1$. Conversely, $\mathrm{topdim}(A) \leq 1$ only implies $\mathrm{sr}(A) \leq 2$, see Theorem 8.7, and the Toeplitz algebra shows that this is the best estimate. The Toeplitz algebra is not (residually) stable finite, and we will show that this is the only obstruction that a type I C^*-algebra with topological dimension one also has stable rank one, see Theorem 8.17. It follows that $\mathrm{dr}(A) \leq 1$ implies $\mathrm{sr}(A) = 1$, and these implications are summarized in Theorem 8.21.

To prove these results we show that certain index maps in K-theory vanish, see Theorem 8.15. It is the vanishing of these index maps that also proves Theorem 8.23, which states that the K_0-group of a stable rank one, type I C^*-algebra is torsion-free.

Proposition 8.11. *Let A be a type I C^*-algebra. Then the following are equivalent:*

(1) $\mathrm{rr}(A) = 0$,
(2) $\mathrm{dr}(A) = 0$,
(3) $\dim_{nuc}(A) = 0$,
(4) $\mathrm{topdim}(A) = 0$,
(5) A is an LF-algebra (i.e., A is approximated by finite-dimensional sub-C^*-algebras).

Proof. Note that $\mathrm{topdim}(A) = \mathrm{topdim}^{\sim}(A)$, so that the equivalence of (2)-(5) follows from the more general result in Proposition 5.11. The implication (5)⇒(1) also holds in general.

By [Lin97], the implication (1)⇒(5) holds for separable (type I) C^*-algebras. By (D6) for the real rank, a general type I C^*-algebra A with $\mathrm{rr}(A) = 0$ can be approximated by separable sub-C^*-algebras A_λ (necessarily of type I) with $\mathrm{rr}(A_\lambda) = 0$. Then each A_λ is an AF-algebra, and so A is LF. □

Let us recall two facts with striking similarity. In both cases the vanishing of a K-theoretic obstruction implies that some property of an extension algebra is already determined by the behavior of the ideal and quotient.

See 1.5 for the definition of (residual) stable finiteness.

Proposition 8.12 (Spielbgerg, [Spi88, Lemma 1.5], Nistor, [Nis87, Lemma 3]). *Let A be a C^*-algebra, and $I \lhd A$ an ideal. Consider the index map $\delta : K_1(A/I) \to K_0(I)$, and denote the image of δ by $\operatorname{im}\delta$. Then the following hold:*

(1) Assume I and A/I are stably finite.
Then A is stably finite if and only if $\operatorname{im}\delta \cap K_0(I)^+ = \{0\}$.

(2) Assume I and A/I have stable rank one.
Then A has stable rank one if and only if $\delta = 0$.

8.13. If $K_0(I)$ is totally ordered (e.g. $K_0(I) = \mathbb{Z}$ with the usual ordering), then $\operatorname{im}\delta \cap K_0(I)^+ = \{0\}$ already implies $\delta = 0$. This has the following consequence: If A is stably finite, I and A/I have stable rank one, and $K_0(I) \cong \mathbb{Z}$, then δ vanishes, and hence $\operatorname{sr}(A) = 1$.

The following lemma is based on this observation. Then, in Theorem 8.15 we will give sufficient conditions for the vanishing of the index map in a more general setting.

Lemma 8.14. *Let A be a residually stably finite C^*-algebra, and $I \lhd A$ an ideal of continuous-trace and with $\operatorname{topdim}(I) \leq 1$. Then the index map $\delta : K_1(A/I) \to K_0(I)$ is zero.*

Proof. Let us first prove the statement under the additional assumption that I is separable. Set $B := A/I$ and $X := \operatorname{Prim}(I)$. We may assume that I, A and B are stable, in which case $I \cong C_0(X, \mathbb{K})$. Then $K_0(I) \cong \widetilde{H}^0(X_+) \cong [X_+; \mathbb{Z}]_*$. Thus, we may view elements in $K_0(I)$ as maps $f \colon X \to \mathbb{Z}$ that vanish at infinity. Such a map will be in $K_0(I)_+$ if and only if $f \geq 0$, i.e., $f(x) \geq 0$ for all $x \in X$.

Let $\omega \in K_1(B)$ be some element, fixed for now. Set $f := \delta(\omega) \in K_0(I)$, and we think of f as a map in $[X_+; \mathbb{Z}]_*$. We need to show that $f = 0$, i.e., $f(x) = 0$ for all $x \in X$.

