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One-dimensional C*-algebras and their K-theory

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Introduction

The theory of C*-algebras is often considered as non-commutative topology, which is justified by the natural one-one-correspondence between unital, commutative C*-algebras and the category of compact, Hausdorff spaces.

Given this fact, one tries to transfer concepts from commutative topology to C*-algebras. For example, this has been done with great success for K-theory, which is by now one of the most important theories in non-commutative topology. Other concepts that have been generalized to C*-algebras are the theory of shape and dimension, and it is the latter that we will focus on.

In fact there are different non-commutative dimension theories (often called rank), we will consider in particular the real and stable rank, the topological dimension and the decomposition rank. The low-dimensional case of these theories is of most interest. One reason is that the low-dimensional cases often agree for different theories. But more importantly, low dimension (in any dimension theory) is considered as a regularity property.

Such regularity properties are very important for proving the Elliott conjecture, which predicts that separable, nuclear, simple C*-algebras are classified by their Elliott invariant, a tuple consisting of ordered K-theory, the space of traces and a pairing between the two. There are counterexamples to this general form of the conjecture, but if one restricts to certain subclasses that are regular enough, then the conjecture has been verified.

It is interesting that many regularity properties are statements about a certain dimension theories. The obvious example is the requirement of real rank zero, which means that the C*-algebra has many projections. In this setting many classification results have been obtained, although real rank zero is not enough to verify the Elliott conjecture. Let us list a few known results. They all require a certain dimension to be small or at least bounded:

(1) The results of Elliott, Gong and Li (see [**EGL07**]) consider certain algebras of bounded AH-dimension
(2) A result of Lin (see [**Lin08**]), which considers certain algebras with bounded ASH-dimension
(3) A result of Winter (see [**Win07**]), which considers certain real rank zero algebras which have locally finite decomposition rank
(4) A results of Lin [**Lin04**], which considers certain C*-algebras with tracial rank zero

The Elliott conjecture would imply that in principle all properties of a (classifiable) C*-algebra are somehow reflected in its Elliott invariant. Thus, besides the struggle to actually prove the conjecture (which is a field of active research) it is of interest to see how C*-algebraic properties, like the value of a certain dimension theory, can be detected in the Elliott invariant.

In this thesis, we study non-commutative dimension theories and their connection to K-theory. We take a closer look at low dimensions and we will show that this is tightly related to the question for torsion in K-theory. The principle is that low dimensions lead to torsion-freeness (of the K_0- or K_1-group), in some cases it is even equivalent to that. This paper is organized as follows:

1. In the first section we introduce the concept of a (non-commutative) dimension theory. There are in fact different of these theories, and we present prominent examples, namely the real and stable rank, the topological dimension and the decomposition rank.

2. Then , in the next three sections we show how to compute and estimate the dimensions of certain type I C*-algebras. In the second section we start with homogeneous and continuous-trace C*-algebras, which are well-understood in their structure, which makes it possible to compute their dimension theories quite explicitly.

In the appendix be construct an integral Chern character for commutative spaces of dimension ≤ 3. We will us this result to prove the following connection between dimension and K-theory of a σ-unital continuous-trace algebra A:

(1) If $\operatorname{topdim}(A) \leq 2$, then $K_1(A)$ is torsion-free.
(2) If $\operatorname{topdim}(A) \leq 1$, then $K_0(A)$ is torsion-free.

3. In the third section we turn to subhomogeneous C*-algebras, which are more complicated in their structure than homogeneous C*-algebras. Their primitive ideal space can be non-Hausdorff, but they can often be written as an iterated pullback of nicer algebras. We will study two cases: The recursive subhomogeneous C*-algebras and the non-commutative CW-complexes (NCCW-complexes).

The former are iterated pullbacks of homogeneous C*-algebras and these algebras are important because they include almost all subhomogeneous C*-algebras. Thus, it becomes possible to extend results from homogeneous to subhomogeneous C*-algebras.

The NCCW-complexes are very interesting, because they serve as building blocks for the construction of direct limit C*-algebras. The one-dimensional NCCW-complexes are also important because they are semiprojective.

4. In the fourth section we consider general type I C*-algebras. We establish some estimates that seem not to appear in the literature so far. We also take a closer look at low-dimensions, in which it happens that many dimension theories agree. This is known for the case of dimension zero. For dimension one it was shown by Brown and Pedersen showed a type I algebra has topological dimension ≤ 1 if and only if A has generalized stable rank one, i.e. a composition series with quotients of stable rank one. The key example of a type I C*-algebra with $\operatorname{topdim}(A) = 1$ but higher stable rank (or higher decomposition rank) is the Toeplitz algebra. This algebra is not residually stable finite, and we show that this is indeed the only obstruction for stable rank one. Precisely, the following are equivalent for a separable, unital, type I C*-algebra:

(i) $\operatorname{sr}(A) = 1$
(ii) $\operatorname{topdim}(A) = 1$ and A is residually stably finite

For the proof we show that the index map in K-theory vanishes. It also the vanishing of this index map that implies the following result:

1.) If A is a type I C*-algebra with $\operatorname{sr}(A) = 1$, then $K_0(A)$ is torsion-free.

Together with the fact that $\text{topdim}(A) \leq \text{dr}(A)$ for type I C*-algebras we get that type I C*-algebras with decomposition rankn ≤ 1 have torsion-free K_0-group.

5. In the fifth section we introduce approximately (sub)homogeneous C*-algebras, which are just direct limits of certain (sub)homogeneous C*-algebras. We use them to define the concept of AH- and ASH-dimension. These extend the theory of topological dimension (as defined for type I algebras) to a broader class of C*-algebras.

For example, we let $\underline{\text{ASH}}(k)'$ be the class of C*-algebras that are limits of k-dimensional NNCW-complexes, and we let

$$\dim_{ASH'}(A) \leq k :\Leftrightarrow A \in \underline{\text{ASH}}(k)'$$

We will show in section 7 how in the setting of the Elliott conjecture this dimension can be detected in the K-theory.

6. In the sixth section we will give a brief introduction to the Elliott conjecture. This conjecture predicts that certain C*-algebras are completely classified by their Elliott invariant, which would imply that in principle all properties of a (classifiable) C*-algebra are somehow reflected in this invariant.

7. The seventh section contains the main results, which are possibly known to experts, but does not appear in the literature so far. First, we prove a result about the range of the Elliott invariant: Every weakly unperforated, stably finite Elliott invariant with torsion-free K_0-group is realized by a separable, stable, simple C*-algebra A in $\underline{\text{ASH}}(1)'$.

The proof is a modification of ideas due to Elliott. He proved that every weakly unperforated, stably finite Elliott invariant can be realized by an algebra in $\underline{\text{ASH}}(2)'$.

We then study the connection between the dimension and K-theory of a C*-algebra in the setting of the Elliott conjecture and we will show how the ASH-dimension can be detected quite easily by looking for torsion in the K_0-group. Precisely, if the Elliott conjecture holds for nuclear, separable, unital, stably finite, simple, $\mathcal{Z}$-stable C*-algebras, then for any such algebra A we have:

1.) $\dim_{ASH'}(A) \leq 1$ if and only if $K_0(A)$ is torsion-free

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1. Dimension theories for C*-algebras

In this section we review the definitions and basic properties of some dimension theories for C*-algebras, namely the real and (connected) stable rank, the topological dimension and the decomposition rank. This is mainly a review of known results.

Let us begin with an (informal) definition to get a feeling what one would usually want a dimension theory to satisfy:

1.1. Definition: *Let $\underline{C}$ be some class of C*-algebras with $C([0,1]) \in \underline{C}$. A (noncommutative) dimension theory for $\underline{C}$ assigns to each C*-algebra A in $\underline{C}$ a value $d(A) \in \mathbb{N} \cup \{\infty\}$ such that:*

(i) $d(I) \leq d(A)$ *whenever*[1] $I \lhd A$ *is an ideal in* A

(ii) $d(A/I) \leq d(A)$ *whenever* $I \lhd A$

(iii) $d(\lim_k A_k) \leq \underline{\lim}_k d(A_k)$ *whenever* $A = \varinjlim_k A_k$ *is a countable limit*

(iv) $d(A \oplus B) = \max\{d(A), d(B)\}$

(v) $d(C([0,1])) = 1$

1.2. Remark: This definition is not used in the literature. These axioms are all generalizations of well-known facts about commutative dimension theory: For commutative spaces the first two axioms mean that the dimension of an open (or closed) subspace is always smaller than or equal to the dimension of the containing space. (we think of ideals and quotients of a C*-algebra as the generalization of open and closed subsets). The third axiom generalizes the fact that $\dim(\varprojlim X_k) \leq \underline{\lim} \dim(X_k)$ for an inverse system of (compact) spaces. Axiom (v) is a non-degeneracy condition.

1.3. Remark: We will later see that the common dimensions (often called "rank"), are dimension theories in the sense of 1.1: The real rank and the stable rank are dimension theories on the class of all C*-algebras. The topological dimension is a dimension theory on the class of all σ-unital, type I C*-algebras. And the decomposition rank is a dimension theory on the class of separable C*-algebras.

[1] Of course, the following is assumed: For (i) we need $A/I, A \in \underline{C}$. For (ii) we need $I, A \in \underline{C}$, for (iii) $A, A_k \in \underline{C}$) and so on.

1.1. Real and (connected) stable rank.

The first generalization of classical dimension theory to non-commutative spaces was the stable rank as introduced by Rieffel in [**Rie83**]. It is a generalization of the characterization of covering dimension by fragility of maps (see A.8).

Later, Brown and Pedersen defined the real rank in a similar way (see [**BP91**]). Nevertheless the real rank does usually behaves much different from the stable rank, but in some situations the two theories are somehow dual to each other.

1.4: We begin by introducing some notation. Let A be a unital C*-algebra. Then:

$$\mathrm{Lg}_n(A) := \{(a_1, \ldots, a_n) \in A^n \ : \ \sum_{i=1}^{n} a_i^* a_i \in A^{-1}\}$$

$$\mathrm{Lg}_n(A)_{\mathrm{sa}} := \mathrm{Lg}_n(A) \cap (A_{\mathrm{sa}})^n = \{(a_1, \ldots, a_n) \in (A_{\mathrm{sa}})^n \ : \ \sum_{i=1}^{n} a_i^2 \in A^{-1}\}$$

where A^{-1} denotes the set of invertible elements of A. The abbreviation "Lg" stands for "left generators", and the reason is that a tuple $(a_1, \ldots, a_n) \in A^n$ lies in $Lg_n(A)$ if and only if $\{a_1, \ldots, a_n\}$ generate A as a left (not necessarily closed) ideal, i.e. $Aa_1 + \ldots + Aa_n = A$.

1.5. Definition: [**Rie83**, 1.4], [**BP91**]

Let A be a C-algebra. If A is unital, then the **stable rank** of A, denoted by* $\mathrm{sr}(A)$*, is the least integer $n \geq 1$ (or ∞) such that* $\mathrm{Lg}_n(A)$ *is dense in A^n. If A is non-unital, define* $\mathrm{sr}(A) := \mathrm{sr}(\widetilde{A})$.

*If A is unital, then the **real rank** of A, denoted by* $\mathrm{rr}(A)$*, is the least integer $n \geq 0$ (or ∞) such that* $\mathrm{Lg}_{n+1}(A)_{sa}$ *is dense in A_{sa}^{n+1}. If A is non-unital, define* $\mathrm{rr}(A) := \mathrm{rr}(\widetilde{A})$.

1.6. Remark: There is a subtlety of indices in the above definition. Just to make things clear, we have:

(1) $\mathrm{sr}(A) \leq n :\Leftrightarrow \mathrm{Lg}_n(\widetilde{A}) \subset (\widetilde{A})^n$ is dense
(2) $\mathrm{rr}(A) \leq n :\Leftrightarrow \mathrm{Lg}_{n+1}(\widetilde{A}) \subset (\widetilde{A}_{\mathrm{sa}})^{n+1}$ is dense

The smallest possible value of the stable rank is one, which happens exactly if the invertible elements in $\widetilde{A}$ are dense in $\widetilde{A}$. However, the smallest possible value of the real rank is zero, which analogously happens precisely if the *self-adjoint* invertible elements in $\widetilde{A}$ are dense in $\widetilde{A}_{\mathrm{sa}}$.

Let $\mathrm{Gl}_m(A)$ denote the invertible elements in $M_m(A) = A \otimes M_m$, and $\mathrm{Gl}_m(A)_0$ the connected component of $\mathrm{Gl}_m(A)$ containing the identity. Note that $\mathrm{Gl}_m(A)$ acts on A^n by multiplication (say, from the left), and this action send $\mathrm{Lg}_m(A)$ to Lg_m. It is important to know whether this action is transitive, and we define:

1.7. Definition: [**Rie83**, 4.7.]
Let A be a unital C-algebra. The* ***connected stable rank*** *of A, denoted $\mathrm{csr}(A)$, is the least integer $n \geq 1$ (or ∞) such that $\mathrm{Gl}_m(A)^0$ acts transitively on $\mathrm{Lg}_m(A)$ for all $m \geq n$. If A is not unital, define $\mathrm{csr}(A) := \mathrm{csr}(\widetilde{A})$.*

We have the following general estimates:

1.8. Proposition: [**BP91**, 1.2], [**Nis86**, 2.4], [**Rie83**, 7.2]
Let A be a C-algebra. Then:*

$$\begin{aligned} \mathrm{rr}(A) &\leq 2\,\mathrm{sr}(A) - 1 \\ \mathrm{csr}(A) &\leq \mathrm{sr}(C([0,1]) \otimes A) \\ \mathrm{sr}(C([0,1]) \otimes A) &\leq \mathrm{sr}(A) + 1 \end{aligned}$$

1.9. Remark: The connected stable rank is homotopy invariant. Even more, if A is homotopy dominated[2] by B, then $\mathrm{csr}(A) \leq \mathrm{csr}(B)$. (see [**Nis86**, Lma 2.8])

Thus, $\mathrm{csr}(C(I^k) \otimes A) = \mathrm{csr}(A)$ for any C*-algebra A (and I^k is the k-dimensional cube) (see [**Nis86**, Cor 2.9]). In particular $\mathrm{csr}(C(I^k)) = \mathrm{csr}(\mathbb{C}) = 1$, while $\mathrm{sr}(C(I^k)) > 1$ (see 2.5).

The connected stable is most interesting in relation with the stable rank. It is often used to obtain better estimates for the stable rank, for example the following:

1.10. Proposition: [**Rie83**, 4.3., 4.4., 4.11.], [**EH95**, 1.4]
Let A be a C-algebra, and $I \lhd A$ an ideal. Then:*

$$\begin{aligned} &\max\{\mathrm{sr}(I), \mathrm{sr}(A/I)\} \leq \mathrm{sr}(A) \leq \max\{\mathrm{sr}(I), \mathrm{sr}(A/I), \mathrm{csr}(A/I)\} \\ &\max\{\mathrm{rr}(I), \mathrm{rr}(A/I)\} \leq \mathrm{rr}(A) \end{aligned}$$

Note that the result for the stable rank is much better than for the real rank. It is actually a principal that the stable rank is much easier to compute than the real rank.

Note also that we can combine the two above results to get for $I \lhd A$:

$$\max\{\mathrm{sr}(I), \mathrm{sr}(A/I)\} \leq \mathrm{sr}(A) \leq \max\{\mathrm{sr}(I), \mathrm{sr}(A/I) + 1\}$$

However, the connected stable rank itself behaves much different with respect to extensions. Instead of $\mathrm{csr}(I), \mathrm{csr}(A/I) \leq \mathrm{csr}(A)$ we have the following:

$$\mathrm{csr}(A) \leq \max\{\mathrm{sr}(I), \mathrm{sr}(A/I)\}$$

We see that the connected stable rank is not a dimension theory in the sense of 1.1. (it satisfies all conditions excepts (i) and (ii)) but the real and stable rank are:

[2] This means that there are $*$-homomorphisms $\varphi : A \to B$ and $: B \to A$ such that $\circ \varphi \simeq \mathrm{id}_A$.

1.11. Proposition: *The real and stable rank are dimension theories in the sense of 1.1 for the class of all C*-algebras.*

Proof:
The difficult part is to show the estimates for ideals and quotients, which is 1.10. For the limit see [**Rie83**, 5.1.], and the same idea works for the real rank. The rest is also easy. □

The real and stable rank behave well with respect to pullbacks. This will be useful when computing the dimension of homogeneous and subhomogeneous C*-algebras.

1.12. Proposition: ***Pullbacks*** [**NOP01**, 1.6], [**BP07**, 4.1]
Let B, C, D be unital C-algebras, $\varphi : C \to D$ a surjective $*$-homomorphism, $\quad : B \to D$ a unital $*$-homomorphism. Let $A = B \oplus_D C$ be the pullback along $\quad$ and φ. Then:*

$$\begin{aligned} \mathrm{sr}(A) &\leq \max\{\mathrm{sr}(B), \mathrm{sr}(C)\} \\ \mathrm{rr}(A) &\leq \max\{\mathrm{rr}(B), \mathrm{rr}(C)\} \end{aligned}$$

1.2. Topological dimension.

Since the dual space of $C(X)$ is again X, one might wonder if it makes sense to define a dimension theory for C*-algebras simply via $\dim(\widehat{A})$ or rather $\dim(\mathrm{Prim}(A))$. This will however run into problems if the primitive ideal space is non-Hausdorff.

One way of dealing with this problem is to restrict to (nice) Hausdorff subsets of $\mathrm{Prim}(A)$, and taking the supremum over the dimension of these Hausdorff subsets. (see 1.15) This will only be a good theory if there are enough such "nice" Hausdorff subsets and we define:

1.13. Definition: [**BP07**, 2.2 (iv)]
Let X be a topological space.

1. *A subset $C \subset X$ is called **locally closed** if there is a closed set $F \subset X$ and an open set $G \subset X$ such that $C = F \cap G$*
2. *X is called **locally Hausdorff** if every non-empty closed subset F contains a non-empty relatively open subset $F \cap G$ (so $F \cap G$ is locally closed in X) which is Hausdorff*

1.14. Remark: We consider locally closed, compact subsets as "nice" subsets. Then being locally Hausdorff means having enough "nice" Hausdorff subsets.

For a C*-algebra A, the locally closed subsets of $\mathrm{Prim}(A)$ correspond to ideals of quotients of A (equivalently to quotients of ideals of A) up to canonical isomorphism. (see [**BP07**, 2.2(iii)]). Therefore, the primitive ideal space of every type I C*-algebra is almost Hausdorff, since every non-zero quotient contains a non-zero ideal with continuous trace (see [**Ped79**, 6.2.11]).

1.15. Definition: [**BP07**, 2.2(v)]
Let A be a C-algebra. If* Prim(A) *is locally Hausdorff, then the* ***topological dimension*** *of A is*

$$\operatorname{topdim}(A) = \sup_K \dim(K)$$

where the supremum runs over all locally closed, compact, Hausdorff subsets $K \subset$ Prim(A).

1.16. Remark: The topological dimension is invariant under tensoring with M_n or $\mathbb{K}$, i.e. $\operatorname{topdim}(A) = \operatorname{topdim}(M_n(A)) = \operatorname{topdim}(A \otimes \mathbb{K})$. If $A = C(X)$ is commutative, then $\operatorname{topdim}(A) = \operatorname{locdim}(X)$, where $\operatorname{locdim}(X)$ is the so-called local dimension of X, defined as the least integer $n \geq 0$ such that each point in X has a closed neighborhood of covering dimension $\leq n$ (see [**Nag70**, Def 11-6, p.61]). We always have $\operatorname{locdim}(X) \leq \dim(X)$, and if X is locally compact, Hausdorff then

$$\begin{aligned} \operatorname{locdim}(X) &= \dim(\alpha X)) \\ &= \sup_K \dim(K) \end{aligned}$$

where αX is the one-point compactification of X, and the supremum runs over all compact subsets $K \subset X$. If X is paracompact (e.g. σ-compact and locally compact, Hausdorff), then $\operatorname{locdim}(X) = \dim(X) = \operatorname{topdim}(C(X))$, and then the topological dimension does (as expected) agree with the covering dimension.

In general, for any C*-algebra A with locally Hausdorff primitive ideal space we have

$$\operatorname{topdim}(A) = \sup_F \operatorname{locdim}(F)$$

where the supremum runs over all locally closed, Hausdorff subspaces $F \subset \operatorname{Prim}(A)$. (see [**BP07**, 2.2(ii)])

We also have the following useful result, which is implicit in the papers of Brown and Pedersen, e.g. [**Bro07**, Thm 3.6].

1.17. Proposition: *Let A be a C*-algebra, and* $B \leq A$ *a hereditary sub-C*-algebra. If* Prim(A) *is locally Hausdorff, then so is* Prim(B), *and then* $\operatorname{topdim}(B) \leq \operatorname{topdim}(A)$. *If B is even full hereditary, then* $\operatorname{topdim}(B) = \operatorname{topdim}(A)$.

Proof:
By D.2 Prim(B) is homeomorphic to an open subset of Prim(A) and therefore locally Hausdorff if Prim(A) is. Also, every locally closed, compact subset $K \subset \operatorname{Prim}(B)$ is also locally closed, compact in Prim(A). It follows $\operatorname{topdim}(B) \leq \operatorname{topdim}(A)$.

If B is full, then $\operatorname{Prim}(B) \cong \operatorname{Prim}(A)$ and the statement follows. □

1.18. Proposition: [**BP07**, Prop 2.6]

Let $(I_\alpha)_{\alpha \leq \mu}$ *be a composition series for a C*-algebra A. Then* Prim(A) *is almost Hausdorff if and only each* $\operatorname{Prim}(I_{\alpha+1}/I_\alpha)$ *is almost Hausdorff, and if this is so, then:*

$$\operatorname{topdim}(A) = \sup_{\alpha<\mu} \operatorname{topdim}(I_{\alpha+1}/I_\alpha)$$

For a σ-unital, continuous-trace C*-algebra A we have the Dixmier-Douady theory. If $\operatorname{topdim}(A) < \infty$, then the bundle for $A \otimes \mathbb{K}$ has finite type. This means we can find a finite open covering, over which the bundle is trivial, which proves the following result:

1.19. Corollary: *Let A be a σ-unital, type I C*-algebra, $n \in \mathbb{N}$. Then* $\operatorname{topdim}(A) \leq n$ *if and only if A has a composition series $(I_\alpha)_{\alpha \leq \mu}$ with continuous-trace quotients of* topdim $\leq n$.

In particular, if $\operatorname{topdim}(A) < \infty$, *then A has a composition series $(I_\alpha)_{\alpha \leq \mu}$ with continuous trace quotients that are stably equivalent to a commutative C*-algebra, i.e.:*

$$(I_{\alpha+1}/I_\alpha) \otimes \mathbb{K} \cong C_0(\operatorname{Prim}(I_{\alpha+1}/I_\alpha), \mathbb{K})$$

In particular, $A \otimes \mathbb{K}$ has a composition series with quotients of the form $I_{\alpha+1}/I_\alpha \cong C_0(\operatorname{Prim}(I_{\alpha+1}/I_\alpha), \mathbb{K})$.

Let us see wether the topological dimension is a dimension theory (in the sense of 1.1). We might first extend the definition to all C*-algebras by simply setting $\operatorname{topdim}(A) = \infty$ if $\operatorname{Prim}(A)$ is not locally Hausdorff. Then it is clear from 1.18 that properties (i) and (ii) of 1.1 are fulfilled. Also, $\operatorname{topdim}(C[0,1]) = 1$.

It is however not clear if the property of having a locally Hausdorff primitive ideal space is preserved under limits. Since the topological dimension is mostly used for type I C*-algebras, the following result is still good enough:

1.20. Theorem: *The topological dimension is a dimension theory in the sense of 1.1 for σ-unital, type I C*-algebras.*

Proof:

It remains to show that topdim is compatible with countable limits (within the class of σ-unital, type I C*-algebras). Thus, let $A, A_1, A_2, \ldots$ be such algebras with $A = \varinjlim A_k$, and set $n := \underline{\lim}_k \operatorname{topdim}(A_k)$. We may assume A (and A_k) are stable, since all properties are invariant under stabilization. We want to show $\operatorname{topdim}(A) \leq n$. If $n = \infty$ there is nothing to show, so assume $n < \infty$.

By 1.19 there exists a composition series $(I_\alpha)_{\alpha \leq \mu}$ for A with continuous-trace quotients $Q_\alpha := I_{\alpha+1}/I_\alpha \cong C_0(X_\alpha, \mathbb{K})$, where $X_\alpha := \operatorname{Prim}(I_{\alpha+1}/I_\alpha)$. Then $Q_\alpha \cong \varinjlim B_\alpha^k$ where B_α^k is a quotient of an ideal of A_k. Note that $\operatorname{topdim}(B_\alpha^k) \leq \operatorname{topdim}(B_k) \leq n$.

We want to show $\operatorname{topdim}(Q_\alpha) \leq n$. Let $e_1 \in M(Q_\alpha)$ (the multiplier algebra of Q_α) be the constant function on X to a central rank-one projection. Let $H_\alpha := e_1 Q_\alpha e_1$, and $C_\alpha^k := e_1 B_\alpha^k e_1$. We get $H_\alpha = e_1 Q_\alpha e_1 = \varinjlim e_1 B_\alpha^k e_1 = \varinjlim C_\alpha^k$.

Note that $H_\alpha \cong C_0(X_\alpha)$ is commutative, and so are the C_α^k, say $C_\alpha^k \cong C_0(X_\alpha^k)$. Then $X_\alpha \cong \varprojlim X_\alpha^k$. Since H_α is a full hereditary C*-subalgebra of Q_α, we have $\operatorname{topdim}(Q_\alpha) = \operatorname{topdim}(H_\alpha) = \dim(X_\alpha)$. Similarly we get $\dim(X_\alpha^k) = \operatorname{topdim}(C_k^\alpha) \leq \operatorname{topdim}(B_k^\alpha) \leq n$. Then:

$$\begin{aligned} \operatorname{topdim}(Q_\alpha) &= \dim(X_\alpha) \\ &\leq \underline{\lim}_k \dim(X_\alpha^k) \\ &= \underline{\lim}_k \operatorname{topdim}(C_\alpha^k) \leq n \end{aligned}$$

By 1.18 we get $\mathrm{topdim}(A) \leq n$, as desired. □

Remark: It is noted in [**BP07**, Rmk 2.7 (v)] that the theorem above would follow from [**Sud04**]. However, the statement is formulated as an axiom there, and it is not clear that the formulated axioms are consistent and give a dimension theory that agrees with the topological dimension.

We also note that the topological dimension behaves well with respect to tensor products (within the class of type I C*-algebras):

$$\mathrm{topdim}(A \otimes B) = \mathrm{topdim}(A) + \mathrm{topdim}(B)$$

in particular $\mathrm{topdim}(A \otimes C([0, 1])) = \mathrm{topdim}(A) + 1$.

1.3. Decomposition rank.

The decomposition rank as introduced by Kirchberg and Winter in [**KW04**] defines a dimension theory in terms of completely positive (c.p.) approximations that fulfill a certain decomposability condition.

Recall that a C*-algebra is nuclear if it has the completely positive approximation property (CPAP):

(1) For any $b_1, \ldots, b_m \in A$, and $\varepsilon > 0$, there exists a vs-finite-dimensional[3] C*-algebra F and completely positive contractive (c.p.c.) maps $\quad : A \to F$, $\varphi : F \to A$ such that $\|\varphi \circ \quad (b_i) - bi\| < \varepsilon$ for each $i = 1, \ldots, m$.

If $\quad, \varphi$ are as above, then we say $(F, \quad, \varphi)$ is a c.p. approximation for F within ε. If we place some condition on φ in the CPAP, then we end up with the definition of decomposition rank. We need some definitions:

1.21. Definition: [**Win03**, Def 3.1]
Let A be a C-algebra, F a vs-finite-dimensional C*-algebra and $\varphi : F \to A$ a c.p. map, and $n \in \mathbb{N}$. Then the **strict order** of φ is $\leq n$, denoted by $\mathrm{ord}(\varphi) \leq n$, if for any set $\{e_0, \ldots, e_{n+1}\}$ of pairwise orthogonal, minimal projections in F there are some indices $i, j \in \{0, \ldots, n+1\}$ such that $\varphi(e_i)$ and $\varphi(e_j)$ are orthogonal[4].*

The case of strict order zero is of special interest. There are nice characterizations of this case (see [**Win03**, Prop 4.1.1(a)]).

1.22. Definition: [**KW04**, Def 2.2]
Let A be a C-algebra, and $\varphi : \oplus_{i=1}^{s} M_{r_i} \to A$ a completely positive map, and $n \in \mathbb{N}$. Then φ is n**-decomposable** if there exists a decomposition $\{1, 2, \ldots, s\} = \coprod_{j=0}^{n} I_j$ such that the restriction of φ to $\oplus_{i \in I_j}$ has strict order zero.*

[3]Recall that we call a C*-algebra "vs-finite-dimensional" if its dimension as a vector space (vs) is finite-dimensional.

[4]Two elements a, b of a C*-algebra are said to be orthogonal if $ab = a^*b = ab^* = a^*b^* = 0$

1.23. Definition: [**KW04**, Def 3.1]
Let A be a separable C-algebra. The* ***decomposition rank*** *of A, denoted by* $\mathrm{dr}(A)$, *is the smallest integer $n \geq 0$ (or ∞) such that the following holds:*

(1) For any $b_1, \ldots, b_m \in A$, and $\varepsilon > 0$, there is a c.p. approximation $(F, \ ,\varphi)$ for $b_1, \ldots, b_m$ within ε such that φ is n-decomposable.

It is shown in [**KW04**, Rmk 3.2] that the decomposition rank behaves well with respect to ideals, quotients, direct sums and limits. (as required by 1.1) and even more is invariant under tensoring with M_n or $\mathbb{K}$. We also have $\mathrm{dr}(C([0,1])) = 1$ and therefore:

1.24. Proposition: *The decomposition rank is a dimension theory in the sense of 1.1 for the class of all separable C*-algebras.*

We end this section with some questions.

1.25. Question: Does the decomposition rank and connected stable rank behave well with respect to pullbacks?

Precisely: If B, C, D are unital C*-algebras, $\varphi : C \to D$ a surjective $*$-homomorphism, $: B \to D$ a unital $*$-homomorphism. Does the following hold:

$$\mathrm{dr}(B \oplus_D C) \leq \max\{\mathrm{dr}(B), \mathrm{dr}(C)\}$$
$$\mathrm{csr}(B \oplus_D C) \leq \max\{\mathrm{csr}(B), \mathrm{csr}(C)\}$$

From 1.12 and 1.18 we see that real and stable rank as well as topological dimension enjoy this property.

2. Homogeneous and continuous-trace C*-Algebras

Homogeneous and continuous-trace C*-algebras are well-understood and in their structure pretty close to commutative C*-algebras. We will see that this makes it possible to compute their dimension theories quite explicitly. Then we will use the integral Chern character for low-dimensional spaces as constructed in the appendix to show that low (topological) dimension leads to torsion-free K-theory. Let us begin with the basic notions:

2.1. Definition: [**Fel61**, 3.2]
Let A be a C-algebra and $n \geq 1$. Then A is called n-**homogeneous** if all its irreducible representations are n-dimensional. We further say that A is **homogeneous** if it is n-homogeneous for some n.*

Let us recall a general construction: Assume $\mathfrak{B} = (E \xrightarrow{p} X)$ is a locally trivial fibre bundle (over a locally compact, Hausdorff space X) whose fiber has the structure of a C*-algebra. Let

$$\Gamma_0(\mathfrak{B}) = \{f : X \to E \, : \, p \circ f = id_X, (x \to \|f(x)\|) \in C_0(X)\}$$

be the sections of $\mathfrak{B}$ that vanish at infinity. Then $\Gamma_0(\mathfrak{B})$ has a natural structure of a C*-algebra, with the algebraic operations defined fibrewise, and norm $\|f\| := \sup_{x \in X} \|f(x)\|$.

If the bundle has fibre M_n (a so-called M_n-bundle), then $A := \Gamma_0(\mathfrak{B})$ is n-homogeneous and $\widehat{A} \cong X$. Thus, every M_n-bundle defines an n-homogeneous C*-algebra. The converse does also hold:

2.2. Proposition: [**Fel61**, Thm 3.2]
Let A be a C-algebra, $n \in \mathbb{N}$. Then the following are equivalent:*

(1) A is n-homogeneous
(2) $A \cong \Gamma_0(\mathfrak{B})$ for a locally trivial M_n-bundle $\mathfrak{B}$

If $A \cong \Gamma_0(\mathfrak{B})$, then the primitive ideal space of A is (canonically) homeomorphic to the base space of the bundle, a Hausdorff space. Thus, the primitive ideal space of a homogeneous C*-algebra is Hausdorff. Let us consider another natural class of C*-algebras with Hausdorff primitive ideal space, the so-called continuous-trace algebras. We need to briefly recall some concepts:

Recall that for each Hilbert-space H there is a trace-map $\mathrm{tr} : B(H) \to \mathbb{C}$ (which might take value ∞). It has the property that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for any $a, b \in B(H)$, in particular $\mathrm{tr}(uau^*) = \mathrm{tr}(a)$ for a unitary $u \in A$. For a representation $\pi : A \to B(H)$ we can consider the map $\mathrm{tr}_\pi := \mathrm{tr} \circ \pi : A \to \mathbb{C}$. It is well defined up to unitary equivalence, i.e. for each $[\pi] \in \widehat{A}$ we get a well-defined map $\mathrm{tr}_{[\pi]}$. (see [**Bla06**, IV.1.4.8, p.333]).

We say a that a positive element $a \in A_+$ has continuous-trace if the map $\hat{a} : \widehat{A} \to \mathbb{C}$ defined as $[\pi] \mapsto \mathrm{tr}_{[\pi]}(a) = \mathrm{tr}(\pi(a))$ is finite, bounded and continuous. We let $\mathfrak{m}_+(A)$ denote all these elements. They form the positive cone of an ideal which is denote by $\mathfrak{m}(A)$ (see [**Bla06**, IV.1.4.11, p.333]).

2.3. Definition: [**Bla06**, IV.1.4.12, p.333]
A C-algebra A is of* ***continuous-trace*** *if the ideal $\mathfrak{m}(A)$, generated by the positive elements with continuous-trace, is dense in A.*

Homogeneous C*-algebras (and their finite direct sums) are of continuous-trace. Indeed, for a homogeneous C*-algebra A we have $\mathfrak{m}_+(A) = A_+$ and $\mathfrak{m}(A) = A$. Conversely not every continuous-trace algebra is homogeneous, in fact not even $\mathfrak{m}_+(A) = A_+$ does imply that. For example $C_0(X, \mathbb{K})$ is of continuous-trace with $\mathfrak{m}_+(A) = A_+$, but A is not homogeneous. The reason is simply that all irreducible representations are infinite-dimensional. But this algebra is the section algebra of a bundle (with fibre $\mathbb{K}$), and some authors call these algebras $\aleph_0$-homogeneous.

Another example is

$$A = \{f \in C([0,1], M_2) \ : \ f(0), f(1) \in \left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset M_2$$

which is more obviously not homogeneous, but still of continuous-trace. This example should not be confused with the so-called dimension-drop algebras, which are not of continuous-trace.

2.4. Proposition: [**Bla06**, IV.1.4.18, p.335]
Let A be a C-algebra. Then the following are equivalent:*

(1) A is of continuous-trace
(2) $\widehat{A}$ is Hausdorff, and A satisfies Fell's condition, i.e. for every $\pi \in \widehat{A}$ there is a neighborhood $U \subset \widehat{A}$ of π and some $x \in A$ such that $\rho(x)$ is a rank-one projection for each $\rho \in U$.

2.1. Dimension of homogeneous and continuous-trace C*-algebras.

The easiest homogeneous C*-algebras are the commutative C*-algebras. So, to get some feeling let us first recall how non-commutative dimension theories see the commutative world.

2.5. Proposition: *(1.16),* [**Rie83**, 1.7]*,* [**Nis86**, 2.5]*,* [**BP91**, 1.1]*,* [**KW04**, 3.3]
Let $A = C_0(X)$ be a commutative C-algebra. Then:*

$$\begin{aligned} \operatorname{topdim}(A) &= \operatorname{locdim}(X) \\ \operatorname{sr}(A) &= \left\lfloor \frac{\operatorname{locdim}(X)}{2} \right\rfloor + 1 \\ \operatorname{csr}(A) &\leq \left\lfloor \frac{\operatorname{locdim}(X)+1}{2} \right\rfloor + 1 \\ \operatorname{rr}(A) &= \operatorname{locdim}(X) \end{aligned}$$

If A is separable (i.e. X second-countable), then $\operatorname{locdim}(X) = \dim(X)$ *and:*

$$\operatorname{dr}(A) = \dim(X)$$

Thus, the real rank, topological dimension and decomposition rank agree for separable, commutative C*-algebras. This is no longer the case for non-commutative homogeneous C*-algebras. Let us first consider the case $A = C_0(X, M_n)$ (say X second-countable), which is the easiest n-homogeneous C*-algebra.

We may view A as the sections (vanishing at infinity) of the trivial bundle $X \times M_n \to X$. Now things change. While still $\operatorname{topdim}(A) = \operatorname{dr}(A) = \dim(X)$, the (connected) stable rank and the real rank "break down". For the stable rank there is a general formula expressing $\operatorname{sr}(A \otimes M_n)$ in terms of $\operatorname{sr}(A)$.

2.6. Theorem: [**Rie83**, 6.1], [**Nis86**, 2.10]
Let A be a C-algebra,* $n \geq 1$*. Then:*

$$\operatorname{sr}(M_n(A)) = \left\lceil \frac{\operatorname{sr}(A) - 1}{n} \right\rceil$$

$$\operatorname{csr}(M_n(A)) \leq \left\lceil \frac{\operatorname{csr}(A) - 1}{n} \right\rceil$$

For the real rank no such general result is known (and it cannot be expected), however we can say something:

2.7. Theorem: [**BE91**, 3.2]
Let X be a compact, Hausdorff space, and $n \geq 1$*. Then:*

$$\operatorname{rr}(C(X, M_n)) = \left\lceil \frac{\dim(X)}{2n - 1} \right\rceil$$

For the stable rank we combine 2.6 and 2.5 to get:

$$\operatorname{sr}(C_0(X, M_n)) = \left\lceil \left\lfloor \frac{\operatorname{locdim}(X)}{2} \right\rfloor / n \right\rceil + 1 = \left\lceil \frac{\dim(X) + 2n - 1}{2n} \right\rceil$$

Now, by 2.2 a general n-homogeneous C*-algebra A is nothing else than the section algebra of a locally trivial M_n-bundles over $X = \widehat{A} = \operatorname{Prim}(A)$ with a possible twist (i.e. the bundle need not be globally trivial). Recall that a bundle has finite type if it can be covered by finitely many open sets over which the bundle is trivial, which is for example the case if X is compact or finite-dimensional and paracompact. In this case we can use the above results and the fact that real and stable rank behave well with respect to pullbacks (see 1.12) to get:

2.8. Proposition: *Let A be a n-homogeneous C^*-algebra with spectrum $\hat{A} = X$. Then:*

$$\operatorname{topdim}(A) = \operatorname{topdim}(C_0(X, M_n)) = \operatorname{locdim}(X)$$

If A is σ-unital (i.e. X σ-compact), then $\operatorname{locdim}(X) = \dim(X)$ *and:*

$$\begin{aligned}
\operatorname{topdim}(A) &= \operatorname{topdim}(C_0(X, M_n)) = \dim(X) \\
\operatorname{sr}(A) &= \operatorname{sr}(C_0(X, M_n)) = \left\lceil \frac{\dim(X) + 2n - 1}{2n} \right\rceil \\
\operatorname{csr}(A) &\leq \left\lceil \frac{\dim(X) + 2n}{2n} \right\rceil = \left\lceil \frac{\dim(X)}{2n} \right\rceil + 1 \\
\operatorname{rr}(A) &= \operatorname{rr}(C_0(X, M_n)) = \left\lceil \frac{\dim(X)}{2n - 1} \right\rceil
\end{aligned}$$

If A is even separable (i.e. X second-countable), then:

$$\operatorname{dr}(A) = \operatorname{dr}(C_0(X, M_n)) = \dim(X)$$

Proof:
For the topological dimension everything is clear. For the real and stable rank we use pullbacks as indicated above. For the decomposition rank see [**KW04**, 3.10.]. For the connected stable rank use 1.8. □

Thus, if $\mathfrak{B}_1, \mathfrak{B}_2$ are M_n-bundles over some (σ-compact, locally compact, Hausdorff) space X, then the considered dimension theories cannot distinguish between $\Gamma_0(\mathfrak{B}_1)$ and $\Gamma_0(\mathfrak{B}_2)$, i.e. they cannot see the twist of the M_n-bundles.

If A has continuous-trace, then $\widehat{A}$ is Hausdorff, and consequently $\operatorname{topdim}(A) = \operatorname{locdim}(\widehat{A})$ (see 1.16). Below we will see a general result computing the real and stable rank for CCR algebras (which includes the continuous-trace algebras). Therefore we only note the following:

2.9. Proposition: [**KW04**, Cor 3.10]
Let A be a separable, continuous-trace algebra with spectrum $\hat{A} = X$. Then:

$$\begin{aligned}
\operatorname{topdim}(A) &= \dim(X) \\
\operatorname{dr}(A) &= \dim(X)
\end{aligned}$$

2.2. Low dimensions and K-theory.

Let A be a σ-unital C*-algebras of continuous-trace (which includes σ-unital, homogeneous C*-algebras). We will see that $K_1(A)$ is torsion-free whenever $\mathrm{topdim}(A) \leq 2$, and that $K_0(A)$ is torsion-free whenever $\mathrm{topdim}(A) \leq 1$. For this we relate the K-theory of A to that of $\widehat{A} \cong \mathrm{Prim}(A)$.

2.10. Lemma: *Let A be a σ-unital C*-algebra of continuous-trace. If* $\mathrm{topdim}(A) \leq 2$*, then:*

$$A \otimes \mathbb{K} \cong C_0(\widehat{A}, \mathbb{K})$$

Proof:
This follows from the Dixmier-Douady theory for continuous-trace C*algebras algebras. Note that $\widehat{A}$ is paracompact if A is σ-unital. If $\dim(\widehat{A}) \leq 2$, then $H^3(\widehat{A}) = 0$. Hence all continuous trace algebras with spectrum $\widehat{A}$ are stably isomorphic, in particular stably isomorphic to the trivial one. □

This is of course well-known. It means that for spaces of dimension ≤ 2 there is no twisted K-theory. We get the following: If A is a σ-unital, continuous-trace C*-algebra with primitive ideal space X and $\mathrm{topdim}(A) = \dim(X) \leq 2$, then

$$K_*(A) \cong K_*(C_0(X))$$

Note that $K_0(C_0(X))$ is for compact X isomorphic to $K^0(X)$, and for non-compact X isomorphic to $\widetilde{K}^0(\alpha X)$. We may unify the two cases by considering the space X_+ which is the one-point compactification αX if X is non-compact, and otherwise $X_+ = X \cup \{\mathrm{pt}\}$ (one disjoint point adjoint). The basepoint is in each case the adjoined point. For $K_1(A)$ the situation is easier. Together we have:

$$K_0(A) \cong K_0(C_0(X)) \cong \widetilde{K}^0(X_+)$$
$$K_1(A) \cong K_1(C_0(X)) \cong K^1(\alpha X)$$

Next, using the integral Chern character as constructed in the appendix we can relate K-theory to cohomology.

2.11. Theorem: *Let A be a σ-unital, continuous-trace C*-algebra with primitive ideal space X and* $\mathrm{topdim}(A) \leq 2$*. Then:*

$$K_0(A) \cong \widetilde{K}^0(X_+) \cong \widetilde{H}^0(X_+) \oplus H^2(\alpha X)$$
$$K_1(A) \cong K^1(\alpha X) \cong H^1(\alpha X)$$

2.12. Corollary: *Let A be a unital, continuous-trace C*-algebra.*

(1) If $\mathrm{topdim}(A) \leq 2$*, then $K_1(A)$ is torsion-free.*
(2) If $\mathrm{topdim}(A) \leq 1$*, then $K_0(A)$ is torsion-free.*

3. Subhomogeneous C*-Algebras

Subhomogeneous C*-algebras are in some sense one step further away from commutative C*-algebras than homogeneous C*-algebras. Their primitive ideal space need not be Hausdorff, but as we will see they have a nice description as iterated pullbacks of homogeneous C*-algebras.

The dimension theories of subhomogeneous C*-algebras can be computed quite explicitly. We will in particular see that subhomogeneous C*-algebras of topological dimension ≤ 1 have stable rank one. From a more general result in the next section, we get that the K_0-group is torsion-free.

Let us begin with the basic notions:

3.1. Definition: [**Bla06**, IV.1.4.1, p.330]
Let A be a C-algebra and $n \in \mathbb{N}$. Then A is called n-**subhomogeneous** if all its irreducible representations are at most n-dimensional. We further say a C*-algebra is **subhomogeneous** if it is n-subhomogeneous for some n.*

3.2. Proposition: [**Bla06**, IV.1.4.3, p.331]
Let A be a C-algebra. Then the following are equivalent:*

(1) A is subhomogeneous
(2) A is a C-subalgebra of a (sub)homogeneous C*-algebra*
(3) A is a C-subalgebra of $C(X, M_n)$ (for some X and n)*

We introduce the following notion of writing a C*-algebra as an iterated pullback. This idea was used by Phillips for recursive subhomogeneous C*-algebras [**Phi07**] and by Pedersen (and others) for noncommutative CW-complexes [**Ped99**].

3.3. Definition: *Let A be a C*-algebra. An **iterated pullback structure** for A consists of the following data:*

(1) a finite sequence of C-algebras $A_0, A_1, \dots, A_l$ with $A_l \cong A$*
(2) for $k = 1, \dots, l$ two unital C-algebras C_k and B_k together with a surjective, unital morphism $\partial_k : C_k \to B_k$*
(3) for $k = 1, \dots, l$ attaching maps $\gamma_k : A_{k-1} \to B_k$

such that the following is a pullback (for $k = 1, \dots, l$):

$$\begin{array}{ccc} A_k & \longrightarrow & A_{k-1} \\ \downarrow & & \downarrow{\scriptstyle \gamma_k} \\ C_k & \xrightarrow{\partial_k} & B_k \end{array}$$

This is often written as $A_k = A_{k-1} \oplus_{B_k} C_k$. The iterated pullback structure is called unital if all attaching maps are unital.

*An **iterated pullback** C*-algebra is a C*-algebra together with a (fixed) iterated pullback structure for it.*

3.4. Remark: Note that an iterated pullback C*-algebra can have many different iterated pullback structures, just as a CW-complex can have different cell-decompositions. Also, the definition is fairly general. Actually every C*-algebras is (trivially) an iterated pullback algebra in this sense.

Interesting classes of C*-algebras are defined if we restrict upon the C_k and ask them to be of some special form. For example, if the pullback is unital and all C_k are required to be (trivial) homogeneous C*-algebras $C(X_k, M_{n_k})$, then we have defined the recursive subhomogeneous C*-algebras. If the C_k are required to be of the form $C(\mathbb{I}^{m_k}, M_{n_k})$ (and ∂_k is evaluation at the boundary of $\mathbb{I}^{m_k}$), then we have defined the noncommutative CW-complexes.

We will often write $A = (A_l, \ldots, A_0)$ or $A_k = A_{k-1} \oplus_{B_k} C_k$, $k = 1, \ldots, l$ for an iterated pullback C*-algebra. Note that A is unital if it has a unital iterated pullback structure.

3.1. Recursive subhomogeneous C*-algebras.

Phillip introduced these algebras in [**Phi07**]. They are iterated pullbacks of homogeneous C*-algebras, and most subhomogeneous have this form. Here come the definitions:

3.5. Definition: *(compare* [**Phi07**, Def 1.1]*)*
A ***RSH-structure*** *is an iterated pullback structure $A_k = A_{k-1} \oplus_{B_k} C_k$, $k = 1, \ldots, l$ where:*

(i) $A_0 = C(X_0)$ *for some compact space* X_0
(ii) $C_k = C(X_k, M_{n_k})$ *for some compact space* X_k
(iii) *the attaching maps* $\gamma_k : A_{k-1} \to B_k$ *are unital*

A ***recursive subhomogeneous*** *C*-algebra is a C*-algebra A together with a fixed RSH-structure for it.*

3.6. Remark: The spaces X_k are called base spaces. Since B_k is a quotient of $C_k = C(X_k, M_k)$ we have $B_k \cong C(Y_k, M_{n_k})$ for some closed subset $Y_k \subset X_k$, and ∂_k is the restriction map, i.e. a function $f \in C(X_k, M_{n_k})$ gets mapped to $\partial_k(f) = f_{|Y_k}$.

Thus, a RSH-structure is nothing else than a unital iterated pullback structure $A_k = A_{k-1} \oplus_{B_k} C_k$ with each C_k trivial homogeneous, i.e. successively attaching trivial homogeneous C*-algebras.

3.7. Remark: There is a subtlety about the (natural) definition of dimension for a recursive subhomogeneous C*-algebra. If we use the notation as in 3.5, then it is natural to consider the number $\max_k \dim(X_k)$ and we will call this the dimension of the *RSH-structure* for A.

However, a recursive subhomogeneous C*-algebra can in general have many different RSH-structure, and the dimension of two RSH-structures for the same C*-algebra do not necessarily agree. The reason for this is rather trivial:

If $A_1 = A_0 \oplus_{B_1} C_1$ is a pullback along $\partial_1 : C_1 \to B_1$ and $\gamma_1 : A_0 \to B_1$ with $B_1 \cong C(Y_1, M_{n_1})$, then we can for any compact space Z "smuggle in" the C*-algebra $C(Z, M_{n_1})$ and still obtain the same pullback: Simply define maps $\partial_1' : C_1 \oplus D \to$

$B_1 \oplus D$ as $\partial'_1 := \partial \oplus \mathrm{id}_D$, and $\gamma'_1 : A_0 \to B_1 \oplus D$ as $\gamma'_1 := (\gamma, \mathrm{ev}_x \circ \gamma)$ where $\mathrm{ev}_x : B_1 \to M_{n_1}$ is the evaluation map at some (fixed) point $x \in Y_1$ and $\mathrm{ev}_x \circ \gamma$ is the constant function on Z. Then $A_1 \cong A_0 \oplus_{B_1 \oplus D} (C_1 \oplus D)$ along ∂'_1 and γ_1.

This shows that every recursive subhomogeneous C*-algebra can be given a RSH-structures of arbitrarily high RSH-dimension (e.g. ∞).

There are two ways out of this:

(1) Consider the minimum of dimensions of all possible RSH-structures for A
(2) For a given RSH-structure (with notation from 3.5 and 3.6) we consider the number $\max_k \dim(X_k \setminus Y_k)$

In fact, in most cases (e.g. if A is separable) these two values agree and are the same as $\mathrm{topdim}(A)$.

3.8. Theorem: [**Phi07**, Thm 2.16]
Let A be a separable, unital C-algebra, and $d \in \mathbb{N}$. Then the following are equivalent:*

(1) A is subhomogeneous with $\mathrm{topdim}(A) \leq d$
(2) A has a unital iterated pullback structure $A_k = A_{k-1} \oplus_{B_k} C_k$ $(k = 1, \ldots, l)$ where each C_k is separable and homogeneous
(3) A has a RSH-structure with second countable base spaces of dimension $\leq d$

In particular, a separable, unital C-algebra with finite topological dimension is subhomogeneous if and only if it is recursive subhomogeneous.*

Proof:
The equivalence of 1. and 3. is [**Phi07**, Thm 2.16]. The implication from 3. to 2. is clear. For the implication 2. to 3. note that we may apply the theorem to each attached homogeneous C*-algebra C_k (to write it as a pullback of *trivial* homogeneous C*-alebras), and then assemble these pullbacks. □

3.9. Remark: Let A be a non-unital, separable C*-algebras. Then A is subhomogeneous if and only if the smallest unitalization $\widetilde{A}$ is subhomogeneous, and $\mathrm{topdim}(\widetilde{A}) = \mathrm{topdim}(A)$. Thus, we can use the above theorem also in non-unital situations.

The crucial point of the above theorem is that the base spaces are second countable. In fact, we can write a subhomogeneous C*-algebra as iterated extensions with homogeneous ideals. From the theory of Busby-invariant we know that every extension can be written as a pullback (over the multiplier algebra). But multiplier algebras are (almost) never separable.

The result is very useful in dimension theory (but also elsewhere), since often pullbacks can be handled much easier than extensions (see 1.12). This sometimes allows us to extend results for homogeneous C*-algebras to subhomogeneous C*-algebras.

3.2. Noncommutative CW-complexes.

Non-commutative CW-complexes as introduced in [**Ped99**] are the straightforward generalization of finite CW-complexes to the noncommutative world. They are defined in just the same way as CW-complexes, only that instead of commutative C*-algebras $C(I^m)$ we consider $C(I^m, M_{n_m})$.

Recall that by I we denote the interval $[0,1]$ and ∂I^k is the boundary of $I^k = [0,1]^k$.

3.10. Definition: [**Ped99**, Def 11.2]
*A **NCCW-structure** is an iterated pullback structure $A_k = A_{k-1} \oplus_{B_k} C_k$, $k = 1, \dots, l$ where:*

(i) *A_0 is vs-finite-dimensional*
(ii) *$C_k = C(I^k, F_k)$ for some vs-finite-dimensional C*-algebra F_k*
(iii) *$B_k = C(\partial I^k, F_k)$ and $\partial_k : C_k \to B_k$ is induced by the inclusion $\partial I^k \to I^k$*

*A **noncommutative CW-complex** (abbreviated NCCW-complex) is a C*-algebra A together with a fixed NCCW-structure for it.*

3.11. Remark: For a NCCW-structure there is only one (natural) dimension: the length of the decomposition, i.e. the number l such that $A = A_l$. In contrast to the situation for recursive subhomogeneous C*-algebras, this concept does not depend on the NCCW-structure of a noncommutative CW-complex. Further, this concept agrees with with the topological dimension topdim.

In particular all noncommutative CW-complexes have finite topological dimension, and NCCW-complexes are only a generalization of *finite* CW-complexes. A general CW-complexes can have an infinite cell-decomposition with the weak topology. This would somehow amount to taking the inverse limit of an (infinite) iterated pullback. We will not work out this idea.

Note also that NCCW-complexes are subhomogeneous, and if unital even recursive subhomogeneous by 3.8.

3.12. Remark: The zero-dimensional NCCW-complexes are exactly the vs-finite-dimensional C*-algebras. Further, a one-dimensional noncommutative CW-complex is simply a pullback

$$\begin{array}{ccc} A_1 & \longrightarrow & A_0 \\ \downarrow & & \downarrow{\scriptstyle \gamma} \\ C(I,F) & \xrightarrow{\partial} & F \oplus F \end{array}$$

where A_0 and F are finite-dimensional (γ need not be unital).