Let $x \in X$ be some point, and $C \subset X$ the connected component of x. Then C is closed and we may consider the ideal $J \lhd I$ corresponding to the open set $\operatorname{Prim}(I) \setminus C$. The situation is shown in the commutative diagram on the left, where the rows are short exact sequences. Let $\delta' : K_1(B) \to K_0(A/J)$ be the index map for the lower extension. The naturality of the index map implies that the diagram on the right is commutative, where the map $K_0(I) \to K_0(I/J)$ is induced by the quotient morphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I/J & \longrightarrow & A/J & \longrightarrow & B & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} K_1(B) & \xrightarrow{\delta} & K_0(I) & = & [X_+; \mathbb{Z}]_* \\ \| & & \downarrow & & \\ K_1(B) & \xrightarrow{\delta'} & K_0(I/J) & = & [C_+; \mathbb{Z}]_* \end{array}$$

Then A/J is residually finite and $K_0(I/J)$ is totally ordered (it is $\mathbb{Z}$ if $\operatorname{Prim}(I/J) \cong C$ is compact, and 0 otherwise). By the remarks in 8.13, $\delta' = 0$, and therefore $f(x) = 0$. Since this holds for all x, we have $f = 0$ as desired.

Let us prove the non-separable case. Set $B := A/I$. Let $\omega \in K_1(B)$ be some element, fixed from now on. Then there exists a separable sub-C^*-algebra $B_0 \subset B$ and $\omega_0 \in K_1(B_0)$ such that ω_0 gets mapped to ω by the inclusion morphism. Choose some separable sub-C^*-algebra $A_0 \subset A$ such that $A_0/I = B_0$. Applying Lemma 4.9 to the inclusion $A_0 \cap I \subset I$, we get a separable, continuous-trace sub-C^*-algebra $K \subset I$ with $\operatorname{topdim}(K) \leq \operatorname{topdim}(I) \leq 1$ and such that $A_0 \cap I$ is contained in K, and moreover this is a proper inclusion.

Set $D := C^*(K, A_0) \subset A$. By Lemma 4.10, $D \cap I = K$ and $D/K = B_0$. The situation is shown in the commutative diagram on the left, where the rows are short exact sequences. Let $\delta' : K_1(B_0) \to K_0(K)$ be the index map for the lower extension. The naturality of index maps shows that the diagram on the right is commutative, where maps K_0 and K_1 are induced by the inclusion morphisms.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & D & \longrightarrow & B_0 & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{cccc}
\omega & \ni K_1(B) & \xrightarrow{\delta} & K_0(I) \\
 & \uparrow & & \downarrow \\
\omega_0 & \ni K_1(B_0) & \xrightarrow{\delta'} & K_0(K)
\end{array}
$$

Since $K \lhd D$ satisfies the assumptions of the statement, and moreover K is separable, we get that δ' vanishes, and so $\delta(\omega) = \delta'(\omega_0) = 0$. Since ω was arbitrary, this implies that δ vanishes, as desired. □

Theorem 8.15. *Let A be a residually stably finite C^*-algebra, and $I \lhd A$ an ideal of type I with* $\mathrm{topdim}(I) \leq 1$. *Then the index map $\delta : K_1(A/I) \to K_0(I)$ is zero.*

Proof. We may assume A is unital, and set $B := A/I$. Let $x \in K_0(B)$ be some element, fixed from now on. Set $y := \delta(x) \in K_1(I)$. We need to show that $y = 0$.

Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for I with successive quotients of continuous-trace quotients. Note that $\mathrm{topdim}(J_{\alpha+1}/J\alpha) \leq \mathrm{topdim}(I) \leq 1$. For each α we have a 'sub-extension'

$$0 \to I/J_\alpha \to A/J_\alpha \to B \to 0,$$

with index map $\delta_\alpha : K_1(B) \to K_0(I/J_\alpha)$ and we may ask if $\delta_\alpha(x) = 0$.