3.13: Dimension-drop C*-algebras.
An important example of a one-dimensional NCCW-complexes is the so-called **dimension-drop C*-algebra** Int_k, which for any $k \geq 1$ is defined as:

$$\mathrm{Int}_k := \{ f \in C([0,1], M_k) \ : \ f(0), f(1) \in \mathbb{C} \cdot 1_{M_k} \}$$

These algebras are subhomogeneous. Even more they are one-dimensional NCCW-complexes, since the following is a pullback diagram:

$$\begin{array}{ccc} \mathrm{Int}_k & \longrightarrow & \mathbb{C} \oplus \mathbb{C} \\ \big\downarrow & & \big\downarrow {\scriptstyle (\lambda,\mu) \mapsto (\lambda \cdot 1_{M_k}, \mu \cdot 1_{M_k})} \\ C([0,1], M_k) & \xrightarrow{\partial = (f \mapsto (f(0), f(1))} & M_k \oplus M_k \end{array}$$

where the upper horizontal map sends $f \in \mathrm{Int}_k$ to (λ, μ) if $f(0) = \lambda \cdot 1_{m_k}, f(1) = \mu \cdot 1_{m_k}$.

We may compute the K-theory of Int_k using the six-term exact sequence for the ideal $I = \{ f \in \mathrm{Int}_k \ : \ f(0) = f(1) = 0 \}$:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\mathrm{Int}_k) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ \big\uparrow & & & & \big\downarrow {\scriptstyle (a,b) \mapsto ka - kb} \\ 0 & \longleftarrow & K_1(\mathrm{Int}_k) & \longleftarrow & \mathbb{Z} \end{array}$$

which gives $K_0(\mathrm{Int}_k) \cong \mathbb{Z}$ (with positive part $\mathbb{N}$), and $K_1(\mathrm{Int}_k) \cong \mathbb{Z}_k$.

3.14: Unitalization.
Let $A = (A_l, \ldots, A_0)$ be a NNCW complex. Then A_k is unital if and only if A_{k-1} is unital and $\gamma_k : A_{k-1} \to B_k$ is a unital map. This means that $A = A_l$ is unital if and only if all A_k and γ_k are unital.

If this is not the case, then the minimal unitalization $\widetilde{A}$ is still a NCCW-complex as follows: (see [**Ped99**, Remarks after Definition 11.2]) For $k = 0, \ldots, l-1$ define:

(1) $B_k = A_k$ if both A_k and $\gamma_{k+1} : A_k \to A_{k+1}$ are unital[5]
(2) $B_k = \widetilde{A_k}$ if A_k is non-unital
(3) $B_k = A_k \oplus \mathbb{C}$ (forced unitalization) if A_k is unital, but $\gamma_{k+1} : A_k \to B_{k+1}$ is not unital

In the cases where $B_k \neq A_k$, the new linking morphism is the obvious unital extension of the old linking morphism (which is defined since B_{k+1} is unital). At the end, set $B_l = \widetilde{A_l}$. Now $(B_l, \ldots, B_0)$ is a NCCW-structure for $\widetilde{A}$.

The following result makes one-dimensional NCCW-complexes very useful:

3.15. Theorem: [**ELP98**, Thm 6.2.2]
Every NCCW-complex of dimension ≤ 1 *is semiprojective.*

Proof:
If A is zero-dimensional, then it is vs-finite-dimensional, and therefore semiprojective. If A is one-dimensional, then [**ELP98**] proves the unital case. If A is a non-unital, then $\widetilde{A}$ is a unital, one-dimensional NCCW-complex by 3.14, and hence is semiprojective. By [**Lor97**, 14.1.7, p.108] a C*-algebra B is semiprojective if and

[5] Note that there is a small error of indices in the construction of the initialization of a NCCW complex in [**Ped99**] and [**ELP99**]

only if $\widetilde{B}$ is semiprojective. Hence A is semiprojective. □

3.16. Remark: Note that already two-dimensional commutative C*-algebras are not semiprojective. A unital, commutative C*-algebra $C(X)$ is semiprojective in the class of unital, commutative C*-algebras if (and only if) X is an absolute neighborhood retract (ANR), e.g. a polyhedron. But $C(X)$ is only semiprojective in the class of all C*-algebras if X is a one-dimensional ANR.

3.3. Dimension of subhomogeneous C*-algebras.

For a subhomogeneous C*-algebra A we consider the homogeneous parts $\mathrm{Prim}_k(A) \subset \mathrm{Prim}(A)$ consisting of all primitive ideal correpsonding to irreducible representations of dimension k. Since each $\mathrm{Prim}_k(A)$ is locally closed in $\mathrm{Prim}(A)$, there is a C*-algebras A_k corresponding to $\mathrm{Prim}_k(A)$, and this algebra is n-homogeneous. We will see that the dimension theories cannot distinguish A from $\oplus_k A_k$, i.e. they do not see the "faults" in a subhomogeneous C*-algebra.

3.17. Proposition: [**Bro07**, 3.4., 3.9.], [**Win04**, 1.6]
Let A be a subhomogeneous C-algebra. Then:*

$$\begin{aligned}
\mathrm{topdim}(A) &= \max_k \ \mathrm{topdim}(A_k) = \max_k \mathrm{locdim}(\mathrm{Prim}_k(A)) \\
\mathrm{sr}(A) &= \max_k \ \mathrm{sr}(A_k) = \max_k \ \left\lceil \frac{\mathrm{locdim}(\mathrm{Prim}_k(A)) + 2k - 1}{2k} \right\rceil \\
\mathrm{rr}(A) &= \max_k \ \mathrm{rr}(A_k) = \max_k \ \left\lceil \frac{\mathrm{locdim}(\mathrm{Prim}_k(A))}{2k - 1} \right\rceil
\end{aligned}$$

If A is also separable, then

$$\mathrm{dr}(A) = \max_k \ \mathrm{dr}(A_k) = \max_k \ \mathrm{topdim}(A_k) = \mathrm{topdim}(A)$$

Proof:
The result for real and stable rank are [**Bro07**, 3.4., 3.9.] and in the separable case they can be obtained by putting together 3.5, 1.12 and 2.8. For the topological dimension we use 1.18, and the result for the decomposition rank is [**Win04**, 1.6]. □

We may draw the following corollary:

3.18. Corollary: *Let A be a separable, subhomogeneous C*-algebra. Then the following are equivalent:*
(1) $\mathrm{topdim}(A) = 0$
(2) $\mathrm{rr}(A) = 0$
Also, the following are equivalent:
(1) $\mathrm{topdim}(A) \leq 1$
(2) $\mathrm{sr}(A) = 1$

4. Type *I* C*-algebras

In this section we take a look at dimension theories (and the connection to K-theory) of general type *I* C*-algebras. Since these algebras can be fairly complicated, the statements cannot be as explicit as for subhomogeneous C*-algebras. Still, for CCR algebras there is an almost complete picture. Using composition series we can draw some nice conclusions for general type *I* C*-algebras. Then, we will take a closer look at the low-dimensional case and the connections to K-theory.

First, we want to compare the values of different dimension theories. The main ingredient is the following result:

4.1. Theorem: [**Bro07**, 3.10]
Let A be a CCR C-algebra, and* $\operatorname{topdim}(A)<\infty$. *Then:*

(1) *If* $\operatorname{topdim}(A)\leq 1$, *then* $\operatorname{sr}(A)=1$
(2) *If* $\operatorname{topdim}(A)>1$, *then* $\operatorname{sr}(A)=\sup_{k\geq 1}\max\{\left\lceil\frac{\operatorname{topdim}(A_k)+2k-1}{2k}\right\rceil,2\}$
(3) *If* $\operatorname{topdim}(A)=0$, *then* $\operatorname{rr}(A)=0$
(4) *If* $\operatorname{topdim}(A)>0$, *then* $\operatorname{rr}(A)=\sup_{k\geq 1}\max\{\left\lceil\frac{\operatorname{topdim}(A_k)}{2k-1}\right\rceil,1\}$

We may draw the following conclusion:

4.2. Corollary: *Let A be a CCR C*-algebra. Then:*

(1) $\operatorname{sr}(A)\leq\left\lfloor\frac{\operatorname{topdim}(A)}{2}\right\rfloor+1$
(2) $\operatorname{csr}(A)\leq\left\lfloor\frac{\operatorname{topdim}(A)+1}{2}\right\rfloor+1$
(3) $\operatorname{rr}(A)\leq\operatorname{topdim}(A)$

Proof:
If $\operatorname{topdim}(A)=\infty$, then the statements hold. So we may assume $\operatorname{topdim}(A)<\infty$, whence 4.1 applies.
1. If $\operatorname{topdim}(A)\leq 1$, then $\operatorname{sr}(A)=1\leq\lfloor\operatorname{topdim}(A)/2\rfloor+1$. If $d:=\operatorname{topdim}(A)\geq 2$, then set $d_k:=\operatorname{topdim}(A_k)\leq d$ and compute:

$$\begin{aligned}
\operatorname{sr}(A)&=\sup_k\max(\left\lceil\frac{d_k+2k-1}{2k}\right\rceil,2)\\
&\leq\sup_k\max(\left\lceil\frac{d+2k-1}{2k}\right\rceil,2)\\
&\leq\max(\left\lceil\frac{d+1}{2}\right\rceil,2)\\
&\leq\max(\left\lfloor\frac{d}{2}\right\rfloor+1,2) && [\text{ since } d\in\mathbb{N}\ ]\\
&=\left\lfloor\frac{d}{2}\right\rfloor+1
\end{aligned}$$

2. The statement for the connected stable rank follows from 1.8.

3. Again, we use 4.1. If $\text{topdim}(A) = 0$, then $\text{rr}(A) = 0 \leq \text{topdim}(A)$. If $d := \text{topdim}(A) \geq 1$, then set $d_k := \text{topdim}(A_k) \leq d$ and compute:

$$\begin{aligned}
\text{rr}(A) &= \sup_k \max(\left\lceil \frac{d_k}{2k-1} \right\rceil, 1) \\
&\leq \sup_k \max(\left\lceil \frac{d}{2k-1} \right\rceil, 1) \\
&\leq \max(\lceil d \rceil, 1) \\
&\leq d
\end{aligned}$$

□

We can now give general comparison results for type I C*-algebras. It seems that these results do not appear in the literature so far. But first some remark is in order, that might also be of general interest.

4.3. Remark: What makes type I C*-algebra so accessible is the presence of composition series with quotients that are easier to handle (e.g. continuous-trace or CCR). They allow us to prove statements by transfinite induction, for which one has to consider the case of a successor and limit ordinal. Let us see that for statements about dimension theories one only needs to consider successor ordinals.

Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for a C*-algebra. If α is a limit ordinal, then $J_\alpha = [\bigcup_{\gamma<\alpha} J_\gamma]^= = \varinjlim_{\gamma<\alpha} J_\gamma$. For any dimension theory rk we have

$$\begin{aligned}
\text{rk}(J_\alpha) &\leq \varliminf_{\gamma<\alpha} \text{rk}(J_\gamma) \\
&\leq \sup_{\gamma<\alpha} \text{rk}(J_\gamma) \\
&\leq \sup_{\gamma<\alpha} \text{rk}(J_\alpha) && [\text{ since } J_\gamma \lhd J_\alpha \,] \\
&\leq \text{rk}(J_\alpha)
\end{aligned}$$

and thus $\text{rk}(J_\alpha) = \varinjlim_{\gamma<\alpha} \text{rk}(J_\gamma) = \sup_{\gamma<\alpha} \text{rk}(J_\gamma)$.

Then any reasonable estimate about dimension theories that holds for $\gamma < \alpha$ will also hold for α. So, we need only to consider a successor ordinal α, in which case $A = J_\alpha$ is an extension of $B = J_\alpha / J_{\alpha-1}$ by $I = J_{\alpha-1}$. By assumption the result is true for I and has to be proved for A (by using that B is CCR or continuous-trace). This idea is used to prove the next theorem:

4.4. Theorem: *Let A be a type I C*-algebra. Then:*

$$\begin{aligned}
\text{sr}(A) &\leq \left\lfloor \frac{\text{topdim}(A)+1}{2} \right\rfloor + 1 \\
\text{csr}(A) &\leq \left\lfloor \frac{\text{topdim}(A)}{2} \right\rfloor + 2 \\
\text{rr}(A) &\leq \text{topdim}(A) + 2
\end{aligned}$$

If A is also separable, then:

$$\text{topdim}(A) \leq \text{dr}(A)$$

Proof:
Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for A with CCR quotients. Set $d := \text{topdim}(A)$.

1. By 4.3 it is enough to consider short exact sequences of C*-algebras

$$0 \to I \to A \to B \to 0$$

where the quotient B is CCR. By assuming that the inequality is valid for I we show that it holds for A. We have $\mathrm{topdim}(I) \leq d$ and $\mathrm{topdim}(B) \leq d$, and then:

$$\begin{aligned}
\mathrm{sr}(A) &\leq \max\{\mathrm{sr}(I), \mathrm{sr}(B), \mathrm{csr}(B)\} && \text{[by 1.10]} \\
&\leq \max\{\left\lfloor \frac{d+1}{2} \right\rfloor + 1, \left\lfloor \frac{d}{2} \right\rfloor + 1, \left\lfloor \frac{d+1}{2} \right\rfloor + 1\} && \text{[by assumption and 4.2]} \\
&= \left\lfloor \frac{d+1}{2} \right\rfloor + 1
\end{aligned}$$

2. The statement for the connected stable rank and real rank follow from 1.8.
3. Analogously to the stable rank case it is enough to consider a short exact sequences

$$0 \to I \to A \to B \to 0$$

where the quotient B is of continuous-trace. By assuming that the inequality is valid for I we show that it holds for A.

$$\begin{aligned}
\mathrm{topdim}(B) &= \max(\mathrm{topdim}(I), \mathrm{topdim}(Q)) && \text{[by 1.18]} \\
&\leq \max(\mathrm{dr}(I), \mathrm{dr}(Q)) && \text{[by assumption and 2.9]} \\
&\leq \mathrm{dr}(A)
\end{aligned}$$

□

4.5. Remark: We get the following:

(1) $\mathrm{sr}(A) = \left\lfloor \frac{\mathrm{topdim}(A)}{2} \right\rfloor + 1$ if A is commutative
(2) $\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)}{2} \right\rfloor + 1$ if A is CCR
(3) $\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)+1}{2} \right\rfloor + 1$ if A is type I

This also shows that the inequality for the stable rank in 4.2 can not be improved (the same is true for the estimates of real rank and connected stable rank).

Let us see that the estimate of 4.4 for the stable rank can not be improved either: The Toeplitz algebra $\mathcal{T}$ is an extension of $C(S^1)$ by $\mathbb{K}$ and therefore $\mathrm{topdim}(\mathcal{T}) = \max(\mathrm{topdim}(\mathbb{K}), \mathrm{topdim}(C(S^1))) = \max(0, 1) = 1$. But $\mathrm{sr}(\mathcal{T}) = 2$ by [**Rie83**, 4.13]. Thus

$$2 = \mathrm{sr}(\mathcal{T}) \not\leq \left\lfloor \frac{\mathrm{topdim}(\mathcal{T})}{2} \right\rfloor + 1 = 1$$

4.1. Low dimensions and K-theory.

It is shown by Lin in [**Lin97**] that the case of dimension zero agrees for different dimension theories: A separable type I C*-algebra has $\mathrm{rr}(A) = 0$ if and only if $\mathrm{topdim}(A) = 0$, which in turn happens if and only if $\mathrm{dr}(A) = 0$ (and also if and only if A is approximately finite, which can be thought of as a statement about the ASH-dimension as introduced in the next section). Thus, there is a natural concept of "dimension zero" for type I C*-algebras.

This is less clear the concept "dimension one" of (separable) type I C*-algebra A. From 4.4 we get that $\mathrm{topdim}(A) \leq 1$ implies $\mathrm{sr}(A) \leq 2$, $\mathrm{csr}(A) \leq 2$ and $\mathrm{rr}(A) \leq 3$. Also, in [**BP07**, 5.2] it is shown that $\mathrm{topdim}(A) \leq 1$ if and only if A has generalized stable rank one (see [**BP07**, 2.1.(iv)]), i.e. a composition series with quotients of stable rank one.

But such an algebra need not have stable rank one. The easiest example is the Toeplitz algebra, which has $\mathrm{topdim}(\mathcal{T}) = 1$ but $\mathrm{sr}(\mathcal{T}) = 2$ (see 4.5). This algebra is not residually stable finite, and we will show that this is the only obstruction for stable rank one (in the separable case at least), a result which seems not to be in the literature so far. For type I algebras this shows that $\mathrm{dr}(A) \leq 1$ implies $\mathrm{sr}(A) = 1$, and we will also see that stable rank one implies that $K_0(A)$ is torsion-free.

We will call a C*-algebra "unitalized stably finite" if its unitalization is stably finite. The reason is that there is no good notion of stably finiteness for C*-algebras that are not stably unital (see 6.4). Also, we will say a C*-algebra is "unitalized residually stably finite" if its unitalization is residually stably finite, i.e. all its quotients are stably finite.

Before we turn to the case of dimension one, let us recall for completeness the result for dimension zero:

4.6. Proposition: *Let A be a separable, type I C*-algebra. The the following are equivalent:*

- *(1)* $\mathrm{topdim}(A) = 0$
- *(2)* $\mathrm{rr}(A) = 0$
- *(3) A has generalized real rank zero, i.e. a composition series with quotients of real rank zero*
- *(4) A is AF*
- *(5)* $\mathrm{dr}(A) = 0$

Proof:
The equivalence of being AF and $\mathrm{rr}(A) = 0$ (as well as $d(A) = 0$ for some dimension concept equivalent to topdim in dimension zero) was shown by Lin in [**Lin97**]. The equivalence of being AF and $\mathrm{dr}(A) = 0$ holds in general and is shown in [**KW04**, 4.1.] The equivalence of $\mathrm{rr}(A) = 0$ and generalized real rank zero is [**BP07**, 5.1], where it is also shown that some of the statements hold for general type I C*-algebras. □

Now, we turn to "one-dimensional" type I C*-algebras. We begin by recalling two facts with striking similarity. In both cases the vanishing of a K-theoretic obstruction implies that some property of an extension algebra is already determined by the behaviour of the ideal and quotient:

4.7. Proposition: [**Spi88**, Lma 1.5], [**Nis87**, Lma 3]
Let A be a C^*-algebra, $I \lhd A$ an ideal. Consider the index map $\delta : K_1(A/I) \to K_0(I)$. Denote the image of δ by $\operatorname{im}\delta$.

1. *Assume I and A/I are unitalized stably finite Then A is unitalized stably finite if and only if $\operatorname{im}\delta \cap K_0(I)^+ = \{0\}$*
2. *Assume I and A/I have stable rank one.*
   Then A has stable rank one if and only if $\delta = 0$

If $K_0(I)$ is totally ordered (e.g. $= \mathbb{Z}$ with the usual ordering), then $\operatorname{im}\delta \cap K_0(I)^+ = \{0\}$ already implies $\delta = 0$. This has the following consequence: If A is stably finite, I and A/I have stable rank one, and $K_0(I) \cong \mathbb{Z}$, then δ vanishes (and hence $\operatorname{sr}(A) = 1$).

The following lemma is based on this observation. Then, in 4.9 we will give sufficient conditions for the vanishing of the index map in a more general setting, a result which seems not to appear in the literature so far.

4.8. Lemma: *Let A be a unitalized residually stably finite C^*-algebra, and $I \lhd A$ an ideal that is separable, of continuous-trace and with* $\operatorname{topdim}(I) \leq 1$. *Then the index map $\delta : K_1(A/I) \to K_0(I)$ is zero.*

Proof:
Set $B := A/I$ and $X := \widehat{I}$. We may assume that I, A, B are stable, in which case $I \cong C_0(X, \mathbb{K})$. Then $K_0(I) \cong \widetilde{H}^0(X_+) \cong [X_+; \mathbb{Z}]_*$. Thus, we may view elements in $K_0(I)$ as maps $f : X \to \mathbb{Z}$ that vanish at infinity. Such a map will be in $K_0(I)_+$ if and only if $f \geq 0$, i.e. $f(x) \geq 0$ for all $x \in X$.

Let $x \in K_1(B)$ be some element, fixed from now on. Set $f := \delta(x) \in K_0(I)$. We need to show that $f = 0$, i.e. $f(x) = 0$ for all $x \in X$.

So let $x \in X$ be some (fixed) point, and $C \subset X$ the connected component of x. Then C is closed and we may consider the ideal $J \lhd I$ corresponding to the open set $\widehat{I} \setminus C$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I/J & \longrightarrow & A/J & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

It gives an index map $\delta' : K_1(B) \to K_0(A/J)$ for the lower extension. Together we get the following commutative diagram in K-theory:

$$\begin{array}{ccccc} K_1(B) & \xrightarrow{\delta} & K_0(I) & = & [X_+; \mathbb{Z}]_* \\ \downarrow{\scriptstyle \delta'} & & \downarrow & & \\ K_0(I/J) & = & K_0(I/J) & = & [C_+; \mathbb{Z}]_* \end{array}$$

Now, A/J is residually finite and $K_0(I/J)$ is totally ordered (it is $\mathbb{Z}$ if $\operatorname{Prim}(I/J) \cong C$ is compact, and 0 otherwise). By the above remark $\delta' = 0$, and therefore $f(x) = 0$. Since this holds for all x, we have $f = 0$ as desired. $\square$

4.9. Theorem: *Let A be a unitalized residually stably finite C*-algebra, and $I \lhd A$ an ideal that is separable, of type I and with* $\operatorname{topdim}(I) \leq 1$. *Then the index map $\delta : K_1(A/I) \to K_0(I)$ is zero.*

Proof:

We may assume A is unital, and set $B := A/I$. Let $x \in K_0(B)$ be some element, fixed from now on. Set $y := \delta(x) \in K_1(I)$. We need to show that $y = 0$.

Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for I with continuous-trace quotients with (one-dimensional) spectra. For each α we have a ”sub-extension”

$$0 \to I/J_\alpha \to A/J_\alpha \to B \to 0$$

with index map $\delta_\alpha : K_1(B) \to K_0(I/J_\alpha)$ and we may ask if $\delta_\alpha(x) = 0$.

Let $\iota_\alpha : J_\alpha \to I$ be the inclusion map, and $q_\alpha : I \to I/J_\alpha$ the quotient map, then we have extensions:

$$0 \to J_\alpha \xrightarrow{\iota_\alpha} I \xrightarrow{q_\alpha} I/J_\alpha \to 0$$

Note that $\delta_\alpha = K_0(q_\alpha) \circ \delta$, and define

$$\begin{aligned} \beta &:= \inf\{\alpha \leq \mu \ : \ \delta_\alpha(x) = 0\} \\ &:= \inf\{\alpha \leq \mu \ : \ K_0(q_\alpha)(y) = 0\} \end{aligned}$$

We need to show that $\beta = 0$.

Step 1: β is not a limit ordinal. Assume otherwise, in which case the continuity of K-theory gives:

$$K_0(J_\beta) = \varinjlim_{\gamma < \beta} K_0(J_\gamma)$$

Let $\iota_{\beta\gamma} : J_\gamma \to J_\beta$ be the inclusion map. The situation is viewed in the following commutative diagram with exact rows:

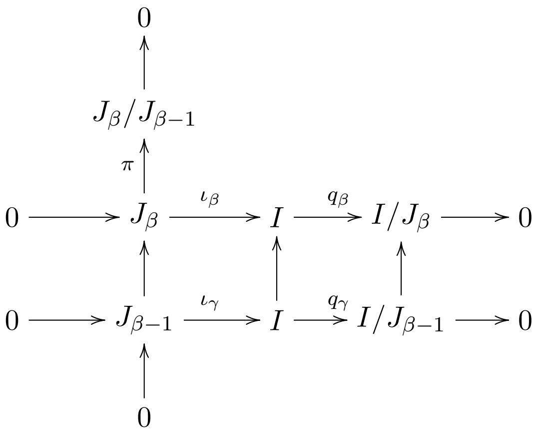
$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_\beta & \xrightarrow{\iota_\beta} & I & \xrightarrow{q_\beta} & I/J_\beta & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \iota_{\beta\gamma}} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & J_\gamma & \xrightarrow{\iota_\gamma} & I & \xrightarrow{q_\gamma} & I/J_\gamma & \longrightarrow & 0 \end{array}$$

In K-theory we get the following commutative diagram (with exact rows):

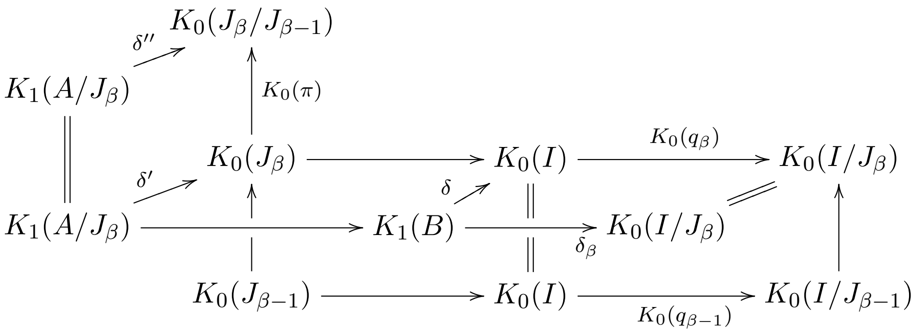
$$\begin{array}{ccccc} K_0(J_\beta) & \xrightarrow{K_0(\iota_\beta)} & K_0(I) & \xrightarrow{q_\beta} & K_0(I/J_\beta) \\ \uparrow{\scriptstyle K_0(\iota_{\beta\gamma})} & & \uparrow & & \uparrow \\ K_0(J_\gamma) & \xrightarrow{K_0(\iota_\gamma)} & K_0(I) & \xrightarrow{K_0(q_\gamma)} & K_0(I/J_\gamma) \end{array}$$

We have $K_0(q_\beta)(y) = 0$ by definition of β. From exactness of the upper row we may find a lift $y' \in K_0(J_\beta)$ for y. By continuity of K-theory we may lift y' to some $y'' \in K_0(J_\gamma)$ (for some $\gamma < \beta$). Then $K_0(\iota_\gamma)(y'') = K_0(\iota_\beta)(K_0(\iota_{\beta\gamma}))(y'') = K_0(\iota_\beta)(y') = y$. The exactness of the lower row shows $K_0(q_\gamma)(y) = 0$, which contradicts the definition of β.

Step 2: β has no predecessor $\beta - 1$. Assume otherwise, in which case we have the following commutative diagram with exact rows and columns:



This gives a commutative diagram in K-theory, and we add index maps for several different extension to the picture:



Since $\delta_\beta(x) = 0 \in K_0(I/J_\beta)$, we may lift x to $x' \in K_1(A/J_\beta)$. Set $y' := \delta'(x') \in K_0(J_\beta)$, which is a lift of $y = \delta(X) \in K_0(I)$.

Consider the extension $0 \to J_\beta/J_{\beta-1} \to A/J_{\beta-1} \to A/J_\beta \to 0$. We have $A/J_{\beta-1}$ (residually) stably finite, and $J_\beta/J_{\beta-1}$ of continuous-trace. Therefore, its index map δ'' is zero by 4.8.

It follows that $K_0(\pi)(y') = 0$, and we can lift y' to $y'' \in K_0(J_{\beta-1})$. But then $K_0(q_{\beta-1})(y) = 0$, which contradicts the definition of β. $\square$

4.10. Corollary: *Let A be a unitalized residually stably finite C*-algebra, and $I \lhd A$ an ideal that is separable and of type I. Assume* $\mathrm{sr}(I) = \mathrm{sr}(A/I) = 1$. *Then* $\mathrm{sr}(A) = 1$.

4.11. Theorem: *Let A be a separable, type 1 C*-algebra. Then the following are equivalent:*

(1) $\mathrm{sr}(A) = 1$
(2) A is unitalized residually stably finite, and $\mathrm{topdim}(A) \leq 1$

Proof:
1) $\Rightarrow$ 2) We have $\mathrm{topdim}(A) \leq 1$ by [**BP07**, Proposition 5.2], and every quotient of A has stable rank one, hence is unitalized stably finite.
2) $\Rightarrow$ 1) Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for I with continuous-trace quotients with one-dimensional spectra. By transfinite induction over $\alpha \leq \mu$ we show that $\mathrm{sr}(J_\alpha) = 1$.

If K is any separable C*-algebra of continuous-trace with $\mathrm{topdim}(K) \leq 1$, then $\mathrm{sr}(K) = \mathrm{sr}(K \otimes \mathbb{K}) = \mathrm{sr}(C_0(\widehat{K}, \mathbb{K})) = \mathrm{sr}(C_0(\widehat{K})) = 1$. In particular $\mathrm{sr}(J_1) = 1$ and

$\mathrm{sr}(J_{\alpha+1}/J_\alpha) = 1$ for every $\alpha < \mu$. Assume now, $\mathrm{sr}(J_\alpha) = 1$ for all $\alpha < \beta$. If β is a limit ordinal, then $J_\beta = \varinjlim_{\alpha<\beta} J_\alpha$, and consequently $\mathrm{sr}(J_\beta) \leq \underline{\lim}_{\alpha<\beta} \mathrm{sr}(J_\alpha) = 1$. If β has a predecessor $\beta - 1$, then J_β is a residually stably finite extension of $J_{\beta-1}$ and $J_\beta/J_{\beta-1}$, which both have stable rank one. Then $\mathrm{sr}(J_\beta) = 1$ by 4.10. □

Let us see that we cannot drop the word "residually" in the above theorem:

4.12. Example: Let S be the unilateral shift (on a separable, infinite-dimensional Hilbert space H). Let $A := C^*(S^* \oplus S)$ be the C*-subalgebra of $B(H \oplus H)$ generated by the operator $S^* \oplus S$ (just as the Toeplitz algebra is $\mathcal{T} = C^*(S)$). Then A is a stably finite, separable, type I C*-algebra A (see [**Bla06**, V.4.2.4.(iii)]) which fits into an extension

$$0 \to \mathbb{K} \oplus \mathbb{K} \to A \to C(S^1) \to 0$$

Hence $\mathrm{topdim}(A) = 1$, and therefore $\mathrm{sr}(A) \leq 2$. But the map $S^* \oplus S \to S$ induces a surjection $A \to \mathcal{T}$. Therefore $\mathrm{sr}(A) \geq \mathrm{sr}(\mathcal{T}) = 2$. The point is that A is not *residually* stably finite, since it has a quotient isomorphic to $\mathcal{T}$.

It is shown in [**KW04**, 5.3] that any separable C*-algebra with finite decomposition rank is strongly quasidiagonal[6]. Quasidiagonal C*-algebras are unitalized stably finite by [**Bla06**, V.4.1.10, p.459]. In particular a separable, type I C*-algebra with $\mathrm{dr}(A) \leq 1$ is unitalized residually stably finite and also $\mathrm{topdim}(A) \leq 1$ by 4.4. Therefore:

4.13. Corollary: *Let A be a separable, type I C*-algebra with* $\mathrm{dr}(A) \leq 1$. *Then* $\mathrm{sr}(A) = 1$ *(and consequently* $\mathrm{rr}(A) \leq 1$*).*

Now, let us show that stable rank one implies torsion-free K_0-group (at least in the σ-unital case), a result which seems not to be in the literature so far.

4.14. Theorem: *Let A be a separable, type I C*-algebra with* $\mathrm{sr}(A) = 1$. *Then* $K_0(A)$ *is torsion-free.*

Proof:
Let $(J_\alpha)_{\alpha\leq\mu}$ be a composition series for A with continuous-trace quotients. By transfinite induction over $\alpha \leq \mu$ we show that $K_0(J_\alpha)$ is torsion-free.

If Q is any separable C*-algebra of continuous-trace with $\mathrm{topdim}(Q) \leq 1$, then $K_0(Q)$ is torsion-free by 2.12. In particular $K_0(J_1)$ and $K_0(J_{\alpha+1}/J_\alpha)$ are torsion-free for every $\alpha \leq \mu$. Assume now, $K_0(J_\alpha)$ is torsion-free for all $\alpha < \beta$. If β is a limit ordinal, then $J_\beta = \varinjlim_{\alpha<\beta} J_\alpha$, and consequently $K_0(J_\beta) = \varinjlim_{\alpha<\beta} K_0(J_\alpha)$ is torsion-free.

If β has a predecessor $\beta - 1$, then J_β is an extension of $J_{\beta-1}$ and $J_\beta/J_{\beta-1}$, and $\mathrm{sr}(J_\beta) = 1$. Then the index map $\delta : K_1(J_\beta/J_{\beta-1}) \to K_0(J_{\beta-1})$ vanishes by 4.7. The six-term exact sequence in K-theory for this extension is:

[6] A C*-algebra A is stongly quasidiagonal if every irreducible representation of A is quasidiagonal. A representation $\pi : A \to B(H))$ is quasdidiagonal, if $\pi(A) \subset B(H)$ is quasidiagonal, and this is the case if every operator $T \in \pi(A)$ is block-diagonal up to the compact operators.

$$\begin{array}{ccccc} K_0(J_{\beta-1}) & \xrightarrow{f} & K_0(J_\beta) & \xrightarrow{g} & K_0(J_\beta/J_{\beta-1}) \\ {\scriptstyle 0}\uparrow & & & & \downarrow \\ K_1(J_\beta/J_{\beta-1}) & \longleftarrow & K_1(J_\beta) & \longleftarrow & K_1(J_{\beta-1}) \end{array}$$

Let H denote the image of g, i.e. the subgroup of $K_0(J_\beta/J_{\beta-1})$ that lies in the kernel of the index map from K_0 to K_1. Note that H is torsion-free (see C.2). Then $K_0(J_\beta)$ is an extension of two torsion-free groups, namely $K_0(J_{\beta-1})$ and H, and therefore torsion-free itself. □

We end this section with some questions.

4.15. Question: Can the estimate of 4.4 for the real rank be improved?

In particular, is the following true for type I C*-algebras:

$$\operatorname{rr}(A) \leq \operatorname{topdim}(A) + 1$$

Note, that we can prove

$$\operatorname{sr}(\mathbb{I}A) \leq \left\lfloor \frac{\operatorname{topdim}(\mathbb{I}A)}{2} \right\rfloor + 1$$

along the lines of 4.4 using $\operatorname{csr}(\mathbb{I}^k A) = \operatorname{csr}(A)$ ([**Nis86**, 2.8., 2.9]). (Recall that we use the notation $\mathbb{I}B = C([0,1]) \otimes B$). More generally $\operatorname{sr}(\mathbb{I}^k A) \leq \left\lfloor \frac{\operatorname{topdim}(\mathbb{I}^k A)}{2} \right\rfloor + 1$ for $k \geq 1$. Then $\operatorname{rr}(\mathbb{I}^k A) \leq \operatorname{topdim}(A) + 1 + k$ for $k \geq 1$, and the question is whether this also holds for $k = 0$.

4.16. Question: Does the following hold for separable, type I C*-algebras:

$$\operatorname{rr}(A) \leq \operatorname{dr}(A)$$

For the cases $\operatorname{dr}(A) = 0, 1$ it is true by 4.6 and 4.13.

It is possible that the inequality is even true for all (separable) C*-algebras (see [**Win03**, 5.1.5.] and [**KW04**, 4.11.]).

4.17. Question: Can 4.13 be generalized to all separable C*-algebras? That is, does every separable C*-algebra with $\operatorname{dr}(A) \leq 1$ have stable rank one?

Note that in [**Win08**, 5.1.] it is shown that a simple, separable, unital C*-algebra with finite decomposition rank is $\mathcal{Z}$-stable. This in turn implies that $\operatorname{sr}(A) = 1$ by [**Rr04**].

4.18. Question: Does the converse of 4.13 hold? That is, does a separable, type I C*-algebra with stable rank one already have $\operatorname{dr}(A) \leq 1$?

There are related questions: Is a separable, type 1 C*-algebra with stable rank one already strongly quasidiagonal? Maybe, it can be even approximated by one-dimensional subhomogeneous C*-algebras.

5. Approximately (sub)homogeneous C*-algebras

In this section we introduce approximately (sub)homogeneous C*-algebras, and show how they can be used to define the concept of AH- and ASH-dimension.

If $\mathcal{P}$ is some property that C*-algebras might enjoy, then one can consider the class of C*-algebras that are direct limits of C*-algebras with the given property. These limit algebras are often called "approximately $\mathcal{P}$" (or similar).

Next, recall that a class $\mathcal{A}$ of sub-C*-algebras are said to locally approximate A if for every finite subset $F \subset A$ and $\varepsilon > 0$ there exists some algebra $B \in \mathcal{A}$ such that $F \subset_\varepsilon B$. One can consider the class of C*-algebras that can be locally approximated by sub-C*-algebras with a given property $\mathcal{P}$, and one often calls these algebras "locally $\mathcal{P}$" (or similar).

For example, if $\mathcal{P}$ means being vs-finite-dimensional, then we get the so-called "approximately finite" (AF) C*-algebras, which are just the limits $A = \varinjlim A_k$ where each $A_k \cong \oplus_{i=1}^{r_k} M_{[k,i]}$ for some natural numbers $[k,i] \geq 1$. AF-algebras were first considered by Bratteli[7] in [**Bra72**]. They were classified (vie the ordered K_0-group) by Elliott in [**Ell76**], which inspired the later more ambitious quest to classify (real rank zero) C*-algebras (see the next chapter on the Elliott conjecture). The reader is referred to [**Bla98**] for more information.

Also, the C*-algebras that are locally approximated by vs-finite-dimensional C*-algebras turn out to be already AF-algebras. The reason is that vs-finite-dimensional C*-algebras are semiprojective. In general, if $\underline{C}$ is a class of (finitely presented) semiprojective C*-algebras, and $\mathcal{P}$ means "is in $\underline{C}$", then "locally $\mathcal{P}$" algebras are already "approximately $\mathcal{P}$". (see [**Lor97**, Lma 15.2.2, p.119])

We want to apply these ideas to the property of being (sub)homogeneous (see 2.1, 3.1). For subhomogeneous C*-algebras this works fine, but for homogeneous C*-algebras the situation is a little bit more complicated, since we need to allow for finite direct sums of (unital) homogeneous C*-algebras:

5.1. Definition: *Let A be a separable C*-algebra. Then A is called* ***approximately homogeneous*** *(resp.* ***locally homogeneous****) if it is a direct limit of a sequence (resp. locally approximated by) separable C*-algebras that are finite direct sums of* unital *homogeneous C*-algebras.*

Further, A is called ***approximately subhomogeneous*** *(resp.* ***locally subhomogeneous****) if it is a direct limit of a sequence (resp. locally approximated by) separable C*-algebras that are subhomogeneous.*

5.2. Remark: We will abbreviate "approximately homogeneous" (resp. "approximately subhomogeneous") by AH (resp. ASH) and speak of AH-algebras etc. Similarly we abbreviate by LH (resp. LSH).

We have to be careful when speaking about "unital" AH-algebras. Since the connecting maps are not assumed to be unital, an AH-algebra can be non-unital although all then building blocks are unital (e.g. $\mathbb{K}$ is an AH-algebra). However,

[7]Special classes of AF-algebras were studied earlier, e.g. by Glimm and Dixmier. In the definition of Bratteli the connecting maps of the limit are required to be unital. We do not make this assumption (as is usual today), and for example the compact operators $\mathbb{K}$ form an AF-algebra.

the minimal unitalization $\widetilde{A}$ of an AH-algebra is again approximately homogeneous, and we can force the connecting maps to be unital (here it is of course needed that we allow for direct sums of unital homogeneous C*-algebras).

In contrast, if we were looking at the class of C*-algebras that are limits of direct sum of (not necessarily unital) homogeneous C*-algebras, then this class would not be closed under unitalization. The problem is that the unitalization of a (non-unital) homogeneous C*-algebra is not homogeneous unless it is commutative.

This is easier for ASH-algebras, since the unitalization of a subhomogeneous C*-algebra is again subhomogeneous. Therefore we do not need to assume the subhomogeneous C*-algebras in the above definition to be unital.

5.3. Remark: There are different definitions for AH-algebras in the literature. Let us clarify the situation: In general, AH-algebras are defined as limits of some C*-algebras, called building blocks, which are finite direct sums of (special) homogeneous C*-algebras. For this, let us introduce some concepts for a unital C*-algebra A:

(1) A is **trivial homogeneous** if $A \cong C(X, M_n)$ for some compact space X, and integer $n \geq 1$
(2) A is **stably trivial homogeneous** if $A \cong pC(X, M_n)p$ for some compact space X, integer $n \geq 1$ and projection $p \in C(X, M_n)$

Note that a (unital) homogeneous C*-algebra A is stably trivial if and only if $A \otimes \mathbb{K} \cong C(\widehat{A}, \mathbb{K})$, and this happens precisely if $A \cong \mathrm{End}(V)$ for some vector bundle V.

Finite direct sums of homogeneous C*-algebras are sometimes called "locally homogeneous", which collides with our remarks above. We will use the word "semi" to indicate that finite direct sums are allowed, i.e. a unital C*-algebra A is **semi-homogenous** (resp. **trivial semi-homogeneous**, **stably trivial semi-homogeneous**) if it is a finite direct sum of unital homogeneous (resp. trivial homogeneous, stably trivial homogeneous) C*-algebras.

Now, there are (a priori) different definitions of AH-algebras according to which building blocks we allow. The natural choices for the building blocks are:

(1) unital, semi-homogeneous C*-algebras [**Bla93**, 2.1.]
(2) unital, stably trivial semi-homogeneous C*-algebras [**Rør02**, Chapter 3]

Note that the latter building blocks are of the form

$$A_k = \oplus_{i=1}^{r_k} p_{n,i} C(X_{n,i}, M_{[n,i]}) p_{n,i}$$

for some compact, Hausdorff spaces $X_{n,j}$, some integers $[n,j] \geq 1$, and projections $p_{n,i} \in C(X_{n,i}, M_{[n,i]})$. Additionally, it is sometimes assumed that the spaces $X_{n,j}$ are connected and/or polyhedra.

It is shown in [**Bla93**, 2.3.] that these possible definitions are equivalent for separable C*-algebras and (unital) building blocks, i.e. every separable limit of unital, semi-homogeneous C*-algebras is already a direct limit of unital, stably trivial semi-homogeneous C*-algebras whose spectra are finite, connected polyhedra. For all this, the connecting maps in the limit are not assumed to be unital. In some sense, we should call these algebra "approximately semi-homogeneous", but the term "approximately homogeneous" is more common.

Elliott, Gong and Li show in [**EGL05**] that every AH-algebra A can even be written as a direct limit $A \cong \varinjlim A_k$ with *injective* connecting maps (and the A_k are unital, stably trivial semi-homogeneous C*-algebras whose spectra are finite, connected polyhedra).

5.4. Remark: What would the statement of [**Bla93**, 2.3.] that all definitions of AH-algebras are equivalent mean for ASH-algebras? As NCCW-complexes are the subhomogeneous analogues of homogeneous C*-algebras whose spectra are CW-complexes, it would mean the following: Every ASH-algebra is already the limit of NCCW-complexes. This is not known, and likely to be false in general. However, there is some hope that it is true for simple ($\mathcal{Z}$-stable) C*-algebras. We will see that it follows from the Elliott conjecture.

5.1. The AH- and ASH-dimension.

It is a general concept of dimension theory to extend a well-defined notion of dimension for "nice" spaces to more general spaces as follows: We say the dimension of the general space is $\leq n$ if the space can be "approximated" by "nice" space of dimension $\leq n$.

For example, we could consider polyhedra to be "nice" spaces with a natural dimension concept, and we extend this to general spaces by shape theoretic ideas (so called expansions, or resolutions), and the result is exactly the covering dimension of spaces.

We can do a similar thing for C*-algebras. Our "nice" C*-algebras (the building blocks) are the unital, semi-homogeneous (resp. subhomogeneous) C*-algebras, for which there is a natural topological dimension. Then, we extend this dimension concept to C*-algebras that can be approximated (e.g. as limits or locally) by these nice algebras. Let us make this precise:

5.5. Definition: *For an integer $n \geq 0$ we define the following classes of separable C*-algebras: We let $\underline{\mathrm{H}}(n)$ be the class of unital, separable, semi-homogeneous C*-algebras of topological dimension $\leq n$, and $\underline{\mathrm{H}}(n)'$ the class of unital, separable, semi-homogeneous C*-algebras whose spectrum is a polyhedron of dimension $\leq n$. Further:*

(1) $\underline{\mathrm{SH}}(n)$:= all separable, subhomogeneous C-algebras of topological dimension $\leq n$*

(2) $\underline{\mathrm{SH}}(n)'$:= all separable NCCW-complexes of topological dimension $\leq n$

For a class $\underline{\mathrm{C}}$ of C-algebras, we denote by $\underline{\mathrm{AC}}$ the limits of algebras in $\underline{\mathrm{C}}$, and by $\underline{\mathrm{LC}}$ the C*-algebras that are locally in $\underline{\mathrm{C}}$. In particular we get the classes $\underline{\mathrm{ASH}}(n)$, $\underline{\mathrm{ASH}}(n)'$ $\underline{\mathrm{AH}}(n)$, $\underline{\mathrm{LSH}}(n)$ and so on.*

5.6. Remark: We have the following obvious relations:

$$\underline{\mathrm{SH}}(n)' \subset \underline{\mathrm{SH}}(n)$$
$$\underline{\mathrm{AH}}(n)' \subset \underline{\mathrm{ASH}}(n)' \subset \underline{\mathrm{ASH}}(n)$$
$$\underline{\mathrm{AH}}(n) \subset \underline{\mathrm{ASH}}(n)$$

and of course every "approximately $\mathcal{P}$" algebra is also "locally $\mathcal{P}$" (if property $\mathcal{P}$ is preserved by quotients), which gives for example $\underline{\mathrm{ASH}}(n) \subset \underline{\mathrm{LSH}}(n)$ and so on.

In low dimensions we can do better:

5.7. Proposition: *The following relations hold:*

$$\underline{\mathrm{SH}}(0)' = \underline{\mathrm{F}} \subset \underline{\mathrm{SH}}(0) \subset \underline{\mathrm{AF}}$$
$$\underline{\mathrm{ASH}}(0)' = \underline{\mathrm{ASH}}(0) = \underline{\mathrm{AH}}(0) = \underline{\mathrm{AH}}(0)' = \underline{\mathrm{AF}}$$
$$\underline{\mathrm{LSH}}(0)' = \underline{\mathrm{LSH}}(0) = \underline{\mathrm{LH}}(0) = \underline{\mathrm{LH}}(0)' = \underline{\mathrm{AF}}$$

The above result follows from the fact that vs-finite-dimensional C*-algebras are semiprojective. Since one-dimensional NCCW-complexes are also semiprojective (see 3.15), we also get the following result:

5.8. Proposition: *The following relations hold:*

$$\underline{\mathrm{LH}}(1) = \underline{\mathrm{AH}}(1) = \underline{\mathrm{LH}}(1)' = \underline{\mathrm{AH}}(1)'$$
$$\underline{\mathrm{LSH}}(1)' = \underline{\mathrm{ASH}}(1)'$$

5.9. Remark: It is shown in [**DE99**] that there exists an algebra A which is a limit of algebras in $\underline{\mathrm{AH}}(3)$ (so in particular A is in $\underline{\mathrm{LH}}(3)$), but such that A is not an AH-algebra. This implies that $\underline{\mathrm{LH}}(n) \neq \underline{\mathrm{AH}}(n)$ for all $n \geq 3$. It is also expected that $\underline{\mathrm{LSH}}(n) \neq \underline{\mathrm{ASH}}(n)$ and $\underline{\mathrm{LSH}}(n)' \neq \underline{\mathrm{ASH}}(n)'$, at least for n big enough.

The situation for $\underline{\mathrm{LSH}}(1)$ seems to be open, and it might very well be that $\underline{\mathrm{LSH}}(1) = \underline{\mathrm{ASH}}(1)$. Also, for $\underline{\mathrm{AH}}(2)$ the situation is unclear.

5.10. Definition: *For a C*-algebra A we define:*

$$\dim_{ASH}(A) \leq n :\Leftrightarrow A \in \underline{\mathrm{ASH}}(n)$$
$$\dim_{ASH'}(A) \leq n :\Leftrightarrow A \in \underline{\mathrm{ASH}}(n)'$$
$$\dim_{AH}(A) \leq n :\Leftrightarrow A \in \underline{\mathrm{AH}}(n)$$

*and we call the least integer $n \geq 0$ such the the respective dimension is $\leq n$ the ASH-**dimension** (resp. ASH'-**dimension**, AH-**dimension**) of A.*

5.11. Remark: Note that these dimension concepts are (probably) not dimension theories in the sense of 1.1. They indeed behave well with respect to quotients, ideals and direct sums. It is also easy to see that for a commutative C*-algebra $A = C_0(X)$ we have $\dim_{ASH}(A) = \dim_{ASH'}(A) = \mathrm{locdim}(X)$ and if X is compact these also agree with $\dim_{AH}(A)$. Thus, the only property of 1.1 that is in question is whether the AS(H)-dimension behaves well with respect to limits.

It follows from 5.9 that indeed the AH-dimension does not. And it is expected to be false for the other dimension concepts as well.

There are two ways out of this:

(1) restrict the class of considered C*-algebras (e.g. to simple algebras, or to $\mathcal{Z}$-stable algebras)
(2) restrict to low dimensions

In fact, both ideas are related. First, it is unknown whether the $A(S)H$-dimensions behave well with respect to limits of *simple* C*-algebras (which are automatically simple again). A more natural question (as noted in 5.9) is whether a simple C*-algebra in $\underline{\mathrm{LSH}}(n)$ has $\dim_{ASH}(A) \leq n$ (and similar for the ASH'-, AH-dimension).

The second possibility is to restrict to low dimensions. We could define a dimension as $\widetilde{\dim}_{ASH'}(A) := \min\{\dim_{ASH'}(A), 2\}$ and $\widetilde{\dim}_{AH}(A) := \min\{\dim_{AH}(A), 2\}$, i.e.

$$\widetilde{\dim}_{ASH'}(A) = \begin{cases} 0 & \text{,if } A \text{ is AF} \\ 1 & \text{,if } A \text{ is in } \underline{\mathrm{ASH}}(1)' \text{ (and not AF)} \\ 2 & \text{,otherwise} \end{cases}$$

$$\widetilde{\dim}_{AH_1}(A) = \begin{cases} 0 & \text{,if } A \text{ is AF} \\ 1 & \text{,if } A \text{ is in } \underline{\mathrm{AH}}(1) \text{ (and not AF)} \\ 2 & \text{,otherwise} \end{cases}$$

which define dimension theories in the sense of 1.1 for the class of all (separable) C*-algebras.

6. The Elliott conjecture

In this section we give a short introduction to the Elliott conjecture, which states that separable, nuclear, simple C*-algebras are classified (up to $*$-isomorphism) by their so-called Elliott invariant. This invariant consists of the ordered K-theory, the space of traces and a pairing between both. There are counterexamples to this general form of the conjecture, but if we restrict so certain subclasses of C*-algebras, then the conjecture is known to be true.

Let us start by introducing the basic notions needed to define the Elliott invariant. First, recall that the K_0-group of a unital C*-algebra A is given as

$$K_0(A) = \{[p]_0 - [q]_0 \ : \ p, q \in \mathrm{Proj}(A \otimes \mathbb{K})\}$$

and for a general C*-algebra $K_0(A) := \ker(K_0(A^+) \to K_0(\mathbb{C}))$. In any case there is a map $\mathrm{Proj}(A \otimes \mathbb{K}) \to K_0(A)$ and besides $K_0(A)$ we we attach the following invariants to any C*-algebra A (compare [**Ell95**, 7.]):

$$\begin{aligned}
K_0(A)^+ &= \{[p]_0 \ : \ p \in \mathrm{Proj}(A \otimes \mathbb{K})\} \subset K_0(A) \\
\Sigma(A) &= \{[p]_0 \ : \ p \in \mathrm{Proj}(A)\} \subset K_0(A)^+ \\
K_1(A) &= \{[u]_1 \ : \ u \text{ is a unitary in } \widetilde{A} \otimes M_k \text{ for some } k \geq 1\} \\
T(A) &= \text{ densely defined, lower semicontinuous (positive) traces on } A
\end{aligned}$$

For every C*-algebra, $K_0(A)$ and $K_1(A)$ are abelian groups, and $K_0(A)^+$ defines a pre-order on $K_0(A)$ (see C.5 for definitions). If A is stably unital (see 6.4 below), then $K_0(A)^+ - K_0(A)^+ = K_0(A)$, and if moreover A is stably finite, then $(K_0(A), K_0(A)^+)$ is an ordered group. Recall that $\Sigma(A)$ is called the dimension range of A. (compare [**Bla98**] for precise definitions relating to K-theory).

For traces (and tracial states) we use the definitions in [**Bla06**, II.6.8.1, p.121], i.e. a trace τ on A is a weight[8] (which is automatically positive, but possibly unbounded) satisfying the trace condition, i.e. $\tau(x^*x) = \tau(xx^*)$ (for all $x \in A$). It is densely defined if $\{x \in A_+ \ : \ \tau(x) < \infty\}$ is dense in A_+. A densely-defined, lower semicontinuous trace is automatically semifinite (see [**Bla06**, II.6.8.8, p.122]). We give $T(A)$ the topology of pointwise convergence, in which it is a topological convex cone. (the sum of two lower semicontinuous traces is again a again a lower semicontinuous trace by [**Bla06**, II.6.7.2(iii), p.119]).

Assume $A \cong B \otimes \mathbb{K}$ for a unital C*-algebra B, and $\tau \in T(A)$. Then $\tau(1_B)$ is finite (since τ is densely defined). Let $\tau' : B \to \mathbb{C}$ be defined as $\tau'(x) := \tau(x)/\tau(1_B)$ on positive elements $x \in B$ (and then extended to B). This is a tracial state[9] on B. Further, τ is completely determined by τ' as follows: Let $\mathrm{tr}_{M_k} : M_k \to \mathbb{C}$ be the usual (bounded) trace $\mathrm{tr}((a_{ij})) = \sum_i a_{ii}$. This extends to an (unbounded) trace $\mathrm{tr}_{\mathbb{K}}$ on $\mathbb{K}$. Then $\tau' \otimes \mathrm{tr}_{\mathbb{K}}$ defines a trace on $B \otimes \mathbb{K}$, which can be shown to agree with τ.

Thus any $\tau \in T(A)$ takes finite values on projections $p \in A$. If p is full (e.g. if A is simple), then by Brown's stabilization theorem (see [**Bro77**]), τ is completely determined by its values on pAp.

[8] A weight on a C*-algebra A is an additive function $\phi : A_+ \to [0, \infty]$ such that $\phi(0) = 0$ and $\phi(\lambda a) = \lambda \phi(a)$ (for $\lambda > 0$)

[9] A tracial state on a C*-algebra A is a bounded, positive, linear functional $\tau : A \to \mathbb{C}$ with $\|\tau\| = 1$ and which satisfies the trace condition $\tau(ab) = \tau(ba)$ for all $a, b \in A$

Now, the pairing $< .,. >_A$: $K_0(A) \times T(A) \to \mathbb{R}$ is defined as follows: If $p \in \widetilde{A} \otimes M_k$ is a projection and $\tau \in T(A)$, then first extend τ to $\widetilde{A} \otimes M_k$ by extending it to $\widetilde{A}$ and then tensoring with tr_{m_k} (by abuse of notation this is also called τ). Then evaluate at p, i.e. $< [p], \tau >:= \tau(p) = (\tau \otimes \mathrm{tr})(p)$, which will be a finite (well-defined) value.