Let $\iota_\alpha : J_\alpha \to I$ be the inclusion map, and $q_\alpha : I \to I/J_\alpha$ the quotient map, then we have extensions:

$$0 \to J_\alpha \xrightarrow{\iota_\alpha} I \xrightarrow{q_\alpha} I/J_\alpha \to 0.$$

Note that $\delta_\alpha = K_0(q_\alpha) \circ \delta$, and define

$$\beta := \inf\{\alpha \leq \mu \mid \delta_\alpha(x) = 0\} \quad = \inf\{\alpha \leq \mu \mid K_0(q_\alpha)(y) = 0\}.$$

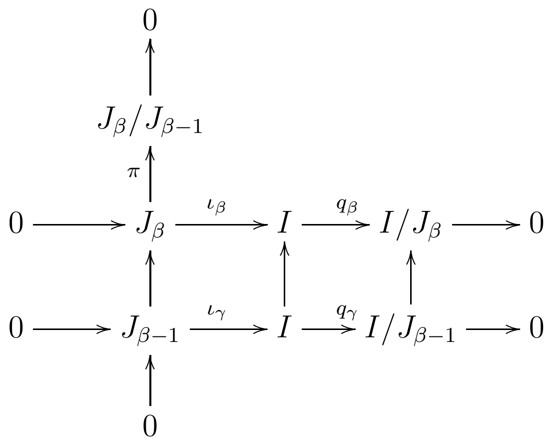
We need to show that $\beta = 0$.

Step 1: We show that β is not a limit ordinal. Assume otherwise, in which case the continuity of K-theory gives $K_0(J_\beta) = \varinjlim_{\gamma<\beta} K_0(J_\gamma)$. Let $\iota_{\beta\gamma} : J_\gamma \to J_\beta$ be the inclusion map. The situation is shown in the following diagram on the left, which is commutative with exact rows. In K-theory we get a commutative diagram, which is shown on the right (with exact rows):

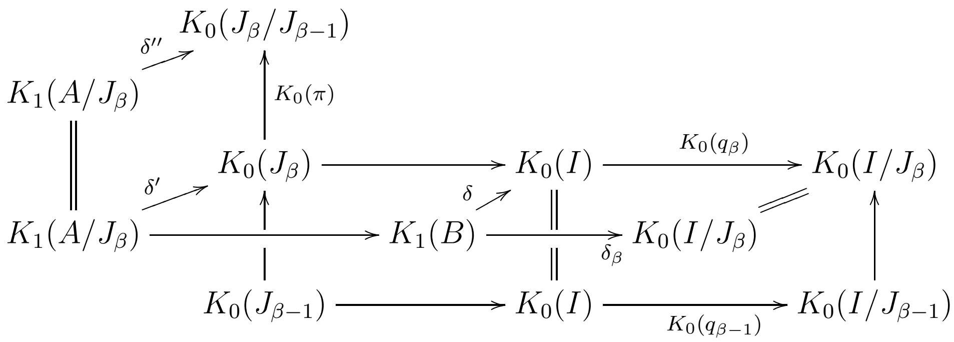
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J_\beta & \xrightarrow{\iota_\beta} & I & \xrightarrow{q_\beta} & I/J_\beta & \longrightarrow & 0 \\
 & & {\scriptstyle\iota_{\beta\gamma}}\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & J_\gamma & \xrightarrow{\iota_\gamma} & I & \xrightarrow{q_\gamma} & I/J_\gamma & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccc}
K_0(J_\beta) & \xrightarrow{K_0(\iota_\beta)} & K_0(I) & \xrightarrow{q_\beta} & K_0(I/J_\beta) \\
{\scriptstyle K_0(\iota_{\beta\gamma})}\uparrow & & \uparrow & & \uparrow \\
K_0(J_\gamma) & \xrightarrow{K_0(\iota_\gamma)} & K_0(I) & \xrightarrow{K_0(q_\gamma)} & K_0(I/J_\gamma)
\end{array}
$$

We have $K_0(q_\beta)(y) = 0$ by definition of β. From exactness of the upper row we may find a lift $y' \in K_0(J_\beta)$ for y. By continuity of K-theory we may lift y' to some $y'' \in K_0(J_\gamma)$ (for some $\gamma < \beta$). Then $K_0(\iota_\gamma)(y'') = K_0(\iota_\beta)(K_0(\iota_{\beta\gamma}))(y'') = K_0(\iota_\beta)(y') = y$. The exactness of the lower row shows $K_0(q_\gamma)(y) = 0$, which contradicts the definition of β.