6.1. Definition: *Let A be a nuclear C*-algebra. The **Elliott invariant** of A is the tuple:*

$$\mathrm{Ell}(A) := (K_0(A), K_0(A)^+, \Sigma(A), K_1(A), T^+(A), < .,. >_A)$$

6.2: The stable case.
If A is stable, then $\Sigma(A) = K_0(A)^+$ and we may simplify the Elliott invariant to be the tuple

$$(K_0(A), K_0(A)^+, K_1(A), T(A), < .,. >_A)$$

6.3: The unital case.
If A is unital, then the situation can be simplified even more. Firstly, the dimension range can be recovered from the position of the unit [1] in $K_0(A)$ as follows:

$$\Sigma(A) = \{x \in K_0(A)_+ \ : \ x \leq [1]\}$$

Secondly, all (densely defined, lower semicontinuous) positive traces are bounded. Thus, $T(A)$ can be recovered from the set $TS(A)$ of tracial states on A. The topology of pointwise convergence on $TS(A)$ is then the weak*-topology, and $TS(A)$ is a compact, convex set (possibly empty). It is even a even a so-called Choquet simplex (see [**Sak98**, 3.1.18, p.130]). See [**Alf71**, II.3, p.84ff] for the definition and properties of Choquet simplices. If A is separable, then $TS(A)$ is metrizable: In general, if X is a separable Banach space, then a norm-bounded subset of the dual space X^* is metrizable in the weak*-topology (see [**Con90**, V.5.1, p.134]). This applies to $TS(A)$.

Further, we could restrict the pairing $< .,. >$ to $TS(A)$. Instead, one usually considers the map

$$r_A : TS(A) \to S(K_0(A))$$

with $r_A(\tau) = K_0(\tau) : K_0(A) \to \mathbb{R}$ defined as $K_0(\tau)([p]_0 - [q]_0) = \tau(p) - \tau(q)$. ([**Rør02**, 1.1.10, p.10]), where $S(K_0(A))$ is the compact, convex set of states on $K_0(A)$ (see C.6).

Altogether, for a unital C*-algebra A we may consider the tuple

$$(K_0(A), K_0(A)^+, [1], K_1(A), TS(A), r_A)$$

which is often defined to be the Elliott invariant for unital C*-algebras.

We need some more definitions:

6.4. Definition: [**Rør02**, 1.1.1, 1.1.6, p.7], [**Bla98**, 5.5.4, p.31]
Let A be a C-algebra. Then we say:*

(1) *A is **stably projectionless** if and only if $A \otimes \mathbb{K}$ contains no projection*
(2) *A is **stably unital** if and only if $A \otimes \mathbb{K}$ contains an approximate unit of projections*
(3) *If A is stably unital or simple, then we say A is **stably finite** if and only if $A \otimes \mathbb{K}$ contains no infinite projection.*

Assume A is stably isomorphic to a unital C*-algebra B, i.e. $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$. Then A is stably unital, and A and B behave alike:

(1) A is stably finite if and only if B is stably finite
(2) A is purely infinite if and only if B is purely infinite

Note also that the unital C*-algebras B is stably finite if and only if the unit in $B \otimes M_k$ is a finite projection for all $k \geq 1$ (this is often used as a definition for stably finite, *unital* C*-algebras e.g. in [**Bla06**, V.2.1.5, p.419]). For simple C*-algebras the situation is even easier:

6.5: The three cases: stably projectionless, stably finite, infinite.
If A is simple, then A is either stably projectionless or stably unital, and the latter case occurs precisely if A is stably isomorphic to a unital C*-algebra ([**Rør02**, 2.2, p.26]).

Now, let A be a simple, nuclear C*-algebra. To shorten notation let $K_0^+ := K_0(A)^+$ and $K_0 := K_0(A)$. Then there are three disjoint (and exhaustive) possibilities ([**Rør02**, 2.2.1, p.26], [**Ell95**, 10.]):

(F_0) $K_0^+ = 0$ and $T(A) \neq 0$
(F_1) $K_0^+ \cap -K_0^+ = 0$, $K_0^+ - K_0^+ = K_0 \neq 0$ and $T(A) \neq 0$
(Inf) $K_0^+ = K_0$ and $T(A) = 0$

Case (F_0) occurs precisely if A is stably projectionless, and case (F_1) occurs precisely if A is stably unital and stably finite (in that case $K_0(A)$ is an ordered abelian group). The last case occurs precisely if $A \otimes \mathbb{K}$ contains an infinite projection.

If A is simple, nuclear and *unital*, then the case (F_0) is impossible, and A fulfills one of the following two cases:

(F_1) $TS(A) \neq 0$ and $(K_0(A), K_0(A)^+)$ is a non-empty, ordered, abelian group
(Inf) $TS(A) = 0$ and $K_0^+ = K_0$

6.6. Remark: Injectivity and surjectivity of r_A
Let A be a unital, nuclear C*-algebra. Then r_A is surjective.

Furthermore, r_A is injective if and only if projections in $A \otimes \mathbb{K}$ separate traces[10] and this holds in particular if $TS(A)$ is empty or contains only one element. It also holds if A has real rank zero. (see [**Rør02**, 1.1.11, 1.1.12, p.10f])

If r_A is a bijection (hence homeomorphism), then we may simplify the Elliott invariant to be the tuple

$$(K_0(A), K_0(A)^+, [1], K_1(A))$$

For example, this applies to the case (Inf) considered above.

[10] This means that two tracial states $\tau, \tau' \in TS(A)$ are equal whenever $\tau(p) = \tau'(p)$ for all projections $p \in A \otimes \mathbb{K}$, so this is just a reformulation for r_A being injective.

Besides the above, not much can be said in general about the Elliott invariant. To summarize: If A is a separable, simple, unital, nuclear C*-algebra, then $K_0(A)$ is a simple, countable, pre-ordered group, $K_1(A)$ is a countable group, $TS(A)$ is a metrizable Choquet simplex, and r_A is surjective.

6.7: Functoriality of the Elliott invariant.
The Elliott invariant does not only attach to each C*-algebra A a tuple $\mathrm{Ell}(A)$, but it defines in some sense a functor. To make this precise, let $\varphi : A \to B$ be a $*$-homomorphism.

On K-theory we get induced (covariant) maps with $K_0(\varphi)(K_0(A)^+) \subset K_0(B)^+$ and $K_0(\varphi)(\Sigma(A)) \subset \Sigma(B)$. On the cones of traces we get an induced (contravariant) map $T(\varphi) : T(B) \to T(A)$ defined as $\tau \mapsto \tau \circ \varphi$. This is compatible with the paring, i.e.

$$< \tau, K_0(\varphi)(\alpha) >_B = < T(\varphi)(\tau), \alpha >_A$$

for $\tau \in T(B)$ and $\alpha \in K_0(A)$.

To make this into a proper covariant functor, we should consider the opposite category for the cones of traces. This is only a technicality, and we will just write $\mathrm{Ell}(\varphi) : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ to mean the induced maps

$$\begin{aligned} K_0(\varphi) :& (K_0(A), K_0(A)^+, \Sigma(A)) \to (K_0(B), K_0(B)^+, \Sigma(B)) \\ K_1(\varphi) :& K_1(A) \to K_1(B) \\ T(\varphi) :& T(B) \to T(A) \end{aligned}$$

and the compatibility with the pairings.

If A, B and φ are unital, then $K_0(\varphi)([1_A]) = [1_B]$, and we get a map

$$K_0(\varphi) : (K_0(A), K_0(A)^+, [1_A]) \to (K_0(B), K_0(B)^+, [1_B])$$

which induces a dual map

$$\widehat{K_0(\varphi)} : S(K_0(B)) \to S(K_0(A))$$

by $\chi \mapsto \chi \circ \varphi$ (for $\chi \in S(K_0(B))$).

Now, the compatibility of the induced maps with the pairing of K-theory and traces can be rephrased by saying that the following diagram commutes (see [**Rør02**, 2.2.5, p.28]):

$$\begin{array}{ccc} TS(B) & \xrightarrow{T(\varphi)} & TS(A) \\ \downarrow{\scriptstyle r_B} & & \downarrow{\scriptstyle r_A} \\ S(K_0(B)) & \xrightarrow{\widehat{K_0(\varphi)}} & S(K_0(A)) \end{array}$$

We say that $\mathrm{Ell}(\varphi)$ is an isomorphism if the induced maps are group-isomorphisms (identifying the positive part and dimension range of the K_0-groups) resp. a homeomorphism on traces. More generally we can say that two Elliott invariants $\mathrm{Ell}(A)$ and $\mathrm{Ell}(B)$ are isomorphic if there are group-isomorphisms (again, identifying the positive part and dimension range of the K_0-groups) and a homeomorphism on traces compatible with the pairing. Such an abstract isomorphism of Elliott invariants need (a priori) not be of the form $\mathrm{Ell}(\varphi)$ for some $*$-isomorphism $\varphi : A \to B$. However, Elliott made the following conjecture:

6.8. Conjecture: ***(The Elliott conjecture)***
Let A, B be separable, nuclear, simple C-algebras. Then A and B are isomorphic if and only if* $\mathrm{Ell}(A)$ *and* $\mathrm{Ell}(B)$ *are isomorphic. Moreover if* $\Phi : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ *is an isomorphism of Elliott invariants, then there exists a* $*$*-isomorphism* $\varphi : A \to B$ *such that* $\Phi = \mathrm{Ell}(\varphi)$*.*

There are in fact counterexamples to this general form of the conjecture. In [**Rør03**, Cor 7.9] it is shown that there are counterexamples for case (Inf) , and in [**Tom05**] counterexamples for case (F_1) are found (see also [**Tom08**]).

Therefore we need to restrict the conjecture to subclasses: If $\underline{C}$ is some class of separable, simple, nuclear C*-algebras, then we say that the Elliott conjecture (EC) holds for $\underline{C}$ if the assertions of the general Elliott conjecture holds for all C*-algebras $A, B \in \underline{C}$.

In fact there are broad classes[11] of C*-algebras for which the Elliott conjecture is known (a good overview is in [**Rør02**]). For the infinite case (Inf) the conjecture is true for:

(1) purely infinite C*-algebras that satisfy the Universal coefficient theorem (UCT) (see [**Rør02**, Thm 8.4.1, p.128])

For the stably unital, stably finite case (F_1) we mention only a view results. There is much progress right now in extending these results to broader classes:

(1) unital, AH-algebras with no dimension growth (see [**Rør02**, Thm 3.3.7, p.60], and [**EGL07**])
(2) unital, inductive limits of subhomogeneous C*-algebras with Hausdorff spectrum (and no dimension growth) (see [**Lin08**])
(3) unital, real rank zero, $\mathcal{Z}$-stable algebras which have locally finite decomposition rank (e.g. ASH-algebras), satisfying the UCT (see [**Win07**])

[11] We will always assume that they are subclasses of the class the separable, simple, nuclear C*-algebras. The Elliott conjecture can also we formulated for non-simple C*-algebras, and it indeed holds in some cases of non-simple C*-algebras (e.g. AF-algebras).

7. The range of the Elliott invariant

Besides the aim for classification results in the Elliott program it has always been of interest which Elliott invariants can be realized by C*-algebras, i.e. what the range of the Elliott invariant is. It has turned out that the noition of weak unperforation is an important regularity property, and it was shown in [**Ell96**] that all weakly unperforated Elliott invariants can be realized.

If we consider the stably finite case, then every weakly unperforated Elliott invariant can even be realized by a C*-algebra in $\underline{\text{ASH}}(2)'$ (see 7.7).

In this section we prove that every weakly unperforated Elliott invariant with torsion-free K_0-group can be realized by a C*-algebra in $\underline{\text{ASH}}(1)'$ (see 7.8). The proof is a modification of ideas due to Elliott.

The converse is also true: Every (not necessarily simple) C*-algebra in $\underline{\text{ASH}}(1)$ has torsion-free K_0-group. This allows us to draw some nice conclusions, assuming that the Elliott conjecture is true for nuclear, separable, unital, stably finite, simple, $\mathcal{Z}$-stable C*-algebras: For such an algebra A we have $\dim_{ASh}(A) = \dim_{ASH'}(A) \leq 2$, and we can detect the ASH-dimension in an easy way in the K-theory: the algebra is one-dimensional if and only if there is no torsion in $K_0(A)$.

To construct a (simple) C*-algebra with a prescribed Elliott invariant $\mathcal{E}$ we use roughly the following strategy (due to Elliott):

(1) decompose $\mathcal{E}$ as a direct limit $\cong \varinjlim(\mathcal{E}^k, \theta_{k+1,k})$ where the $\mathcal{E}^k$ are basic
(2) construct C*-algebras A_k and $*$-homomorphisms $\varphi_{k+1,k} : A_k \to A_{k+1}$ such that $\text{Ell}(A_k) = \mathcal{E}_k$ and $\text{Ell}(\varphi_{k+1,k}) = \theta_{k+1,k}$.
(3) the limit $A := \varinjlim_k A_k$ already has $\text{Ell}(A) = \mathcal{E}$, but is not necessarily simple. Deform the connecting maps $\varphi_{k+1,k}$ such that the limit gets simple (while the invariant is unchanged)

The construction of the A_k is possible, since the invariants $\mathcal{E}_k$ are basic (i.e. easier to handle). The algebras A_k are called building blocks. Let us first see how the K-theory can be decomposed in certain nice situations.

7.1. Decomposing the Elliott invariant.

The first part of the following theorem is sometimes called the theorem of Effros, Handelmann and Shen. It states that every Riesz group is a so-called dimension group, i.e. a direct limit of finitely generated, free abelian groups. The second and third parts of the theorem are generalizations by Elliott. The fourth part is another variant which seems not to appear in the literature so far.

7.1. Theorem: *Let G be a countable, ordered group. Then:*

1.) *G is unperforated and has the Riesz interpolation property* $\Leftrightarrow G \cong_{ord} \varinjlim_k G_k$ *where each* $G_k = \mathbb{Z}^{r_k} = \bigoplus_{i=1}^{r_k}(\mathbb{Z})$

2.) *G is weakly unperforated and has the Riesz interpolation property* $\Leftrightarrow G \cong_{ord} \varinjlim_k G^k$ *and each* $G^k = \bigoplus_{i=1}^{r_k}(\mathbb{Z}\oplus\mathbb{Z}_{[k,i]})$ *(for some numbers $[k,i] \geq 1$)*

Let $G_* = G_0 \oplus_{str} G_1$ be a countable, graded, ordered group. Then:

3.) *G_* is weakly unperforated and has the Riesz interpolation property* $\Leftrightarrow G_* \cong_{ord} \varinjlim_k G_*^k$ *and each* $G_*^k = \bigoplus_{i=1}^{r_k}((\mathbb{Z}\oplus\mathbb{Z}_{[k,i]}) \oplus_{str} (\mathbb{Z}\oplus\mathbb{Z}_{[k,i]}))$ *(for some numbers $[k,i] \geq 1$)*

4.) *G_* is weakly unperforated, has the Riesz interpolation property and G_0 is torsion-free* $\Leftrightarrow G_* \cong_{ord} \varinjlim_k G_*^k$ *and each* $G_*^k = \bigoplus_{i=1}^{r_k}((\mathbb{Z}) \oplus_{str} (\mathbb{Z}_{[k,i]}))$ *(for some numbers $[k,i] \geq 1$)*

Proof:

1.) This is [**EHS80**, Thm 2.2].

2, 3.) This is [**Ell90**, Thm 3.2] and [**Ell90**, Thm 5.2].

4.) "$\Leftarrow$": First, by 3.) G_* is weakly unperforated and has the Riesz interpolation property. Then $G_0 \cong \varinjlim_k G_0^k$ and $G_0^k = \mathbb{Z}^{r_k}$ is free (in particular torsion-free). By C.2 we have G_0 torsion-free.

"$\Rightarrow$": This is a modification of the proof of [**Ell90**, Thm 5.2]. Since we do not require the even and odd part of building blocks to be isomorphic, the proof is in fact a little bit less complicated. We adopt the following terminology:

(1) A *basic building block* is a graded, ordered groups of the form $((\mathbb{Z}) \oplus_{\mathrm{str}} (\mathbb{Z}_l))$ (for some natural number $l \geq 1$, and for $l = 1$ we mean $\mathbb{Z}_1 = \mathbb{Z}$).

(2) A *building block* is a finite direct sum of basic building blocks, i.e. of the form $\bigoplus_{i=1}^{r_i}((\mathbb{Z}) \oplus_{\mathrm{str}} \mathbb{Z}_{[\alpha,i]})$). Here (as in the theorem), the "big sum" is understood as a direct order sum (as defined above in C.10)

(3) A *generalized building block* is a graded, ordered group of the form $\bigoplus_{i=1}^{r_i}((\mathbb{Z}) \oplus_{\mathrm{str}} F_i)$ for some finitely generated, abelian groups F_i.

Firstly, thanks to Shen (see [**She79**, Thm 3.1]) it is enough to verify the following local criterion: Let $G_* = G_0 \oplus_{\mathrm{str}} G_1$ be a weakly unperforated, graded, ordered group which has the Riesz interpolation property and such that G_0 is torsion-free. Let further G'_* be a building block and $\varphi : G'_* \to G_*$ a morphism (by which we mean a graded, order-homomorphism). Then there exists a building block G''_* and morphisms making the following diagram commute

$$\begin{array}{ccc} G'_* & \xrightarrow{\varphi} & G_* \\ {\scriptstyle\alpha}\downarrow & \nearrow_{\beta} & \\ G''_* & & \end{array}$$

and such that $\ker(\varphi) = \ker(\alpha)$.

Step 1. As in [**Ell90**, Thm 5.2] we first show that it is enough to consider the local criterion for generalized building blocks. In fact, we show that each morphism $H_* \to G_*$ with H_* a generalized building block can be factored through some building block.

We may consider each summand of H_* separately, i.e. we assume $H_* = \mathbb{Z} \oplus_{\text{str}} F$ with F a finitely generated group, and let $\varphi : H_* \to G_*$ be a morphism. Let $F = \mathbb{Z}^l \oplus K_1 \oplus \ldots \oplus K_m$ with the K_i singly generated, finite groups, i.e.:

$$H_* := \mathbb{Z} \oplus_{\text{str}} (\bigoplus_{j=1}^{l} \mathbb{Z} \oplus \bigoplus_{k=1}^{m} K_k)$$

For each of the l $\mathbb{Z}$-summands fix a generator u_j and let $\bar{u}_j := \varphi(u_j)$ denotes its image in G_*. Also, for each K_k fix a generator s_k and let $\bar{s}_j := \varphi(s_j) \in G_*$ (the $\bar{u}_j$ and $\bar{s}_k$ are all in G_1). Further, fix a positive generator a in the $\mathbb{Z}$-summand in the even part of H_*, and let $\bar{a}$ be its image in G_* (it will land in G_0).

The map φ is positive, and therefore $\bar{u}_1, \ldots, \bar{u}_l, \bar{s}_1, \ldots, \bar{s}_m \leq \bar{a}$ in G_*. Since G_* is a weakly unperforated group with the Riesz decomposition property, from [**Ell90**, Cor 6.6] we get $\bar{b}_j, \bar{c}_k \in G_*^+$ such that $\bar{u}_k \leq \bar{b}_j$, $\bar{s}_k \leq \bar{c}_k$ ($j = 1, \ldots, l$ and $k = 1, \ldots, m$) and $\bar{a} = \bar{b}_1 + \ldots + \bar{b}_l + \bar{c}_1 + \ldots + \bar{c}_m$. Set

$$H'_* := \bigoplus_{j=1}^{l} (\mathbb{Z} \oplus_{\text{str}} \mathbb{Z}) \oplus \bigoplus_{k=1}^{m} (\mathbb{Z} \oplus_{\text{str}} K'_k)$$

where $K'_k \cong K_k \cong \mathbb{Z}_{[k]}$. Denote a typical element by $(x_1, x'_1, \ldots, x_l, x'_l, y_1 | y'_1, \ldots, y_m, y'_m)$ (so $x_j, x'_j, y_k \in \mathbb{Z}, y'_k \in \mathbb{Z}_{[k]}$). Define maps $\alpha : H_* \to H'_*$ as follows:

(1) The positive generator a is mapped to the sum of the positive generators of the even part of H'_*, i.e. to $(1, 0, 1, 0, \ldots, 1, 0 | 1, 0, \ldots, 1, 0)$.
(2) The subgroup $\mathbb{Z}^l$ of H_1 is mapped isomorphically onto the $\mathbb{Z}^l$-summand in H'_1, i.e. the generator u_1 gets mapped to $(0, 1, 0, 0 \ldots, 0 | 0, \ldots, 0)$ and so on.
(3) The subgroup K_k of H_1 is mapped isomorphically onto the summand K'_k in H'_1, i.e. the generator s_1 gets mapped to $(0, \ldots, 0 | 0, 1, 0, 0 \ldots, 0)$ and so on.

One can easily check that this map is a graded order-homomorphism.

Define also a map $\beta : H'_* \to G_*$ as follows:

(1) The positive generators of $H'_0 = \mathbb{Z}^l \oplus \mathbb{Z}^m$ are mapped (in order) to $\bar{b}_1, \ldots, \bar{b}_l$ and $\bar{c}_1, \ldots, \bar{c}_m \in G_*^+$.
(2) The subgroup $\mathbb{Z}^l$ of H'_1 is isomorphic to $\mathbb{Z}^l \leq H_1$ (via a fixed isomorphism, e.g. using α), and we map it to G_* just like φ does, i.e. $(0, 1, 0, 0 \ldots, 0 | 0, \ldots, 0)$ is mapped to $\bar{u}_1$, and so on.
(3) The summand K'_k of H'_1 is isomorphic to $K_k \leq H_1$ (via a fixed isomorphism, e.g. using α), and we map K'_k by using this and the map φ, i.e. $(0, \ldots, 0 | 0, 1, 0, 0 \ldots, 0)$ is mapped to $\bar{s}_1$ and so on.

Let us check that β is an order-homomorphism, i.e. maps positive elements to positive elements. We may check it on each summand separately. Consider the j-th $\mathbb{Z} \oplus_{\text{str}} \mathbb{Z}$ summand, which gets mapped to G_* as $(m, n) \mapsto m\bar{b}_j + n\bar{u}_j$. Certainly the image of $(0, 0)$ is positive, and the other positive elements of $\mathbb{Z} \oplus_{\text{str}} \mathbb{Z}$ are (m, n) with $m > 0$. But $m\bar{b}_j + n\bar{u}_j$ is positive for $m > 0$ since $\bar{u}_j \in G_1$, $\bar{b}_j \in G_*^+$ and $\bar{u}_j \leq \bar{b}_j$ (and thus, either $\bar{b}_j = 0$ and then $\bar{u}_j = 0$, or $\bar{b}_j > 0$ and then $m\bar{b}_j + n\bar{u}_j > 0$). The same argument shows the positivity on the summand $\mathbb{Z} \oplus_{\text{str}} K'_k$.

We are done since φ factors as $\beta \circ \alpha$.

Step 2. We proceed as in the proof of [**Ell90**, Thm 5.2]. If $\varphi : H_* \to G_*$ is a morphism with H_* a generalized building block, then we may restrict this map to the even part to get $\varphi_0 : H_0 \to G_0$. Note that $H_0 \cong \mathbb{Z}^r$ for some r. We want to factorize φ_0 through some $H_0' = \mathbb{Z}^s$ (which is to be thought of as the even part of another generalized building block) via maps $\alpha_0 : H_0 \to H_0'$ and $\beta_0 : H_0' \to G_0$ with $\ker(\varphi_0) = \ker(\alpha_0)$:

$$\begin{array}{ccc} H_0 & \xrightarrow{\varphi_0} & G_0 \\ {\scriptstyle \alpha_0}\downarrow & \nearrow {\scriptstyle \beta_0} & \\ H_0' & & \end{array}$$

This can be done since G_0 is torsion-free (by assumption), and therefore a Riesz-group. We may use the local criterion (in its original form) from [**She79**, Thm 3.1].

Step 3. In the last step we want to verify the local criterion for generalized building blocks. By step 2. we already have a factorization of the even part. Now step 3. in the proof of [**Ell90**, Thm 5.2] finds the "odd parts", i.e. a group H_1' and maps $\alpha_1 : H_1 \to H_1'$ and $\beta_1 : H_1' \to G_1$ such that $H' := H_0' \oplus H_1'$ is a generalized building block, and we get a factorization

$$\begin{array}{ccc} H_0 \oplus H_1 & \xrightarrow{\varphi_0 \oplus \varphi_1} & G_0 \oplus G_1 \\ {\scriptstyle \alpha_0 \oplus \alpha_1}\downarrow & \nearrow {\scriptstyle \beta_0 \oplus \beta_1} & \\ H_0' \oplus H_1' & & \end{array}$$

with $\ker(\varphi_0 \oplus \varphi_1) = \ker(\alpha_0 \oplus \alpha_1)$.

This proves the local criterion for generalized building blocks, and together with step 1. the local criterion for building blocks. So we are done. □

7.2. Building block C*-algebras.

Let us recall the construction of the building block[12] C*-algebras used by Elliott in [**Ell96**]. These algebras are in fact special two-dimensional NCCW-complexes, constructed out of the algebras $C(\mathbb{T})$, the dimension-drop algebras Int_k (see 3.13), and their tensor products. We begin by recalling the K-theory of these ingredients:

A	$K_0(A)$	$K_1(A)$
Int_k	$\mathbb{Z}$	$\mathbb{Z}_k$
$C(\mathbb{T})$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{I} = \mathbb{I}_1$	$\mathbb{Z}$	0
$\mathbb{I}_k \otimes C(\mathbb{T})$	$\mathbb{Z} \oplus_{\mathrm{str}} \mathbb{Z}_k$	$\mathbb{Z} \oplus \mathbb{Z}_k$

[12] The term "building block" is used for C*-algebras and for groups. If necessary to avoid ambiguity we will speak of "building block algebras" and "building block groups"

7.2: Grade one building blocks. (see [**Ell96**])
First, recall that a general one-dimensional NCCW-complex is given as a pullback

$$\begin{array}{ccc} B & \longrightarrow & C \\ \big\downarrow & & \big\downarrow {\scriptstyle (\lambda,\rho)} \\ C([0,1],D) & \xrightarrow{\ \partial\ } & D \oplus D \end{array}$$

with C and D vs-finite-dimensional C*-algebras, say $C = \bigoplus_{i=1}^{p} M_{n_i}$, $D = \bigoplus_{j=1}^{l} M_{m_j}$ and $\lambda, \rho : C \to D$ any $*$-homomorphisms. It has a canonical ideal

$$I = \{f \in C([0,1],D) \ : \ f(0) = f(1) = 0\}$$

which gives the following six-term exact sequence:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(B) & \longrightarrow & K_0(B/I) \cong \mathbb{Z}^p \\ \big\uparrow & & & & \big\downarrow {\scriptstyle \partial_0} \\ 0 & \longleftarrow & K_1(B) & \longleftarrow & K_1(I) \cong \mathbb{Z}^l \end{array}$$

As noted by Elliott, $K_0(B)$ is not only a subgroup of $K_0(B/I)$, but the pre-order on $K_0(B)$ is determined by the pre-order on $K_0(B/I)$, which means the following:

$$K_0(B)^+ = K_0(B) \cap K_0(B/I)^+$$

Now the "grade one building blocks" are defined as the one-dimensional NCCW-complexes with surjective boundary map $\partial_0 : K_0(B/I) \to K_1(I)$, This implies that $K_1(B)$ is zero, and that $K_0(B)$ is a subgroup of $K_0(B/I)$ with a (torsion-)free quotient. The latter means exactly that $K_0(B)$ is a relatively divisible[13] subgroup of $K_0(B/I)$. Note also that for any inclusion $\iota : G_0 \subset \mathbb{Z}^p$ (as a relatively divisible subgroup) we can find a grade one building block B such that $K_0(B) \to K_0(B/I)$ is isomorphic to ι.