Step 2: We show that β has no predecessor $\beta - 1$. Assume otherwise, in which case we have the following commutative diagram with exact rows and columns:



This gives a commutative diagram in K-theory, and we add index maps for several different extension to the picture:



Since $\delta_\beta(x) = 0 \in K_0(I/J_\beta)$, we may lift x to $x' \in K_1(A/J_\beta)$. Set $y' := \delta'(x') \in K_0(J_\beta)$, which is a lift of $y = \delta(X) \in K_0(I)$.

Consider the extension $0 \to J_\beta/J_{\beta-1} \to A/J_{\beta-1} \to A/J_\beta \to 0$. We have $A/J_{\beta-1}$ (residually) stably finite, and $J_\beta/J_{\beta-1}$ of continuous-trace. Therefore, its index map δ'' is zero by Lemma 8.14.

It follows that $K_0(\pi)(y') = 0$, and we can lift y' to $y'' \in K_0(J_{\beta-1})$. But then $K_0(q_{\beta-1})(y) = 0$, which contradicts the definition of β.

Altogether we have $\beta = 0$, or put differently $y = \delta(x) = 0$. Since x was arbitrary, we get that δ vanishes, as desired. □

Corollary 8.16. *Let A be a residually stably finite C^*-algebra, and $I \lhd A$ an ideal of type I. Assume* $\mathrm{sr}(I) = \mathrm{sr}(A/I) = 1$. *Then* $\mathrm{sr}(A) = 1$.

Proof. By [BP09, Proposition 5.2], $\mathrm{sr}(I) = 1$ implies $\mathrm{topdim}(I) = 1$. Thus, Theorem 8.15 applies and we obtain that the index map δ vanishes. It follows that $\mathrm{sr}(A) = 1$, see [Nis87, Lemma 3], see also Proposition 8.12. □

Theorem 8.17. *Let A be a type I C^*-algebra. Then the following are equivalent:*
(1) $\mathrm{sr}(A) = 1$,
(2) $\mathrm{topdim}(A) \leq 1$, *and A is residually stably finite.*

Proof. '(1) $\Rightarrow$ (2)': We have $\mathrm{topdim}(A) \leq 1$ by [BP09, Proposition 5.2], and every quotient of A has stable rank one, hence is stably finite.

'(2) $\Rightarrow$ (1)': Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for I with successive quotients of continuous-trace. Note that $\mathrm{topdim}(J_{\alpha+1}/J\alpha) \leq \mathrm{topdim}(I) \leq 1$. For any CCR algebra (in particular, every continuous-trace C^*-algebra) B, $\mathrm{topdim}(B) \leq 1$ implies $\mathrm{sr}(B) = 1$, see [Bro07, Theorem 3.10], see also Theorem 8.4. In particular $\mathrm{sr}(J_1) = 1$ and $\mathrm{sr}(J_{\alpha+1}/J_\alpha) = 1$ for every $\alpha < \mu$.

By transfinite induction over $\alpha \leq \mu$ we show that $\mathrm{sr}(J_\alpha) = 1$. Assume $\mathrm{sr}(J_\alpha) = 1$ for all $\alpha \leq \beta$. If β is a limit ordinal, then $J_\beta = \varinjlim_{\alpha<\beta} J_\alpha$, and consequently $\mathrm{sr}(J_\beta) \leq \varliminf_{\alpha<\beta} \mathrm{sr}(J_\alpha) = 1$. If β has a predecessor $\beta - 1$, then J_β is a residually stably finite extension of $J_{\beta-1}$ and $J_\beta/J_{\beta-1}$, which both have stable rank one. Then $\mathrm{sr}(J_\beta) = 1$ by Corollary 8.16.

It follows $\mathrm{sr}(A) = \mathrm{sr}(J_\mu) = 1$, as desired. □

Let us see that we cannot drop the word 'residually' in the above theorem:

Remark 8.18. We cannot drop the word 'residually' in the above Theorem 8.17. Indeed, for the algebra $A := C^*(S^* \oplus S)$ from Example 9.2, we have $\mathrm{sr}(A) = 2$ while $\mathrm{topdim}(A) = 1$. The point is that A is not *residually* stably finite, since it has a quotient isomorphic to the Toeplitz algebra (which an infinite C^*-algebra).

Corollary 8.19. *Let A be a type I C^*-algebra with* $\mathrm{dr}(A) \leq 1$. *Then* $\mathrm{sr}(A) = 1$, *and consequently also* $\mathrm{rr}(A) \leq 1$.