We want to construct simple C*-algebras as direct limits of building blocks. With grade one building blocks we can only construct such algebras with zero K_1-group and torsion-free K_0-group. It is also not enough to consider arbitrary one-dimensional NCCW-complexes, since these cannot provide us with torsion in the K_0-group. We have to consider two-dimensional NCCW-complexes, and we will recall the construction of these "grade two building blocks":

7.3: Grade two building blocks. (see [**Ell96**])
Let $B = C([0,1],D) \oplus_{D\oplus D} C$ be a grade one building block (we use the notation from above), which comes with a surjective map $\pi : B \to C$. If we tensor everything with M_2, we get a surjective map $M_2(\pi) : M_2(B) \to M_2(C)$. Let B_1 be the following sub-C*-algebra of $M_2(B)$:

$$B_1 := \{f \in M_2(B) \ : \ M_2(\pi)(f) \in \left(\begin{smallmatrix} * & \cdot \\ \cdot & * \end{smallmatrix}\right) \subset M_2(C)\}$$

That means B_1 is the preimage under $M_2(\pi)$ of the diagonal matrices in $M_2(C)$. We have a surjective map $B_1 \to C \oplus C$, and in fact B_1 is again a one-dimensional NCCW-complex as follows

[13]A subgroup $H \leq G$ of a group G is called relatively divisible (in H), if the following implications holds (for all $n \geq 1$, $g \in G$): $ng \in H \Rightarrow g \in H$

$$\begin{array}{ccc} B_1 & \xrightarrow{M_2(\pi)} & C \oplus C \\ \big\downarrow & & \Big\downarrow \left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right), \left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu \end{smallmatrix}\right)\right) \\ C([0,1], M_2(D)) & \xrightarrow{\partial} & M_2(D) \oplus M_2(D) \end{array}$$

where ∂ is the map $f \mapsto (f(0), f(1))$. Note that $C \oplus C = \bigoplus_{i=1}^{p}(M_{n_i} \oplus M_{n_i})$.

Consider the two-dimensional NCCW-complex $M_{n_i} \otimes C(\mathbb{T}) \otimes \mathrm{Int}_{k_i}$, which we view as

$$\{g : \mathbb{T} \times [0,1] \to M_{n_i} \otimes M_{k_i} \ : \ g(t,0), g(t,1) \in M_{n_i} \otimes 1_{M_{k_i}} \text{ for all } t \in \mathbb{T}\}$$

Define a map $\gamma_i : M_{n_i} \otimes C(\mathbb{T}) \otimes \mathrm{Int}_{k_i} \to M_{n_i} \oplus M_{n_i}$ by $g \mapsto (a,b)$ if $g(1,0) = a \otimes 1_{M_{k_i}}$ and $g(1,1) = b \otimes 1_{M_{k_i}}$ (where $1 \in \mathbb{T}$ is some fixed point).

Now, we attach this space to the pair $(M_{n_i} \oplus M_{n_i}) \in C \oplus C$ via a pullback over the map γ_i. We do this for every i, i.e. for some numbers $k_1, \ldots, k_p \geq 1$, and the resulting pullback A is:

$$\begin{array}{ccc} A & \twoheadrightarrow & \bigoplus_{i=1}^{p}(M_{n_i} \otimes C(\mathbb{T}) \otimes \mathrm{Int}_{k_i}) =: A/I \\ \big\downarrow & & \big\downarrow \bigoplus_i \gamma_i \\ B_1 & \xrightarrow[]{M_2(\pi)} \!\!\!\!\twoheadrightarrow & \bigoplus_{i=1}^{p}(M_{n_i} \oplus M_{n_i}) \end{array}$$

It can be checked that this pullback is a two-dimensional NCCW-complex, using that B_1 is also a NCCW-complex.

As indicated in the diagram, there is a (canonical) ideal I, which is the kernel of the map from A to the attached spaces, which in turn is isomorphic to the kernel of the map $M_2(\pi)$, i.e.

$$I \cong \{f \in B_1 \subset C([0,1], D) \ : \ f(0) = f(1) = 0\}$$

and this gives (again) a six-term exact sequence:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \cong \bigoplus_{i=1}^{p}(\mathbb{Z}^p \oplus_{\mathrm{str}} \mathbb{Z}_{k_i}) \\ \big\uparrow & & & & \big\downarrow \partial_0 \\ K_1(A/I) \cong \bigoplus_{i=1}^{p}(\mathbb{Z}^p \oplus \mathbb{Z}_{k_i}) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \cong \mathbb{Z}^l \oplus \mathbb{Z}^l \end{array}$$

Here we used that $K_0(C(\mathbb{T}) \otimes \mathrm{Int}_k) \cong \mathbb{Z} \oplus_{\mathrm{str}} \mathbb{Z}_k$ and $K_1(C(\mathbb{T}) \otimes \mathrm{Int}_k) \cong \mathbb{Z} \oplus \mathbb{Z}_k$. We are looking for such algebras A with surjective boundary map $\partial_0 : K_=(A/I) \to K_1(I)$, and we will call them the "grade two building blocks". Then there is a sort exact sequence

$$0 \to K_0(A) \to K_0(A/I) \to K_1(I) \to 0$$

which identifies $K_0(A)$ with a relatively divisible (pre-ordered) subgroup of $\bigoplus_{i=1}^{p}(\mathbb{Z}^p \oplus_{\mathrm{str}} \mathbb{Z}_{k_i})$, and also $K_1(A) \cong K_1(A/I) \cong \bigoplus_{i=1}^{p}(\mathbb{Z}^p \oplus \mathbb{Z}_{k_i})$.

Conversely, for any such data there exists a grade two building block.

7.4: Altered grade two building blocks. (compare [**Ell96**])
For our purposes we need a variant of the above grade two building blocks. Namely, instead of $M_{n_i} \otimes C(\mathbb{T}) \otimes \mathrm{Int}_{k_i}$ we want to attach $M_{n_i} \otimes C(\mathbb{T})$ or $M_{n_i} \otimes \mathrm{Int}_{k_i}$ to B_1. For the indices $I_1 \subset I = \{1, \dots, p\}$ where we attach $M_{n_i} \otimes C(\mathbb{T})$ we do not want to split the "point" M_{n_i} in C. For this we can either change the definition of B_1, or define the map $\gamma_i : M_{n_i} \otimes C(\mathbb{T}) \to M_{n_i} \oplus M_{n_i}$ as being evaluation at $1 \in \mathbb{T}$ followed be the diagonal embedding $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. The results are isomorphic, and define a one-dimensional NCCW-complex

$$\begin{array}{ccc} A & \twoheadrightarrow & \bigoplus_{i \in I_1}(M_{n_i} \otimes C(\mathbb{T})) \oplus \bigoplus_{i \in I_2}(M_{n_i} \otimes \mathrm{Int}_{k_i}) =: A/I \\ \downarrow & & \downarrow \bigoplus_{i \in I_2} \gamma_i \oplus \bigoplus_{i \in I_2} \gamma_i \\ B_1 & \xrightarrow{M_2(\pi)} & \bigoplus_{i \in I_1}(M_{n_i} \oplus M_{n_i}) \bigoplus_{i \in I_2}(M_{n_i} \oplus M_{n_i}) \end{array}$$

where $I = I_1 \sqcup I_2$.

The canonical ideal I defines a six-term exact sequence in K-theory and an index map. The "altered grade two building blocks" are the algebras just described with surjective index map.

To summarize, the building block algebras have the following K-theory (where $k_i \geq 0$, and $\mathbb{Z}_0 = \mathbb{Z}$):

building block	$K_0(A)$ is rel. divisible subgroup of	$K_1(A)$
grade one	$\bigoplus_{i=1}^p(\mathbb{Z}) = \mathbb{Z}^p$	0
grade two	$\bigoplus_{i=1}^p(\mathbb{Z} \oplus_{\mathrm{str}} \mathbb{Z}_{k_i})$	$\bigoplus_{i=1}^p(\mathbb{Z} \oplus \mathbb{Z}_{k_i})$
altered grade two	$\bigoplus_{i=1}^p(\mathbb{Z}) = \mathbb{Z}^p$	$\bigoplus_{i=1}^p(\mathbb{Z} \oplus \mathbb{Z}_{k_i})$

7.3. Detecting the ASH-dimension in the Elliott invariant.

We focus on the non-infinite case, which means we assume that the cone of traces is non-zero. The infinite case is uninteresting for us, since these algebras will never have finite ASH-dimension.

7.5: Admissible (stable) Elliott invariant. (compare [**Ell96**] and [**Ell95**])
Let us clarify which Elliott invariants we are looking for. There are of course necessary conditions, namely all (known) properties that the Elliott invariant of a separable, stable, simple, nuclear C*-algebras enjoys. Besides these we only add the property of "weak unperforation". We formulate this property for the pairing (and not the K_0-group). As pointed out by Elliott, this has the advantage that we can study at the same time stably projectionless C*-algebras and stably unital, stably finite C*-algebras.

First, to a pairing $< .,. >: G_0 \times C \to \mathbb{R}$ we may define maps $r : C \to \mathrm{Fct}(G_0)$ and $\rho : G_0 \to \mathrm{Fct}(C)$ (where by $\mathrm{Fct}(X)$ we just mean all functions $X \to \mathbb{R}$) simply by $r(\tau) = (g \mapsto < g, \tau >)$ and $\rho(g) = (\tau \mapsto < g, \tau >)$.

We say $(G_0, G_1, C, < .,. >)$ is an **admissible stable** Elliott invariant if:

(1) G_0 is a countable, simple, pre-ordered, abelian group
(2) the dimension range is equal to G_0^+ (this is why we call it *stable* Elliott invariant)

(3) G_1 is a countable, abelian group
(4) C is a non-empty, topological convex cone with a compact, convex base that is a metrizable Choquet simplex
(5) ρ is an order-homomorphism to $\mathrm{Aff}_0(C)$, the continuous affine maps $C \to \mathbb{R}$ that are zero at $\tau = 0$. Recall that $\mathrm{Aff}_0(C)$ has the strict ordering, i.e. for any $g \in G_0^+ \setminus \{0\}$ we have not only $\rho(g)(\tau) \geq 0$ for all $\tau \in C$ but even $\rho(g)(\tau) > 0$, i.e. $\rho(g) \gg 0$
(6) r is a continuous, affine map to $\mathrm{Pos}(G_0)$, the space of order-homomorphisms $G_0 \to \mathbb{R}$ with the topology of pointwise convergence. If $G_0^+ \neq 0$ (the stably unital case (F_1)), then r is assumed surjective

Besides these necessary conditions we also impose the following:

(7) the pairing is weakly unperforated in the sense that $\rho(g) \gg 0$ implies $g \geq 0$ for all $g \in G_0$. Together with the above this means $G_0^+ = \rho^{-1}(\mathrm{Aff}_0(C)^+)$

7.6. Remark: Weak unperforation.
As noted by Elliott, the above definition of "weak unperforation" for the pairing generalizes the concept of "weak unperforation" for (simple) ordered groups. In this way we can treat all stably finite, simple C*-algebras (including the stably projectionless algebras) at once. Indeed, "weak unperforation" of the pairing means exactly that the order on G is determined by the map

$$\rho : G \to \mathrm{Aff}_0(C)$$

But $\mathrm{Aff}_0(C)$ is unperforated. Since G is simple, $\rho(g) \gg 0$ for each $g > 0$. Let $g \in G$, $n \geq 1$ with $ng > 0$. Then $\rho(ng) = n\rho(g) \gg 0$, which implies $\rho(g) \gg 0$, whence $g > 0$. Since G is simple, this means it is weakly unperforated.

The converse is also true: We may suppose G has a order unit u. The order of G is induced by that on G/G_{tor}, so we may also assume G is unperforated group. By [**EHS80**, Lma 1.4] the order for simple, unperforated groups is induced by the map $G \to \mathrm{Aff}\, S_u(G)$ (with $S_u(G)$ the state space of G), and the map ρ factors through $\mathrm{Aff}\, S_u(G)$.

The next theorem shows that all admissible Elliott invariants are in the range of the Elliott invariant, i.e. they can be realized by some C*-algebra.

7.7. Theorem: [**Ell96**, Thm 5.2.3.2]
Let $\mathcal{E}$ be an admissible Elliott invariant. Then there exists a separable, stable, simple C-algebra A in $\underline{\mathrm{ASH}}(2)'$ such that $\mathrm{Ell}(A) = \mathcal{E}$.*

Note that these algebras are automatically nuclear (since nuclearity is preserved by direct limits, and subhomogeneous C*-algebras are nuclear). In the stably unital case they are also $\mathcal{Z}$-stable. To see this, let A be a separable, stable, simple C*-algebra A in $\underline{\mathrm{ASH}}(2)'$, and $p \in A$ a projection. Let $B := pAp$, which is a unital, separable, simple C*-algebra with finite decomposition rank (namely $\mathrm{dr}(B) \leq \mathrm{dr}(A) \leq 2$), and then [**Win08**, Thm 5.1] implies that B is $\mathcal{Z}$-stable. But then with Brown's stabilization theorem we get $A \otimes \mathcal{Z} \cong (pAp) \otimes \mathbb{K} \otimes \mathcal{Z} \cong (pAp) \otimes \mathbb{K} \cong A$, hence A is also $\mathcal{Z}$-stable.

7.8. Theorem: *Let $\mathcal{E}$ be an admissible Elliott invariant with G_0 torsion-free. Then there exists a separable, stable, simple C*-algebra A in* $\underline{\mathrm{ASH}}(1)'$ *such that* $\mathrm{Ell}(A) = \mathcal{E}$.

Proof:
This is a modification of the proof of [**Ell96**, Thm 5.2.3.2] using altered grade two building blocks instead of grade two building blocks.

As explained in the beginning of this section, the idea is as follows: We first decompose the invariant into a direct limit of altered building bock groups (using 7.1). Then for each such group we construct an altered grade two building block algebra and assembly them into a direct system of C*-algebras. Finally we deform the limit so that it becomes simple.

Step 1. We cannot apply 7.1 directly to $G := G_0 \oplus_{\mathrm{str}} G_1$, since it may not have the Riesz interpolation property. Instead we embed G into such a group $H = H_0 \oplus_{\mathrm{str}} H_1$, then decompose H into a direct limit, and pull this decomposition back in order to decompose G.

To construct this group H we use the map $\rho : G_0 \to \mathrm{Aff}_0 C$. We factor it through a group H_0 that has dense image in $\mathrm{Aff}_0 C$:

$$\begin{array}{ccc} G_0 & \xrightarrow{\rho} & \mathrm{Aff}_0 C \\ {\scriptstyle\alpha}\downarrow & \nearrow{\scriptstyle\beta} & \\ H_0 & & \end{array}$$

Such an H_0 exists. For example set $H_0 := G_0 \oplus K$ (as a group, so far without order), where K is a dense countable subgroup of $\mathrm{Aff}_0 C$, and let $\beta : H_0 \to \mathrm{Aff}_0 C$ be defined as ρ on G_0 and as the inclusion map ι on K. Order H_0 using the map β, i.e

$$\begin{aligned} H_0^+ &:= \beta^{-1}(\mathrm{Aff}_0(C)^+) \\ &= \{(g,k) \in G_0 \oplus K \ : \ \rho(g) + \iota(k) \gg 0\} \cup \{(0,0)\} \end{aligned}$$

Since the pairing of the Elliott invariant is weakly unperforated, the order on G_0 is determined by ρ. It follows that it is also determined by α, i.e.:

$$G_0^+ = \alpha^{-1}(H_0^+) = G_0 \cap H_0^+$$

By [**EHS80**, Lma 3.1] H_0 is a simple, unperforated group with the Riesz interpolation property. Then $H_* := H_0 \oplus_{\mathrm{str}} G_1$ is a simple, weakly unperforated ordered group with the Riesz interpolation property (by C.12). Note that H_* will be unperforated only if $H_1 = G_1$ is torsion-free.

However G_0 is torsion-free (by assumption), and therefore 7.1 gives us a decomposition $H_* \cong_{\mathrm{ord}} \varinjlim_k H_*^k$ with each H_*^k of the form

$$H_*^k = \bigoplus_{i=1}^{r_k} (\mathbb{Z} \oplus_{\mathrm{str}} \mathbb{Z}_{[k,i]})$$

(for some numbers $[k,i] \geq 1$).

Let $H_*^k = H_0^k \oplus H_1^k$ be the grading (which does not give the ordering as strict order). Recall that there is a map $G_0 \to H_0$ which we now use to pull the limit decomposition of H_0 back to G_0:

$$\begin{aligned} G_0^k &:= \alpha^{-1}(H_0^k) \\ (G_0^k)^+ &:= \alpha^{-1}((H_0^k)^+) \end{aligned}$$

Since the pre-order of G_0 is determined by α, we get that G_0 is order-isomorphic to $\varinjlim_k G_0^k$ and the following diagram commutes:

$$\begin{array}{ccccccc} G_0^1 & \longrightarrow & G_0^2 & \longrightarrow \cdots \longrightarrow & G_0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} \\ H_0^1 & \longrightarrow & H_0^2 & \longrightarrow \cdots \longrightarrow & H_0 \end{array}$$

This shows that each subgroup G_0^k is relatively divisible in H_0^k (since the quotient H_0^k/G_0^k is a subgroup of the torsion-free group $H_0/G_0 \cong K$, and therefore torsion-free itself). We let further $G_1^k := H_1^k$. Then $G_0^k \oplus G_1^k$ is an altered grade two building block group.

Step 2. Next, for each k we can find a corresponding altered grade two building block C*-algebra A_k with canonical ideal I_k such that $K_1(A_k) \cong K_1(A_k/I_k) \cong G_1^k$, and such that the following diagram commutes (and the rows are exact):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(A_k) & \longrightarrow & K_0(A_k/I_k) & \longrightarrow & K_1(I_k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & G_0^k & \longrightarrow & H_0^k \cong \mathbb{Z}^{r_k} & \longrightarrow & H_0^k/G_0^k & \longrightarrow & 0 \end{array}$$

We have $K_*(A_k/I_k) \cong H_*^k$.

As described by Elliott, the A_k can be chosen in such a way that we can also find maps $\gamma_k : A_k \to A_{k+1}$ that map I_k to I_{k+t} and that induce in K-theory maps $K_0(A_k) \to K_0(A_{k+1})$, $K_*(A_k/I_k) \to K_*(A_{k+1}/I_{k+1})$ equal to the maps $G_0^k \to G_0^{k+1}$ and $H_*^k \to H_0^{k+1}$ from the limit decomposition of the invariant.

Let I be the direct limit of the ideals I_k. As shown by Elliott it can be arranged that $T(A) = T(A/I)$ and such that the projections in A/I separate the traces. Then $T(A) = T(A/I) \cong \operatorname{Pos}(K_0(A/I)) \cong C$. Also, A has the desired pairing between $K_0(A)$ and $T(A)$. For this, we may check that the map $\rho_A : K_0(A) \to \operatorname{Aff}_0 T(A)$ has the right form. This can be seen in the following commutative diagram:

$$\begin{array}{ccccc} G_0 & \xrightarrow{\ \alpha\ } & H_0 & \xrightarrow{\ \beta\ } & \operatorname{Aff}_0(C) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ K_0(A) & \longrightarrow & K_0(A/I) & \longrightarrow & \operatorname{Aff}(\operatorname{Pos}(K_0(A/I))) \\ \downarrow{\scriptstyle \rho_A} & & \downarrow{\scriptstyle \rho_{A/I}} & \swarrow{\scriptstyle \cong,\ (r_{A/I})^{\hat{}}} & \\ \operatorname{Aff}_0 T(A) & \xrightarrow{\ \cong\ } & \operatorname{Aff}_0 T(A/I) & & \end{array}$$

Step 3. In the last step we have to deform the maps $\gamma_k : A_k \to A_{k+1}$ in order to turn the direct limit into a simple C*-algebra. This can be done in the same way as Elliott does it. Indeed, he mentions that his procedure works also for grade one building blocks with matrices over $C(\mathbb{T})$ or Int_k attached. But these are exactly our altered grade two building blocks.

At the end, tensor the resulting C*-algebra with $\mathbb{K}$ to make it stable. □

7.9: Admissible unital Elliott invariant.
Let us now turn to the unital versions. We say (G_0, G_1, T, r) is an **admissible unital** Elliott invariant if:

(1) G_0 is a countable, simple, weakly unperforated, ordered, abelian group with a (positive) order unit $u \in G_0$
(2) G_1 is a countable, abelian group
(3) T is a (non-empty) metrizable Choquet simplex
(4) $r : T \to S_u(G_0)$ is a surjective, affine, continuous map

7.10. Theorem: *Let $\mathcal{E}$ be a admissible unital Elliott invariant. Then there exists a separable, unital, simple C*-algebra A in $\underline{\text{ASH}}(2)'$ such that* $\text{Ell}(A) = \mathcal{E}$*. If G_0 is torsion-free, we can even find A in $\underline{\text{ASH}}(1)'$*

Proof:
[**Ell96**, Thm 5.2.3.2] see also [**Rør02**, Thm 3.4.4, p.63] To the given invariant $\mathcal{E} = (G_0, G_0^+, u, G_1, T, r)$ we may construct an admissible stable Elliott invariant $\mathcal{E}' = (G_0', (G_0')^+, G_1', C, < .,. >')$ as follows:

(1) $G_0' = G_0$ (as pre-ordered groups), $G_1 = G_1$
(2) $C = T \times [0, \infty)/T \times \{0\}$
(3) $< g, (\lambda, \tau) >' := \lambda < g, \tau >$

By 7.7 and 7.8 we may construct a stable C*-algebras A with $\text{Ell}(A) = \mathcal{E}'$. Let p be a projection with $[p]_0 = u \in K_0(A)^+ \cong G_0^+$. We may find $p \in A$ since A is stable. Then $B := pAp$ is a separable, unital, simple C*-algebra with $\text{Ell}(B) = \mathcal{E}$. Also, $B \in \underline{\text{ASH}}(n)'$ if $A \in \underline{\text{ASH}}(n)'$. □

7.11. Corollary: *Assume the Elliott conjecture is true for the class $\underline{\text{C}}$ of separable, nuclear, unital, stably finite, simple, $\mathcal{Z}$-stable C*-algebras. Let A be in $\underline{\text{C}}$. Then the following are equivalent:*

(1) A is in $ASH(1)'$
(2) A is in $ASH(1)$
(3) $K_0(A)$ is torsion-free

Proof:
(1)⇒(2) is clear. (2)⇒(3): Let $A = \varinjlim_k A_k$ with $A_k \in \underline{\text{SH}}(1)$. Then $\text{sr}(A_k) = 1$ by 3.18 and then $K_0(A_k)$ is torsion-free by 4.14. Since torsion-freeness is preserved by limits (see C.1), we have $K_0(A)$ torsion-free.

(3)⇒(1): Let $\mathcal{E} := \text{Ell}(A)$, which is an admissable unital Elliott invariant (and G_0 torsion-free). By the above theorem 7.10 there exists a separable, unital, simple C*-algebras $A' \in \underline{\text{ASH}}(1)'$ such that $\text{Ell}(A') \cong \mathcal{E}$. We already remarked that A' will be also nuclear and $\mathcal{Z}$-stable, i.e. $A' \in \underline{\text{C}}$. If the Elliott conjecture is true for $\underline{\text{C}}$, then we will have $A \cong A'$, hence $A \in \underline{\text{ASH}}(1)'$. □

We may draw the following interesting consequence which allows us to read of the ASH-dimension of a (simple, classified) C*-algebra in its K-theory.

7.12. Theorem: *Assume the Elliott conjecture is true for the class* $\underline{\mathrm{C}}$ *of nuclear, separable, unital, stably finite, simple,* $\mathcal{Z}$*-stable C*-algebras. Let* A *be in* $\underline{\mathrm{C}}$*. Then:*

$$\dim_{ASH}(A) = \dim_{ASH'}(A) \leq 2$$

and we can read of the ASH-dimension in the K-theory as follows:

1.) $\dim_{ASH}(A) = 0$ *if and only if* $K_0(A)$ *is a simple dimension group,* $K_1(A) = 0$ *and* r_A *is a homeomorphism*

2.) $\dim_{ASH}(A) \leq 1$ *if and only if* $K_0(A)$ *is torsion-free*

Proof:
From 7.7 we get $\dim_{ASH'}(A) \leq 2$. Then also $\dim_{ASH}(A) \leq \dim_{ASH'}(A) \leq 2$. Hence, the ASH- and ASH'-dimension are at most two.

Now, 7.11 implies that $\dim_{ASH}(A) \leq 1$ if and only if $\dim_{ASH'}(A) \leq 1$, and hence $\dim_{ASH}(A) = 2$ if and only if $\dim_{ASH'}(A) = 2$. We also have $\dim_{ASH}(A) = 0$ if and only if $\dim_{ASH'}(A) = 0$ by 5.7. Together, the ASH- and ASH'-dimension agree.
1.) We have $\dim_{ASH}(A) = 0 \Leftrightarrow A$ is an AF-algebra (see 5.7). For a simple, unital AF-algebra A we have $K_0(A)$ a simple dimension group, $K_1(A) = 0$ and r_A a homeomorphism by [**Rør02**, Prop. 1.4.2, Cor 1.5.4, Prop 1.5.5, p.20ff]). Conversely to any such data exists a simple, unital AF-algebra A, which will be nuclear and $\mathcal{Z}$-stable, hence in $\underline{\mathrm{C}}$.
2.) follows also from 7.11. □

7.13. Remark: Let $\underline{\mathrm{C}}$ be the class of C*-algebras as in 7.12, i.e. the nuclear, separable, unital, stably finite, simple, $\mathcal{Z}$-stable C*-algebras. The Elliott conjecture for $\underline{\mathrm{C}}$ would not only imply that ASH- and ASH'-dimension agree for these algebras, but also that they are dimension theories in the sense of 1.1. This follows from remark 5.11 since the dimension is (a priori) ≤ 2.

Note that we have even more (if the Elliott conjecture holds for $\underline{\mathrm{C}}$), for example: Let A be a C*-algebras in $\underline{\mathrm{C}}$ that is locally approximated by type I C*-algebras with stable rank one. Then $A \in \underline{\mathrm{ASH}}(1)'$. Indeed it is conceivable that every (separable) type I C*-algebra with stable rank one is already itself in $\underline{\mathrm{ASH}}(1)'$ (or just $\underline{\mathrm{ASH}}(1)$).

7.14. Remark: For AH-algebras the situation is more complicated. From 2.12 we can draw the following general conclusions for any C*-algebras A:

(1) If $\dim_{AH}(A) \leq 1$, then $K_0(A)$ and $K_1(A)$ are torsion-free
(2) If $\dim_{AH}(A) \leq 2$, then $K_1(A)$ is torsion-free

It is possible that the converses hold (and that $\dim_{AH}$ is always ≤ 3) for the C*-algebras considered in 7.12, i.e. for the $\mathcal{Z}$-stable case (F_1) of the Elliott conjecture.

7.15. Question: Let A be a nuclear, separable, unital, stably finite, simple, $\mathcal{Z}$-stable C*-algebra. Does the following hold?:

$$\mathrm{dr}(A) = \dim_{ASH}(A)$$

We know that $\mathrm{dr}(A) \leq \dim_{ASH}(A)$ in general. The above results offer the possibility to prove that decomposition rank and ASH-dimension agree by showing that decomposition rank one implies torsion-freeness of the K_0-group. In fact, this was one of the motivations for us to prove the above results.

Appendix

A. Classical dimension and shape theory

In this section we give an introduction to the most important dimension theory for commutative spaces: the covering dimension. It generalizes the combinatorial dimension of polyhedra to all spaces. Then, we will recall some concepts from commutative shape theory that allow us to extend results from CW-complexes to all spaces. This will be used in the next section about the Chern character.

A.1. Dimension theory.

Classical dimension theory extends the intuitive concept of dimension (e.g. that $\mathbb{R}^n$ is n-dimensional) to all topological spaces. In a first step of generalization, there is a natural dimension for every polyhedron. This is extended to all topological spaces by using suitable approximations by polyhedra (usually formalized in terms of open coverings and refinements). It turns out that one obtains different concepts depending on which coverings are used (finite coverings, or numerable coverings) and that both concepts agree for normal spaces.