Proof. Assume A is a type I C^*-algebra with $\mathrm{dr}(A) \leq 1$. Then $\mathrm{topdim}(A) \leq 1$, by Theorem 8.7. Moreover, every quotient A/I of A satisfies $\mathrm{dr}(A/I) \leq 1$. It is shown in [KW04, Theorem 5.1] that any separable C^*-algebra with finite decomposition rank is quasidiagonal. This also holds for non-separable C^*-algebras (using (D6) for the decomposition rank, and that a C^*-algebra is quasidiagonal if it is approximated by quasidiagonal sub-C^*-algebras, see [Bla06, V.4.1.8, p.459]). Hence, A/I is quasidiagonal, and therefore also stably finite, see [Bla06, V.4.2.6, p.459].

Then $\mathrm{sr}(A) = 1$ follows from Theorem 8.17. □

8.20. One can ask if Corollary 8.19 holds for all C^*-algebra, i.e., whether for every C^*-algebra A, $\mathrm{dr}(A) \leq 1$ implies $\mathrm{sr}(A) = 1$, see [KW04, 4.11]. It is true for type I C^*-algebra by the above result.

Let us see that it also holds for simple, unital C^*-algebras. It was shown by Winter, [Win08, Theorem 5.1], that a separable, simple, unital C^*-algebra with finite decomposition rank is $\mathcal{Z}$-stable. Since A is finite, $\mathcal{Z}$-stability implies that $\mathrm{sr}(A) = 1$, see [Rør04]. We can remove the separability condition using (D6) for the decomposition rank together with the theory of separably inheritable properties, see [Bla06, II.8.5, 176ff].

Let us summarize our results Theorem 8.17 and Corollary 8.19 as follows:

Theorem 8.21. *Let A be a type I C^*-algebra. Consider the following properties:*
(1) $\mathrm{sr}(A) = 1$,

(2) $\mathrm{topdim}(A) = 1$ *and* A *is residually stably finite,*
(3) $\mathrm{dr}(A) \leq 1$.

Then $(1) \Leftrightarrow (2) \Leftarrow (3)$.

8.22. There is another interesting property related to the ones in Theorem 8.21:
(4) A is approximated by subhomogeneous sub-C^*-algebras of stable rank one.

Since for a subhomogeneous C^*-algebra B, we have $\mathrm{sr}(B) = 1$ if and only if $\mathrm{dr}(B) = 1$, we get the implication $(4) \Rightarrow (3)$. There is the interesting possibility that actually all conditions $(1) - (4)$ are equivalent for type I C^*-algebras, but this is not known.

We end this section with an interesting observation which seems not to be in the literature so far.

Theorem 8.23. *Let* A *be a type I* C^**-algebra with* $\mathrm{sr}(A) = 1$. *Then* $K_0(A)$ *is torsion-free.*

Proof. Let $(J_\alpha)_{\alpha\leq\mu}$ be a composition series for A with successive quotients of continuous-trace. If B is any continuous-trace C^*-algebra with $\mathrm{topdim}(B) \leq 1$, then $K_0(B)$ is torsion-free by Corollary 6.17. Note that $\mathrm{topdim}(J_{\alpha+1}/J\alpha) \leq \mathrm{topdim}(I) \leq 1$ for all α. Thus, $K_0(J_1)$ and $K_0(J_{\alpha+1}/J_\alpha)$ are torsion-free for all α.

By transfinite induction over $\alpha \leq \mu$ we show that $K_0(J_\alpha)$ is torsion-free. Assume $K_0(J_\alpha)$ is torsion-free for all $\alpha < \beta$. If β is a limit ordinal, then $J_\beta = \varinjlim_{\alpha<\beta} J_\alpha$, and consequently $K_0(J_\beta) = \varinjlim_{\alpha<\beta} K_0(J_\alpha)$ is torsion-free. (An inductive limit of torsion-free groups is torsion-free, even if the connecting morphisms of the inductive limit are not necessarily injective).

If β has a predecessor $\beta - 1$, then J_β is an extension of $J_{\beta-1}$ and $J_\beta/J_{\beta-1}$, and $\mathrm{sr}(J_\beta) = 1$. Then the index map $\delta : K_1(J_\beta/J_{\beta-1}) \to K_0(J_{\beta-1})$ vanishes by Proposition 8.12. The six-term exact sequence in K-theory for this extension is:

$$\begin{array}{ccccc} K_0(J_{\beta-1}) & \xrightarrow{f} & K_0(J_\beta) & \xrightarrow{g} & K_0(J_\beta/J_{\beta-1}) \\ {\scriptstyle 0}\uparrow & & & & \downarrow \\ K_1(J_\beta/J_{\beta-1}) & \longleftarrow & K_1(J_\beta) & \longleftarrow & K_1(J_{\beta-1}) \end{array}$$

Let H denote the image of g, i.e., the subgroup of $K_0(J_\beta/J_{\beta-1})$ that lies in the kernel of the index map from K_0 to K_1. Note that H is torsion-free. Then $K_0(J_\beta)$ is an extension of two torsion-free groups, namely $K_0(J_{\beta-1})$ and H, and therefore torsion-free itself.

It follows that $K_0(A) = K_(J_\mu)$ is torsion-free. □

9. Examples

Example 9.1. Let $\mathcal{T}$ denote the Toeplitz algebra. It is the universal C^*-algebra generated by an isometry. Let S be the unilateral shift (on a separable, infinite-dimensional Hilbert space H). Then the Toeplitz algebra can be realized as $\mathcal{T} \cong C^*(S) \subset B(H)$. There is a natural short exact sequence

$$0 \to \mathbb{K} \to \mathcal{T} \to C(S^1) \to 0.$$

This allows us to compute $\mathrm{topdim}(\mathcal{T}) = \max\{\mathrm{topdim}(\mathbb{K}), \mathrm{topdim}(C(S^1))\} = \max\{0,1\} = 1$. By [Rie83, Example 4.13], we have $\mathrm{sr}(\mathcal{T}) = 2$. Since $\mathcal{T}$ is not finite, we also get $\mathrm{gsr}(\mathcal{T}) = \mathrm{csr}(\mathcal{T}) = 2$.

We have $\mathrm{dr}(\mathcal{T}) = \infty$, since $\mathcal{T}$ is not quasidiagonal. We also have $\dim_{\mathrm{nuc}}(\mathcal{T}) \leq \dim_{\mathrm{nuc}}(\mathbb{K}) + \dim_{\mathrm{nuc}}(C(S^1)) + 1 = 2$, by [WZ10, Proposition 2.9], see also 5.8. It follows that the Toeplitz algebra has nuclear dimension 1 or 2, and the exact value is not known.

Example 9.2. Let S be the unilateral shift (on a separable, infinite-dimensional Hilbert space H). Let $A := C^*(S^* \oplus S)$ be the sub-C^*-algebra of $B(H \oplus H)$ generated by the operator $S^* \oplus S$ (just as the Toeplitz algebra is $\mathcal{T} = C^*(S)$). Then A is a stably finite, separable, type I C^*-algebra A (see [Bla06, V.4.2.4.(iii), p.461]) which fits into two extensions

$$0 \to \mathbb{K} \oplus \mathbb{K} \to A \to C(S^1) \to 0$$
$$0 \to \mathbb{K} \to A \to \mathcal{T} \to 0.$$

From the first extension we easily get $\mathrm{topdim}(A) = 1$. From the second extension we get $\mathrm{sr}(A) = \mathrm{csr}(A) = 2$, by 3.10 and 3.11. Since A is finite, we have $\mathrm{gsr}(A) = 1$.

We have $\mathrm{dr}(A) = \infty$, since A is not residually stably finite. We get $\dim_{\mathrm{nuc}}(\mathcal{T}) \leq \dim_{\mathrm{nuc}}(\mathbb{K}) + \dim_{\mathrm{nuc}}(\mathcal{T}) + 1 = 3$, by [WZ10, Proposition 2.9], see also 5.8.

We list the dimension theory of some particular C^*-algebras (or classes of C^*-algebras) in the following table:

	AF	$C(S^1)$	$C(D^2)$	$\mathcal{T}$	$C^*(S \oplus S^*)$	$\mathcal{O}_n$	$\mathcal{O}_\infty$
sr	1	1	2	2	2	∞	∞
csr	1	1	1	2	2	∞	2
gsr	1	1	1	2	1	∞	2
rr	0	1	2	1	1	0	0
topdim	0	1	2	1	1	∞	∞
dr	0	1	2	∞	∞	∞	∞
$\dim_{\mathrm{nuc}}$	0	1	2	1,2?	1,2,3?	1	1

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