A.1: Polyhedra.
Let us begin by recalling the definitions and basic properties concerning polyhedra, which are particularly nice spaces since they are given in terms of combinatorial data. They come with a natural concept of combinatorial dimension (compare [**Nag70**, Chap. 1.1] and [**MS82**, Appendix 1.§2, p.289ff]):

1. An **abstract simplicial complex** is a pair (K^0, K) where K^0 is a set (the vertices) and K is a family of *finite* subsets of K^0 (the simplices) such that $\emptyset \in K$ and every subset of an element in K belongs to K (i.e. $s \in K, t \subset S \Rightarrow t \in K$).
2. The **combinatorial dimension** of a simplex s, denoted by $\dim s$, is per definition its cardinality minus one, i.e. a simplex with k vertices has dimension $k-1$. The combinatorial dimension of an abstract simplicial complex K, denoted by $\dim K$, is $\dim K = \max\{\dim s \,:\, s \in K\}$.
3. The **geometric realization** $|K|$ of an abstract simplicial complex K is the set of (formal) points $x = \sum_{v\in s} \lambda_v v$ where the λ_v are nonnegative real numbers such that $\sum_{v\in s} \lambda_v = 1$ and s runs over all simplices $s \in K$. For a simplex $s \in K$ let $|s|$ be the set of points in $|K|$ with $\lambda_v = 0$ for all $v \notin s$. The dimension of a simplex s is per definition its cardinality, i.e. the number of vertices it contains.

4. The **weak topology** on $|K|$ is defined as follows: Choose for any simplex $s \in K$ an enumeration $s = \{v_1^s, \dots, v_n^s\}$ and consider the map $\phi_s : |s| \to \mathbb{R}^n$, $x = \sum_i \lambda_i v_i^s \mapsto \sum_i \lambda_i e_i$ where $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.
   Now, the initial topology[14] induced by the maps ϕ_s is called the weak topology.
5. A **polyhedron** is the geometric realization of an abstract simplicial complex with the weak topology.

The dimension of a polyhedron is (by definition) the combinatorial dimension of the underlying abstract simplicial complex.

A.2: Coverings of a space

Let X be a space. A **covering** (or "cover" for short) of X is a collection $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets $A_i \subset X$ which cover X, i.e. $\bigcup_i A_i = X$. A **refinement** of a cover $\mathcal{B}$ is another cover $\mathcal{A}$ of X s.t. each set in $\mathcal{A}$ is contained in some set in $\mathcal{B}$. This is denoted by $\mathcal{A} \leq \mathcal{B}$, i.e.

$$\mathcal{A} \leq \mathcal{B} \quad :\Leftrightarrow \quad \forall\, A \in \mathcal{A}\ \exists\, B \in \mathcal{B} \,:\, A \subset B$$

Let us recall some definitions concerning coverings (and refinements). They can be defined for any collection of subsets, so assume $\mathcal{A} = \{A_i\}_{i \in I}$ is a collection of subsets of a topological space X.

(1) $\mathcal{A}$ is called **open** (**closed**) if each set in $\mathcal{A}$ is open (closed) (in X). Analogously, $\mathcal{A}$ is called cozero[15] (zero) if each set in $\mathcal{A}$ is cozero (zero).
(2) $\mathcal{A}$ is called **finite** (countable) if the index set I is finite (countable)
(3) Further, $\mathcal{A}$ is **point finite** if each $x \in X$ belongs to only finitely sets in $\mathcal{A}$, and $\mathcal{A}$ is **locally finite** if each $x \in X$ has a neighborhood U that intersects only finitely many sets in $\mathcal{A}$.
(4) The **order** of $\mathcal{A}$, denoted by $\mathrm{ord}(\mathcal{A})$, is the least integer $n \geq 0$ such that the intersection of any $n{+}2$ distinct subsets of $\mathcal{A}$ is empty, i.e. $A_{i_0} \cap A_{i_1} \cap \ldots \cap A_{i_{n+1}} = \emptyset$ whenever $i_0, i_1, \dots, i_{n+1}$ are distinct indices in I.

A.3: Star covering, nerve.

Let $P = |K|$ be a polyhedron, and $v \in K^{(0)}$ a vertex. With view to A.1 we consider the map that assigns to each $x = \sum_{w \in s} \lambda_w w$ the value λ_v. This assignment is continuous and we denote it by $\lambda_v : |K| \to \mathbb{R}$. The open set $\{x \in P \,:\, \lambda_v(X) > 0\}$ is called the star of v, and it is denoted $\mathrm{St}(v, P)$.

The family $\{\mathrm{St}(v, P) \,:\, v \in K^{(0)}\}$ is called the **star covering** of P. It is denoted by $\mathrm{St}(P)$, and provides each polyhedra in a natural way with an open covering. We will now see the converse, i.e. how to associate in a natural way to a covering a polyhedron.

Let X be a space, and $\mathcal{A} = \{A_i\}_{i \in I}$ a collection of subsets of X. The **nerve** of $\mathcal{A}$, denoted by $N(\mathcal{A})$, is the abstract simplicial complex (K^0, K) with vertices $K^0 = I$ and simplices $K = \{s \subset I \,:\, s \text{ is finite and } \bigcap_{i \in s} A_i \neq \emptyset\}$. The geometric realization of $N(\mathcal{A})$, denoted by $|N(\mathcal{A})|$, is also called nerve of $\mathcal{A}$.

---

[14]The initial topology on a set X with respect to mappings $f_i : X \to Y_i$ (to topological spaces Y_i) is the smallest (or weakest) topology such that all f_i become continuous. To be precise, a subset $M \subset X$ is open if for each point $m \in M$ there exists some i and open set $V \subset Y_i$ such that $m \in f_i^{-1}(V) \subset M$.

[15]A subset A of a topological space X is called cozero (zero) if there exists a continuous map $f : X \to [0,1]$ such that $A = f^{-1}((0,1])$ (such that $A = f^{-1}(\{0\})$). Sometimes a cozero-set is called a functionally open set, and a zero-set is called a functionally closed set.

Let $P = |K|$ be a polyhedron. Then there is a natural identification of K with the nerve $N(\mathrm{St}(P))$ of the star covering of P. This gives a natural homeomorphism $P \cong |N(\mathrm{St}(P))|$.

A.4: Partition of unity.
Let X be a space. A **partition of unity** on X is a collection of maps $\varphi_\alpha : X \to [0,1]$ ($\alpha \in A$) such that $\sum_\alpha \varphi_\alpha(x) = 1$ for each $x \in X$.[16]

Let $\Phi = (\varphi_\alpha)_{\alpha\in A}$ be a partition of unity. We assign to it in a canonical way an open covering $\mathcal{U}(\Phi) := \{\varphi_\alpha^{-1}((0,1])\}_{\alpha\in A}$. Then:

(1) Φ is called **locally finite** (**point finite**) if $\mathcal{U}(\Phi)$ is locally finite (point finite)
(2) If $\mathcal{V}$ is an open covering, then Φ is called **subordinate to** $\mathcal{V}$ if $\mathcal{U}(\Phi) \leq \mathcal{V}$.
(3) An open cover $\mathcal{U}$ is called **numerable** if it admits a *locally finite* partition of unity subordinate to it.

We denote by $Cov(X)$ the set of numerable, open coverings of X.

The following are equivalent for an open cover $\mathcal{U}$:

(1) $\mathcal{U}$ can be refined by a locally-finite, cozero covering
(2) $\mathcal{U}$ is numerable, i.e. it admits a subordinate locally-finite partition of unity

There are different definitions of covering dimension in the literature, and at least two of them are also often used. They are all similar in concept and roughly define the dimension of a space X to be $\leq n$ if each "nice" covering of X can be refined by a "nice" covering of order $\leq n+2$.

Depending on what we consider to be the "nice" coverings (e.g. the finite or numerable ones) we get different dimension theories. However, they agree for normal spaces (which includes all paracompact spaces).

A.5. Definition: *Let X be a topological space. The* ***Lebesgue covering dimension*** *of X is the least integer $n \geq 0$ (or ∞) such that every* finite *open cover $\mathcal{U}$ can be refined by a* finite *open cover $\mathcal{V} \leq \mathcal{U}$ such that* $\operatorname{ord}\mathcal{V} \leq n+1$.

The ***covering dimension*** *of X, denoted by* $\dim(X)$*, is the least integer ≥ 0 (or ∞) such that every* numerable *open cover $\mathcal{U}$ can be refined by a* numerable *open cover $\mathcal{V} \leq \mathcal{U}$ such that* $\operatorname{ord}\mathcal{V} \leq n+1$.

A.6. Proposition: *Let X be a normal space. Then the covering dimension of X agrees with its Lebesgue covering dimension.*

A.7. Remark: Many books on dimension theory (e.g. [**Eng95**, Def. 1.6.7, p.42], [**Nag70**, Def. 8-1, p.44]) use the term "covering dimension" to describe what we call Lebesgue covering dimension. It should be remarked however that these books consider only dimension theories of spaces that are at least normal (actually, much dimension theory is about metric spaces).

When considering "less good" spaces (like the dual spaces of C*-algebras) the definition in terms of numerable coverings as suggested by Morita in [**Mor75**] is more natural. It makes it possible to generalize well-known theorems for normal spaces

[16] The sum is understood as the least upper bound of the sums $\sum_{\alpha\in F}\varphi_\alpha(x)$ where $F \subset A$ ranges over all finite subsets of the index set A.

(like the countable sum theorem) to the realm of general spaces (see [**Nag80**]). Further, it provides a natural connection to maps into polyhedra and thus allows a nice characterizations of this dimension theory via (approximate) polyhedral resolutions (see [**Wat91**]).

A.8: Characterization of dim by fragility of maps

There is a characterization of covering dimension in terms of the fragility of maps $X \to \mathbb{R}^n$. In fact, the following are equivalent for a *compact* space X:

(1) $\dim(X) \leq n$
(2) for all maps $f : X \to \mathbb{R}^{n+1}$ and $\varepsilon > 0$ there exists a map $g : X \to \mathbb{R}^{n+1}$ such that $\|f - g\|_\infty < \varepsilon$ and $0 \notin \text{image}(g) = g(X)$

The second condition means that every map $f : X \to \mathbb{R}^{n+1}$ can be approximated arbitrarily closely by maps that miss the origin, i.e. we can perturb f a little bit to miss 0.

Let us look at an example: Take $X = \mathbb{R}$ and $f : X \to \mathbb{R}^1, f(t) = t$. If g is near to f, say $\|f - g\| < \epsilon$, then $g(2\epsilon) \geq f(2\epsilon) - \epsilon = \epsilon > 0$ and similarly $g(-2\epsilon) \leq -\epsilon < 0$. From the intermediate value theorem we get that $0 \in \text{image}(g)$. Thus, we cannot perturb f a little bit to miss 0 and indeed $\dim(\mathbb{R}) \neq 0$.

This characterization of dimension can be generalized to C*-algebras, which was first noted by Rieffel in [**Rie83**]. It leads to the notion of real and stable rank (see 1.5). To see the connection, say to the real rank, recall from 1.4 that

$$\text{Lg}_n(A)_{\text{sa}} := \{(a_1, \ldots, a_n) \in (A_{\text{sa}})^n \ : \ \sum_{i=1}^{n} a_i^2 \in A^{-1}\}$$

If $A = C(X)$, then:

$$\begin{aligned} A_{\text{sa}}^n &= \text{ all maps } f : X \to \mathbb{R}^n \\ \text{Lg}_n(A)_{\text{sa}} &= \text{ all maps } g : X \to \mathbb{R}^n \text{ such that } 0 \notin \text{image}(g) = g(X) \end{aligned}$$

The real rank is (per definition) the least integer such that $Lg_{n+1}(A)_{\text{sa}}$ is dense in A_{sa}^{n+1}. For $A = C(X)$ this value agrees with the covering dimension of X.

A.9: Čech-Cohomology

Since we do not assume our spaces to be CW-complexes, we will work with Čech-cohmology, and we will denote it by H^*. This of course agrees with the "usual" simplicial cohomology for CW-complexes. Let us recall the definitions:

Let X be a space, and G a (discrete) group. For the following compare [**Mor75**, §3]. Denote by $Cov_{\text{lf}}(X)$ the set of locally-finite, open, numerable coverings of X (see A.2 for definitions). We order $Cov_{\text{lf}}(X)$ by refinement. If $\mathcal{U} \in Cov(X)$ is a covering, then denote by ($\mathcal{N}$ the nerve of $\mathcal{U}$, which is a simplicial complex.

If $\mathcal{U}, \mathcal{V} \in Cov_{\text{lf}}(X)$ are two coverings, and $\mathcal{U}$ refines $\mathcal{V}$, denoted $\mathcal{U} \leq \mathcal{V}$, then there exists a surjective simplicial map $\varphi_{\mathcal{U}}^{\mathcal{V}} : \mathcal{N}(\mathcal{U}) \to \mathcal{N}(\mathcal{V})$ with the following property: If $\varphi_{\mathcal{U}}^{\mathcal{V}}$ maps a vertex $U \in \mathcal{U}$ to a vertex $V \in \mathcal{V}$, then $U \subset V$. Any two such maps are homotopic.

The k-th Čech-cohmology group of X with coefficients in G is defined as

$$H^k(X; G) := \varinjlim_{\mathcal{U}} H_s^k(\mathcal{N}(\mathcal{U}); G)$$

where the limit is indexed over $Cov_{\text{lf}}(X)$ ordered by refinement, H_s^n is simplicial cohomology, and for $\mathcal{U} \leq \mathcal{V}$ we map $H^n(\mathcal{N}(\mathcal{U}); G)$ to $H^n(\mathcal{N}(\mathcal{V}); G)$ via $H^n(\varphi_{\mathcal{U}}^{\mathcal{V}})$

(which is well-defined since any two such maps are homotopic, hence induce the same map on cohomology).

If X have finite covering dimension, then $H^n(X;G) = 0$ for $n \geq \dim(X)+1$ and all groups G. (see [**Mor75**, Lma 3.1.]).

We include the easy argument: A locally-finie, numerable cover $\mathcal{U}$ is of order $\leq n+1$ if and only if the nerve $N = \mathcal{N}(\mathcal{U})$ has dimension $\leq n$. In this case we know from simplicial cohomology that $H_s^k(N;G) = 0$ for any $k \geq n+1$ (and any group G).

Let $\dim(X) = n$, and $\mathcal{V} \in Cov_{\mathrm{lf}}(X)$. By A.5 we can find a numerable refinement $\mathcal{V}' \leq \mathcal{V}$ with $\mathrm{ord}(\mathcal{V}) \leq n+1$, by A.4 we can find a locally-finite, numerable refinement $\mathcal{V}'' \leq \mathcal{V}'$ with $\mathrm{ord}(\mathcal{V}'') \leq \mathrm{ord}(\mathcal{V}') \leq n+1$. Thus, we may compute the Čech-cohomology for $k \geq n+1$ as follows:

$$
\begin{aligned}
H^k(X;G) &:= \varinjlim \{H_s^k(\mathcal{N}(\mathcal{U});G) \ : \ \mathcal{U} \in Cov_{\mathrm{lf}}X\} \\
&= \varinjlim \{H_s^k(\mathcal{N}(\mathcal{V});G) \ : \ \mathcal{V} \in Cov_{\mathrm{lf}}X, \mathrm{ord}(\mathcal{U}) \leq n+1\} \\
&= \varinjlim_{\mathcal{V}} 0
\end{aligned}
$$

A.2. Some shape theory.

Shape theory was introduced by Borsuk [**Bor68**] in the late 1960s to study global properties of topological spaces. It was needed since homotopy theory yields useful results only for spaces with good local properties, e.g. CW-complexes. We are interested in this theory since the dual spaces of C*-algebras usually enjoy few properties that would make them applicable to homotopy theory. Indeed, already the dual spaces of commutative C*-algebras provide us with all (pointed) compact, Hausdorff spaces. For more general (non-commuative) C*-algebras it gets worse since their dual spaces need not even be Hausdorff.

The idea of shape theory is to approximate a badly-behaved topological space by nicer spaces, usually polyhedra. The approximation is done using special inverse systems (expansions or resolutions) of the nicer spaces. Shape theory then studies such inverse systems instead of the original space. The interested reader is referred the comprehensive book [**MS82**] by Mardešić and Segal.

A.10: Expansions. (see [**MS82**, I.4.1, p.48])
Let X be a topological space. A resolution of X is an inverse system $(X_i, p_{ij} : X_j \to X_i, I)$ of polyhedra in the homotopy category (i.e. $p_{ij}p_{jk}$ is only homotopic to p_{ik} for $k \geq j \geq i$), together with morphisms $p_i : X \to X_i$ such that for $j \geq i$: $p_{ij}p_j = p_i$, i.e. the following diagram commutes:

$$\begin{array}{ccc} X_j & \xrightarrow{p_{ij}} & X_i \\ {\scriptstyle p_j}\uparrow & \nearrow {\scriptstyle p_i} & \\ x & & \end{array}$$

and such that the following holds:

(i) Let P be a polyhedron, and $h : X \to P$ a map. Then there exists $i \in I$ and a map $f : X_i \to P$ such that the maps $h \simeq fp_i$

(ii) Let P be a polyhedron, and $f, f' : X_i \to P$ two maps such that $fp_i \simeq f'p_i$. Then there exists $j \geq i$ such that $fp_{ij} \simeq f'p_{ij}$.

Now every space has an expansion, and if $\dim(X) \leq n$, then X has an expansion with polyhedra of dimension $\leq n$. The following makes expansions so useful:

A.11. Proposition: *Let X be a topological space, and $(X_i, p_{ij} : X_j \to X_i, I)$ an expansion for X. Then for every space Y that is a polyhedron (or just homotopy equivalent to one), there is a natural isomorphism:*

$$[X; Y] \cong \varinjlim_i [X_i; Y]$$

where for $j \geq i$ we map $[X_i; Y]$ to $[X_j; Y]$ via $f \mapsto fp_{ij}$.

We get interesting applications, if we let Y be a space representing some cohmology theory, for example $Y = BU_\infty$. Then $K^0(X) \cong \varinjlim_i K^0(X_i)$, and similarly $H^k(X) \cong \varinjlim H^k(X_i)$.

B. The Chern character for low-dimensional spaces

In this section we construct a "truncated" Chern character

$$\chi^0 : K^0(X) \to H^0(X) \oplus H^2(X)$$

that is always a surjective group-homomorphism. For spaces of dimension ≤ 3 this map agrees with the usual Chern character and is an isomorphism.

We will also see that for spaces of dimension ≤ 3 the suspended Chern character takes values in $H^3(X)$ (instead of $H^3(X;\mathbb{Z}[1/2])$ or $H^3(X;\mathbb{Q})$) and we define a map

$$\chi^1 : K^1(X) \to H^1(X) \oplus H^3(X)$$

that agrees[17] with the usual Chern character and is also an isomorphism. Thus, for low-dimensional spaces the K-theory and cohomology agree "integrally", and not only up to torsion. Let us make this more precise:

B.1: Construction of the Chern character.

Assume X is a compact space. To a vector bundle ξ over X we can assign so-called characteristic classes (also called Chern classes) $c_k(\xi) \in H^{2k}(X)$ for $k \geq 0$, where by H^k we denote the k-th Čech-cohomology group. For the construction of Chern classes the reader is referred to [**MS74**], or [**Hus93**]. If the rank of ξ is (globally) $\leq n$, then $c_k(\xi) = 0$ for $k \geq n+1$.

The total Chern class of a vector bundle ξ is

$$c(\xi) := 1 + c_1(\xi) + c_2(\xi) + \ldots \in H^{\mathrm{ev}}(X) = \bigoplus_{k\geq 0} H^{2k}(X)$$

where $1 \in H^0(X)$ may also be defined as $c_0(\xi)$. Then the total Chern class maps the direct sum of two vector bundles to the "product" in cohomology, i.e. $c(\xi \oplus \eta) = c(\xi)c(\eta)$. Here the product in $H^{\mathrm{ev}}(X)$ is the graded cup-product, i.e. the formula means $c_k(\xi \oplus \eta) = \sum_{i+j=k} c_i(\xi) \cup c_j(\eta)$.

Now, the Chern character of a vector bundle is defined as

$$\mathrm{ch}(\xi) := \mathrm{rk}(\xi) + \sum_{k\geq 1} s_k(c(\xi))/k! \in H^{\mathrm{ev}}(X;\mathbb{Q}) = \bigoplus_{k\geq 0} H^{2k}(X;\mathbb{Q})$$

where each s_k is a certain polynomial in k variables (called Newton polynomial), and $s_k(c(\xi)) = s_k(c_1(\xi), \ldots, c_k(\xi))$. The first of these polynomials are $s_1(x_1) = x_1$ and $s_2(x_1, x_2) = x_1^2 - 2x_2$ and the reader is again referred to [**MS74**] for more details. Note that $\mathrm{rk}(\xi)$ means the rank of ξ, and we do not require the rank of a vector bundle to be globally constant, so it will be an element in $H^0(X)$ (instead of $\mathbb{N}$).

The construction is just made in the way that the Chern character becomes additive and multiplicative, i.e. $\mathrm{ch}(\xi\oplus\eta) = \mathrm{ch}(\xi)+\mathrm{ch}(\eta)$ and $\mathrm{ch}(\xi\otimes\eta) = \mathrm{ch}(\xi)\,\mathrm{ch}(\eta)$. If we organize the vector bundles over X in $K^0(X)$, then the Chern character is indeed a ring-homomorphism

$$\mathrm{ch}^0 : K^0(X) \to H^{\mathrm{ev}}(X;\mathbb{Q})$$

and it is a ring-isomorphisms after tensoring with $\mathbb{Q}$.

[17] It agrees in the sense that χ^0 composed with the natural map $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Q})$ (induced by the inclusion $\mathbb{Z} \to \mathbb{Q}$) is the usual Chern character $\mathrm{ch}^0 : K^0(X) \to H^{\mathrm{ev}}(X;\mathbb{Q})$

B.2: The suspended Chern character.

Recall that the cone of a space X is defined as $CX = (X\times[0,1])/(X\times\{1\})$. It comes with a canonical embedding $X \to CX$ with sends $x \in X$ to $[(x,0)]$, the image of $(x,0) \in X\times[0,1]$ in CX. The suspension of X is the quotient $SX = CX/(X\times\{0\})$. The pair (CX, X) induces long exact sequences in K-theory and cohomology, and since CX is contractible it gives the following natural isomorphisms

$$\partial_K : K^1(X) \xrightarrow{\cong} K^0(CX, C) \xrightarrow{\cong} K^0(SX)$$

$$\partial_H : H^k(X; G) \xrightarrow{\cong} H^{k+1}(CX, X; G) \xrightarrow{\cong} H^{k+1}(SX; G)$$

The suspended Chern character is defined as $\mathrm{ch}^1 := \partial_H^{-1} \circ \mathrm{ch}^0_{SX} \circ \partial_K$ where ch^0_{SX} denotes the Chern character for SX. It is a group-homomorphism

$$\mathrm{ch}^1 : K^1(X) \to H^{\mathrm{odd}}(X;\mathbb{Q}) = \bigoplus_{k\geq 0} H^{2k+1}(X;\mathbb{Q})$$

which makes the following diagram commute:

$$\begin{array}{ccc} K^1(X) & \xrightarrow[\partial_K]{\cong} & K^0(SX) \\ \downarrow{\scriptstyle \mathrm{ch}^1} & & \downarrow{\scriptstyle \mathrm{ch}^0} \\ H^{\mathrm{odd}}(X;\mathbb{Q}) & \xrightarrow[\partial_H]{\cong} & H^{\mathrm{ev}}(SX;\mathbb{Q}) \end{array}$$

This group-homomorphism also becomes an isomorphism after tensoring with $\mathbb{Q}$. Thus, the Chern character is a map that identifies the K-theory of a space and its cohomology *up to torsion*, i.e. after tensoring with $\mathbb{Q}$. There are spaces where the torsion part of K-theory and cohomology disagree, e.g. in [**Bra00**] it is shown that $K^0(\mathbb{R}P^5) \cong \mathbb{Z}\oplus\mathbb{Z}_4$ while $H^{\mathrm{ev}}(\mathbb{R}P^5) \cong \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$. As we will shorty see, it is no accident that these examples are all high-dimensional.

We will now focus on spaces of low-dimension. In that case A.9 makes life easier. We get that for $\dim(X) \leq 2k$ the Chern classes $c_{k+1}, c_{k+2}, \ldots$ vanish for all vector bundles. In particular for $\dim(X) \leq 3$ all Chern classes except c_0 and c_1 vanish, and in this case the Chern character may be defined with values in $H^{\mathrm{ev}}(X) = H^0(X)\oplus H^2(X)$ instead of $H^{\mathrm{ev}}(X;\mathbb{Q})$. We define in general:

B.3. Definition: *Let X be a compact space. The **truncated Chern character** for X is the map*

$$\chi^0 : K^0(X) \to H^0(X)\oplus H^2(X)$$

defined as $\chi^0 = \mathrm{rk} + c_1$, i.e. $\chi([\xi]-[\eta]) := \mathrm{rk}(\xi) - \mathrm{rk}(\eta) + c_1(\xi) - c_1(\eta)$ for two vector bundles ξ, η.

B.4. Proposition: *Let X be a compact space. Then the truncated Chern character is a surjective group-homomorphism.*

Proof:

The line bundles over X are in one-one-correspondence with $H^2(X) \cong [X : \mathbb{T}]$, and formal differences of trivial bundles (with varying dimension of the fibre) generate all elements of $H^0(X)$. Thus, every element of $H^0(X)\oplus H^2(X)$ can be presented by formal difference of trivial and line bundles. □

The trick proving the following lemma was shown to me by Haija Moustafa. It is probably known to topologists, but we could not find it in the literature.

B.5. Lemma: *Let X be a compact space of dimension ≤ 3, and $a \in H^2(SX)$. Then there exists a unique $b \in H^4(SX)$ such that $a^2 = a \cup a = 2b$.*

Proof:

We will sometimes abbreviate the cup product $w_1 \cup w_2$ by the short notation $w_1 w_2$. For each k we have a short exact sequence

$$0 \to H^k(SX) \to H^k(X \times S^1) \to H^k(X) \to 0$$

and we view $H^k(SX)$ as a subgroup of $H^k(X \times S^1)$.

Let $p_1 : X \times S^1 \to X$ and $p_2 : X \times S^1 \to S^1$ be the obvious projections. The Künneth theorem gives an isomorphism $H^k(X \times S^1) \cong H^k(X) \otimes H^0(S^1) \oplus H^{k-1}(X) \otimes H^1(S^1)$ via the cross product of the projection maps. This means, for $a \in H^2(X \times S^1)$ there are elements $u_1 \in H^2(X), v_1 \in H^0(S^1), u_2 \in H^1(X), v_2 \in H^1(S^1)$ such that $a = w_1 + w_2$ for w_1, w_2 as follows:

$$w_1 = u_1 \times v_1 = p_1^*(u_1) \cup p_2^*(v_1) \in H^2(X \times S^1)$$
$$w_1 = u_2 \times v_2 = p_1^*(u_2) \cup p_2^*(v_2) \in H^2(X \times S^1)$$

Then :

$$\begin{aligned} a^2 &= (w_1 + w_2)(w_1 + w_2) \\ &= w_1^2 + w_1 w_2 + (-1)^{2\cdot 2} w_1 w_2 + w_2^2 \end{aligned}$$

and

$$\begin{aligned} w_1^2 &= p_1^*(u_1) p_2^*(v_1) p_1^*(u_1) p_2^*(v_1) \\ &= (-1)^{2\cdot 0} p_1^*(u_1) p_1^*(u_1) p_2^*(v_1) p_2^*(v_1) \\ &= p_1^*(u_1^2) p_2^*(v_1^2) \\ &= 0 && [\text{ since } u_1^2 \in H^4(X) = \{0\}] \\ w_2^2 &= p_1^*(u_2) p_2^*(v_2) p_1^*(u_2) p_2^*(v_2) \\ &= (-1)^{1\cdot 1} p_1^*(u_1^2) \cup p_2^*(v_1^2) \\ &= 0 && [\text{ since } v_2^2 \in H^2(S^1) = \{0\}] \end{aligned}$$

Therefore:

$$\begin{aligned} a^2 &= w_1^2 + w_1 w_2 + (-1)^{2\cdot 2} w_1 w_2 + w_2^2 \\ &= 2 w_1 w_2 \end{aligned}$$

We set $b = w_1 w_2 \in H^4(X \times S^1)$, which already lies in $H^4(SX)$ since $H^4(X) = \{0\}$.

Note that the decomposition $a = w_1 + w_2$ is unique (the Künneth theorem gives a direct sum), consequently b is unique. □

This means that for compact spaces of dimension ≤ 3 and $a_1 \in H^2(SX), a_2 \in H^4(SX)$ there exists a well-defined $c \in H^4(SX)$ such that $a_1^2 - 2a_2 = 2c$, namely $c := b + a_2$ where b is obtained from the lemma above applied to a_1. We may apply this to Chern classes and define:

B.6. Definition: *Let X be a compact space of dimension ≤ 3. We define the map*

$$\chi^1 : K^1(X) \to H^1(X) \oplus H^3(X)$$

as follows: Let $u \in K^1(X)$, and set $u' = \partial_K(u) \in K^0(SX)$. As explained, there is a well-defined $c \in H^4(SX)$ such that $c_1(u')^2 - 2c_2(u') = 2c$. Then $(c_1(u'), c) \in H^2(X) \oplus H^4(X)$ and we define $\chi^1(u) := \partial_H^{-1} \circ (c_1(u'), c) \in H^1(X) \oplus H^3(X)$.

B.7. Theorem: *Let X be a compact space of dimension ≤ 3. Then:*
1.) $\chi^0 : K^0(X) \to H^0(X) \oplus H^2(X)$ is a group-isomorphism
2.) $\chi^1 : K^1(X) \to H^1(X) \oplus H^3(X)$ is a group-isomorphism

Proof:
Step 1: CW-Complexes. We first prove the statement for CW-complexes by induction over their dimension. We use that CW-complexes are build up by attaching cells. For this we need that χ^0 and χ^1 are isomorphism on pt, S^1, S^2 and S^3. This can be computed directly and also comes out of the theory of the classical Chern character.

The statement is then true for 0-dimensional CW-complexes, since these are just finite sums of S^0. Assume χ^0, χ^1 are isomorphisms on CW-complexes of dimension $\leq l-1$, and let X^l be an l-dimensional CW-complex. The induction proceeds by considering the following commutative diagram, where the rows are exact and induced by the pair (X^l, X^{l-1}):

$$\begin{array}{ccccccccc} K^1(X^{l-1}) & \to & K^0(X^l, X^{l-1}) & \to & K^0(X^l) & \to & K^0(X^{l-1}) & \to & K^1(X^l, X^{l-1}) \\ \downarrow \chi^1 & & \downarrow \chi^0 & & \downarrow \chi^0 & & \downarrow \chi^0 & & \downarrow \chi^1 \\ H^{\mathrm{odd}}(X^{l-1}) & \to & H^{\mathrm{ev}}(X^l, X^{l-1}) & \to & H^{\mathrm{ev}}(X^l) & \to & H^{\mathrm{ev}}(X^{l-1}) & \to & H^{\mathrm{odd}}(X^l, X^{l-1}) \end{array}$$

We also consider its companion diagram where we interchange K^0 and K^1 as well as H^{ev} and H^{odd} (and of course χ^0, and χ^1). Then in both diagrams the most left and second from the right vertical arrows are isomorphisms by assumption of the induction. The second from the left and most right vertical arrows are isomorphisms since they are essentially defined on spheres since $K^i(X^l, X^{l-1}) \cong \widetilde{K}^i(X^l/X^{l-1}) \cong \widetilde{K}^i(\bigvee_{I_l} S^l) \cong \bigoplus_{I_l} \widetilde{K}^i(S^l)$ and similarly for cohomology.

Step 2: General spaces. For a general space X we use an expansion by polyhedra X_i with $\dim(X_i) \leq \dim(X)$ (see A.10). Then $K^0(X) \cong \varinjlim_i K^0(X_i) \cong \varinjlim_i H^0(X_i) \oplus H^2(X_i) \cong H^0(X) \oplus H^2(X)$ and similarly for K^1. □

Remark: Part of the above result may be deduced from the theory of stable rank and unstable K-theory: If $\dim(X) \leq 1$, then $\mathrm{sr}(C(X)) = 1$, and consequently $K_1(C(X)) \cong U_1(C(X))/U_1(C(X))_0 = [X : U_1] \cong H^1(X)$. (see [**Bla06**, V.3.1.26, p.452], and [**Rie83**, 2.10]).

We get the following important corollary. It has applications to the K-theory of low-dimensional homogeneous or continuous-trace C*-algebras, and in this way also for AH-algebras.

B.8. Corollary: *Let X be a compact space.*

(1) If $\dim(X) \leq 2$, then $K^1(X)$ is torsion-free.

(2) If $\dim(X) \leq 1$, then $K^0(X)$ is torsion-free.

Proof:

1. Assume X is a finite CW-complex with $\dim(X) \leq 2$. Then $K^1(X) \cong H^1(X)$ which in turn is isomorphic to $\mathrm{Hom}(H_1(X), \mathbb{Z})$ by the UCT. Since $H_1(X)$ is finitely-generated, $\mathrm{Hom}(H_1(X), \mathbb{Z}) \cong H_1(X)/H_1(X)_{\mathrm{tor}} \cong \mathbb{Z}^k$ for some k. ($H_1(X)_{\mathrm{tor}}$ is the torsion subgroup of $H_1(X)$)

For a general space X we use an expansion by polyhedra X_i with $\dim(X_i) \leq \dim(X)$ (see A.10). Then $K^1(X) \cong \varinjlim_i K^1(X_i)$ is torsion-free.

2. Assume X is a finite CW-complex with $\dim(X) \leq 1$. Then $K^0(X) \cong H^0(X) \cong \mathbb{Z}^k$ where k denotes the (finite) number of connected components of X.

For the general we use an expansion as above. □

B.9. Remark: Reduced version.

There is also a reduced version of the above. For this, let X be a compact space, and let $\pi : X \to \mathrm{pt}$ be the trivial map. By definition, the reduced K-theory (resp. cohomology) is the cokernel of the map $K^*(\pi)$ (resp. $H^*(\pi)$). The **reduced, truncated Chern character** for X is the map

$$\widetilde{\chi^0} : \widetilde{K}^0(X) \to \widetilde{H}^0(X) \oplus H^2(X)$$

defined via the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^0(\mathrm{pt}) & \xrightarrow{K^0(\pi)} & K^0(X) & \longrightarrow & \widetilde{K}^0(X) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \chi^0 & & \downarrow \widetilde{\chi^0} & & \\
0 & \longrightarrow & H^0(\mathrm{pt}) & \xrightarrow{H^0(\pi)} & H^0(X) \oplus H^2(X) & \longrightarrow & \widetilde{H}^0(X) \oplus H^2(X) & \longrightarrow & 0
\end{array}$$

Again, this is always a surjective ring-homomorphism (if we use truncated cup product). If $\dim(X) \leq 3$, then $\widetilde{\chi^0}$ is a ring-isomorphism. Also, if X be a compact space with $\dim(X) \leq 2$, then $\widetilde{K}^0(X)$ is torsion-free.

C. Groups

In this section we first collect some facts about torsion-free groups. Then, we recall the important concepts for (pre-)ordered groups, including (weak) unperforation.

C.1. Torsion-free groups.

Recall that a group G is called torsion-free if $ng = 0$ implies $g = 0$ ($g \in G$ and $n \geq 1$). For a group G let G_{tor} denote the subgroup of torsion-elements. We collect some algebraic facts about torsion-free groups.

C.1: Direct limits.
Let $(G_i, \varphi_{ji})_{i,j\in I}$ be a directed system of groups, and let $G := \varinjlim G_i$ be the direct limit, i.e. the equivalence classes of $\coprod_i G_i$ by the relation (for $g \in G_i, h \in G_j$):

$$g \sim h :\Leftrightarrow \varphi_{ik}(g) = \varphi_{jk}(h) \text{ for some } k \ (\geq i, k)$$

The maps $\varphi_{ji} : G_i \to G_j$ map the torsion subgroup $(G_i)_{\mathrm{tor}}$ into $(G_j)_{\mathrm{tor}}$. Then the subgroups $(G_i)_{\mathrm{tor}}$ also form a directed system, and we get the following commutative diagram:

$$\begin{array}{ccccccc}
(G_i)_{\mathrm{tor}} & \longrightarrow & (G_j)_{\mathrm{tor}} & \longrightarrow & \cdots & \longrightarrow & G_{\mathrm{tor}} \\
\downarrow & & \downarrow & & & & \downarrow \\
G_i & \xrightarrow{\varphi_{ji}} & G_j & \longrightarrow & \cdots & \longrightarrow & G \\
\downarrow & & \downarrow & & & & \downarrow \\
G_i/(G_i)_{\mathrm{tor}} & \longrightarrow & G_j/(G_j)_{\mathrm{tor}} & \longrightarrow & \cdots & \longrightarrow & G/G_{\mathrm{tor}}
\end{array}$$

Further, the limit of the groups $(G_i)_{\mathrm{tor}}$ is isomorphic to G_{tor}, and the limit of the groups $G_i/(G_i)_{\mathrm{tor}}$ is isomorphic to G/G_{tor}.

Thus, if all G_i are torsion-free, then so is their direct limit.

C.2. Lemma:

1.) A subgroup of a torsion-free group is again torsion-free.

2.) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. If A and C are torsion-free, then so is B.

Proof:
1. clear. **2.** Let φ be the map $A \to B$, and be the map $B \to C$. Let A, C be torsion-free, $b \in B$ such that $nb = 0$ for some $n \geq 1$. Then $n\ (b) = \ (nb) = 0$, and $\ (b) = 0$ by torsion-freeness of C. Hence $b \in \mathrm{im}(\varphi) = \ker(\ )$, say $b = \varphi(a)$ for some $a \in A$. Then $\varphi(na) = n\varphi(a) = 0$, and injectivity of φ implies $na = 0$. Torsion-freeness of A gives $a = 0$, and finally $b = \varphi(a) = 0$, which shows that B is torsion-free. □

Remark: The quotient of a torsion-free group need not to be torsion-free, as the following short exact sequence shows:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_{/2} \to 0$$

Thus, torsion-freeness is not a two-out-of-three property.

Recall that K-theory is continuous in the sense that $K_0(\varinjlim A_i) = \varinjlim K_0(A_i)$. Hence, from the previous result we get the following:

C.3. Lemma: *Let $A = \varinjlim A_i$ be a direct limit of C*-algebras A_i with torsion-free $K_0(A_i)$. Then $K_0(A)$ is torsion-free.*

Note that the A_i in the above result are not required to be unital, and the connecting maps are not required to be injective or unital.

We will now give a similar result: If A can be approximated locally by sub-C*-algebras that have torsion-free K_0-group, then $K_0(A)$ is torsion-free. Recall that a collection $(A_i)_{i \in I}$ of sub-C*-algebras of A is said to locally approximate A if for every finite set $F \subset A$, and $\varepsilon > 0$ there is some $i \in I$ such that $F \subset_\varepsilon A_i$.

This seems to include the case of C.3, but there is a subtlety: If $A = \varinjlim A_i$, then there are natural maps $\varphi_i : A_i \to A$ and the sub-C*-algebras $B_i := \varphi_i(A_i)$ locally approximate A. But B_i is a quotient of A_i. Thus, even if $K_0(A_i)$ is torsion-free there may be torsion in $K_0(B_i)$. Hence, C.3 will be a consequence of the next result only if the connecting maps of the limit are injective.

C.4. Lemma: *Let A be a C*-algebra, and $(A_i)_{i \in I}$ a collection of sub-C*-algebras of A that locally approximate A. If each $K_0(A_i)$ is torsion-free, then $K_0(A)$ is torsion-free.*

Proof:

The proof is similar to the proof that $K_0(\varinjlim A_i) = \varinjlim K_0(A_i)$ for an inductive limit. We may assume that A is unital, and that the A_i are unital sub-C*-algebras. Let $\alpha \in K_0(A)$ with $n\alpha = 0$ for some $n \geq 1$.

Then there are projections $p, q \in M_k(A)$ (for some $k \geq 1$) such that $\alpha = [p] - [q]$. The sub-C*-algebras $M_k(A_i)$ locally approximate $M_k(A)$, thus there is some j with $p, q \in_{1/2} M_k(A_j)$. By functional calculus there are projections $p', q' \in M_k(A_j)$ with $\|p - p'\| < 1/2$ and $\|q - q'\| < 1/2$. Then $[p] = [p']$, $[p] = [p']$ and $\alpha = [p'] - [q'] \in K_0(A_j)$ (considered as a subgroup of $K_0(A)$). Since $K_0(A_j)$ is torsion-free, we have that $n\alpha = 0$ implies $\alpha = 0$. □

C.2. (Pre-)ordered groups and perforation.

We will briefly recall the important notions and definitions. See [**She79**].

C.5: Pre-ordered groups.
A **pre-ordered** group is a pair (G, G^+) with G an abelian group and $G^+ \subset G$ a subset such that $0 \in G^+$ and $G^+ + G^+ \subset G^+$ (i.e. G^+ is a sub-monoid of G). If the subset G^+ is understood we will also write G for a pre-ordered group. For elements $x, y \in G$ we write $x \leq y$ if $y - x \in G^+$, and $x < y$ if $y - x \in G^+ \setminus \{0\}$ (i.e. if $x \leq y$ and $x \neq y$). Let (G, G^+), (H, H^+) be pre-ordered, abelian groups. Let us introduce the following notions:

(1) An order-homomorphism (abbreviated by morphism, if the context is clear) $\varphi : (G, G^+) \to (H, H^+)$ is a group-homomorphism $\varphi : G \to H$ such that $\varphi(G^+) \subset H^+$ (i.e. a positive group-homomorphism).
(2) An order-isomorphism is an order-homomorphism $\varphi : (G, G^+) \to (H, H^+)$ such that $\varphi : G \to H$ is a group-isomorphism and $\varphi(G^+) = H^+$.
(3) An **order unit** for (G, G^+) is an element $u \in G^+$ such that for each $h \in G$ there exists some $n \geq 1$ such that $-nu \leq h \leq nu$.
(4) G is called **simple** if every non-zero element $g \in G^+$ is an order unit.

C.6: Ordered groups.
An **ordered** group is a pre-ordered group (G, G^+) such that $G^+ \cap -G^+ = 0$ and $G^+ - G^+ = G$. Note that by our definition all ordered groups are abelian.

There are different definitions of order unit in the literature, which are all equivalent for *ordered* groups. Precisely, let (G, G^+) be an ordered group, and $u \in G^+$ an element. Then the following are equivalent:

(1) u is an order unit (in the sense of the above definition for pre-ordered groups)
(2) for each $h \in G$ there exists some $n \geq 1$ such that $h \leq nu$.
(3) for each $h \in G^+$ there exists some $n \geq 1$ such that $h \leq nu$.

We need to introduce some more concepts. For this, let (G, G^+) be an ordered group. Then:

(1) G is called **unperforated** if $ng \geq 0$ implies $g \geq 0$ (for all $g \in G, n \geq 1$)
(2) G is said to have the **Riesz interpolation property** if for all $g_1, g_2, h_1, h_2 \in G$ with $g_i \leq h_j$ (for $i, j = 1, 2$) there exists $z \in G$ such that $g_1, g_2 \leq z \leq h_1, h_2$.
(3) G is said to have the **Riesz decomposition property** if for all $g_1, g_2, h_1, h_2 \in G^+$ with $g_1 + g_2 = h_1 + h_2$ there exist $c_{ij} \in G$ (for $i, j = 0, 1$) such that $g_i = \sum_j c_{ij}$ and $h_j = \sum_i c_{ij}$
(4) G is called a **Riesz group** if it is unperforated and has the Riesz interpolation property
(5) a **face** is a subset $F \subset G^+$ such that $F + F \subset F$ and F is hereditary, i.e. $0 \leq x \leq a \in F$ implies $x \in F$ (for any $x \in G$)
(6) an **ideal** is a subgroup $I \subset G$ such $I = I^+ - I^+$ (where $I^+ := I \cap G^+$) and I^+ is a face
(7) If u is an order unit for G, then the **state space** of G (w.r.t. u), denoted by $S(G, G^+, u)$ or simply $S(G)$, is the set of all order-homomorphisms $\chi : (G, G^+) \to (\mathbb{R}, \mathbb{R}^+)$ such that $\chi(u) = 1$.

The Riesz interpolation property and the Riesz decomposition property are equivalent. An unperforated group is torsion-free. The state space is a compact convex space in the topology of pointwise convergence.

The faces Faces(G) are in order-preserving, one-one-correspondence with the ideals Idl(G) via the maps

$$\begin{aligned} \mathrm{Faces}(G) &\longrightarrow \mathrm{Idl}(G) & \qquad \mathrm{Idl}(G) &\longrightarrow \mathrm{Faces}(G) \\ F &\longmapsto F - F & I &\longmapsto G^+ \cap I \end{aligned}$$

Further, (G, G^+) is simple if and only if all ideals are trivial (i.e. either $\{0\}$ or G), and this happens precisely if all non-zero, positive elements in G are order units. (This last property is called "strong simplicity" in [**EHS80**], but it is equivalent to simplicity for all ordered groups).

C.7: Weakly unperforated groups.

Unperforated groups are necessarily torsion-free. We now consider a weak form of unperforation as introduced in [**Ell90**, 2.1] which includes also groups with torsion.

Let (G, G^+) be an ordered group. Then G_{tor} and G^+ will be disjoint, and $(G/G_{\mathrm{tor}}, G^+/G_{\mathrm{tor}})$ is an ordered (torsion-free) group. Now, (G, G^+) is called **weakly unperforated** if G/G_{tor} is unperforated and the following holds:

If $g \in G^+, t \in G_{\mathrm{tor}}$, and $ng + mt \in G^+$ for some $n, m \in \mathbb{Z}$ with $n \geq 1$, then there exist $t_1, t_2 \in G_{\mathrm{tor}}$ such that $t = t_1 + t_2$ and $mt_1 = 0$, $g + t_2 \in G^+$

As shown by Elliott this is equivalent to the following three properties:

(i) G/G_{tor} is unperforated
(ii) If $g \in G^+, t \in G_{\mathrm{tor}}$, then $g + t \in G^+$ if and only if t belongs to the ideal of G generated by G
(iii) any ideal of G is a relatively divisible subgroup

Recall that a subgroup $H \leq G$ of a group G is called **relatively divisible** (in H), if the following implications holds (for all $n \geq 1$, $g \in G$): $ng \in H \Rightarrow g \in H$. A simple, ordered groups is weakly unperforated if and only if $ng > 0$ implies $g \geq 0$ (for all $g \in G, n \geq 1$).

C.8: Inductive Limits.

An inductive family of pre-ordered groups is an inductive family in the category of pre-ordered groups and order-homomorphisms, i.e. a family $((G_i, G_i^+)$ of pre-ordered groups indexed overs some directed set I together with order-homomorphisms $\varphi_{ji} : (G_i, G_i^+) \to (G_j, G_j^+)$ (for $i \leq j$) such that $\varphi_{ii} = \mathrm{id}_{G_i}$ and $\varphi_{kj}\varphi_{ji} = \varphi_{ki}$ (for $i \leq j \leq k$). Let $G := \varinjlim G_i$ be the direct limit (of abelian groups), which comes with maps $\varphi_{\infty,i} : G_i \to G$. We define a pre-order on G by

$$G^+ := \bigcup_i \varphi_{\infty,i}(G_i^+)$$

With this pre-order, the maps $\varphi_{\infty,i}$ become order-homomorphisms. If all (G_i, G_i^+) are ordered, then so is (G, G^+) Let us next recall a well-known result showing that direct limits preserve several properties of ordered groups.

C.9. Proposition: *The following properties are preserved by inductive limits of ordered groups, i.e. if all (G_i, G_i^+) have the property, then also the direct limit (G, G^+) has it:*

(1) being simple
(2) being weakly unperforated
(3) being unperforated
(4) having the Riesz interpolation property

Proof:
1. Let $g \in G^+$ and $h \in G$. We need to find $n \geq 1$ s.t. $h \leq ng$. There is some i with $g' \in G_i^+$ that maps to g and $h' \in G_i$ that maps to h. By simplicity of G_i a number n as desired exists for g' and h', and hence also for g, h.

3. and **4.** can be proved in the same way

2. By C.1 we have $G/G_{\mathrm{tor}} \cong \varinjlim G_i/(G_i)_{\mathrm{tor}}$. Since G_i is weakly unperforated, each $G_i/(G_i)_{\mathrm{tor}}$ is unperforated, and then G/G_{tor} is also unperforated by 3. The remaining property is also easy to check.

We give some results which are probably well-known. However we could not find a reference, so we include the straightforward proofs.

C.10: Orders on the direct sum
Let G and H be pre-ordered groups. Then we can define a pre-order on the algebraic directs sum $G \oplus H$, sometimes called the direct sum (pre-)order, as follows:

$$(g, h) \geq_{\mathrm{ord}} 0 :\Leftrightarrow \ g \geq 0 \text{ and } h \geq 0$$

We denote the resulting pre-ordered group by $G \oplus_{\mathrm{ord}} H$. If both G and H are ordered, then also $G \oplus_{\mathrm{ord}} H$ is ordered.

If G is a pre-ordered group, and H an abelian group (not ordered), then we may define an order on $G \oplus H$ by:

$$(g, h) \geq_{\mathrm{str}} 0 :\Leftrightarrow \ g > 0 \text{ or } g = h = 0$$

We call this the **strict order**, and denote the resulting pre-ordered group by $G \oplus_{\mathrm{str}} H$.

C.11. Lemma: [**Ell90**, Lma 6.1]
Let G be an ordered group, and H an abelian group. Then the faces of G and $G \oplus_{str} H$ are in order-preserving one-one-correspondence via the following maps:

$$\begin{aligned} \Phi : \mathrm{Faces}(G) &\longrightarrow \mathrm{Faces}(G \oplus_{str} H) \\ I^+ &\longmapsto I^{++} \oplus H \cup \{(0,0)\} \\ \Psi : \mathrm{Faces}(G \oplus_{str} H) &\longrightarrow \mathrm{Faces}(G) \\ J^+ &\longmapsto G^+ \cap J^+ \end{aligned}$$

where $I^{++} = I^+ \setminus \{0\}$.

Proof:
Firstly, $\Psi \circ \Phi = \mathrm{id}$ is easy to check. To see $\Phi \circ \Psi = \mathrm{id}$, let J^+ be a face in $G \oplus_{\mathrm{str}} H$, and $K^{++} := (G^{++} \cap J^+) \oplus H$. Then $\Phi \circ \Psi(J^+) = K^{++} \cup \{(0,0)\}$, and it is enough to check $J^{++} = K^{++}$. But $J^{++} \subset K^{++}$ is clear. For the converse let $(g, h) \in K^{++}$. Then $g > 0$ and there exists some $h' \in H$ such that $(g, h') \in J^+$. But

then $(g,h) < 2(g,h')$, and since $2(g,h') \in J^+$ also $(g,h) \in J^+$. □

C.12. Proposition: *Let G be an ordered group, and H an abelian group. Then:*

1. *$G \oplus_{str} H$ is simple $\Leftrightarrow$ G is simple*
2. *$G \oplus_{str} H$ has the Riesz interpolation property $\Leftrightarrow$ G has the Riesz interpolation property*
3. *$G \oplus_{str} H$ is weakly unperforated $\Leftrightarrow$ G is weakly unperforated*
4. *$G \oplus_{str} H$ is unperforated $\Rightarrow$ G is unperforated*
5. *$G \oplus_{str} H$ is a Riesz group $\Rightarrow$ G is a Riesz group*

If H is torsion-free, then the converses of (4) *and* (5) *are also true.*

Proof:
To shorten notation we set $K := G \oplus_{\mathrm{str}} H$ and denote elements in K by $\mathbf{a} = (a, a')$.

1. The statement about simplicity follows from C.11

2. "$\Leftarrow$": Assume G has the Riesz interpolation property. Let $\mathbf{a}_i, \mathbf{b}_j \in K$ with $\mathbf{a}_i \leq \mathbf{b}_j$ for all $i,j = 1,2$. We are looking for some $\mathbf{c}$ such that $\mathbf{a}_i \leq \mathbf{c} \leq \mathbf{b}_j$. We have $a_i \leq b_j$ (for $i,j = 1,2$). Hence, there exists some $c \in G$ such that $a_i \leq c \leq b_j$. If c is different form all a_i, b_j, then the inequalities are strict and $(c,0)$ (or just any (c,d)) will be the desired element. If c is equal to some a_i, then $\mathbf{a}_i = (a_i, a_i')$ will do (and analogously if $c = b_j$, then $\mathbf{b}_j$ works).

"$\Rightarrow$": Assume conversely K has the Riesz interpolation property. Let $a_i, b_j \in G$ with $a_i \leq b_j$ for all $i,j = 1,2$. Then $(a_i, 0) \leq (b_j, 0)$, and we get some (c,c') with $(a_i,0) \leq (c,c') \leq (b_j,0)$. Hence $a_i \leq c \leq b_j$, and we are done.

4. Assume K is unperforated, and let $g \in G$ with $ng \geq 0$ for some $n \geq 1$. Then $n(g,0) \geq 0$, which implies $(g,0) \geq 0$, and hence $g \geq 0$.

For the converse, assume H is torsion-free and G is unperforated. Let $\mathbf{g} = (g,g') \in K$ with $n\mathbf{g} \geq 0$ for some $n \geq 1$. Then either $ng = ng' = 0$ or $ng > 0$. In the first case by torsion-freeness of H (and also G) we have $g = g' = 0$ and therefore $\mathbf{g} \geq 0$. In the second case we get $g > 0$ since G is unperforated, and hence $\mathbf{g} \geq 0$.

3. follows directly from 2. and 4.

3. Note that $K_{\mathrm{tor}} = G_{\mathrm{tor}} \oplus H_{\mathrm{tor}}$ and the positive cone in K/K_{tor} is by definition the image of K^+, i.e. K^+/K_{tor}. This is equal to $(G/G_{\mathrm{tor}})^{++} \oplus H/H_{\mathrm{tor}} \cup \{(0,0)\}$, and therefore $K/K_{\mathrm{tor}} = G/G_{\mathrm{tor}} \oplus_{\mathrm{str}} H/H_{\mathrm{tor}}$. Hence K/K_{tor} is unperforated if and only if G/G_{tor} is unperforated. It remains to check the second property of weak unperforation.

Assume G is weakly unperforated. Let $\mathbf{g} \in K^+, \mathbf{t} \in K_{\mathrm{tor}}$ and $n\mathbf{g} + m\mathbf{t} \in K^+$ for some $n, m \in \mathbb{Z}$, $n \geq 1$. We are looking for $\mathbf{t}_1, \mathbf{t}_2 \in K_{\mathrm{tor}}$ with $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$, $m\mathbf{t}_1 = 0$, and $\mathbf{g} + \mathbf{t}_1 \in K^+$. If $g = 0$, then $g' = 0$ and $m\mathbf{t} \in K^+$. Since $\mathbf{t}$ is torsion, this implies $\mathbf{t} = 0$ and we are done.

Otherwise $g > 0$. Then $ng + mt \geq 0$ (and $t \in G_{\mathrm{tor}}$), and since G is weakly unperforated we get some $t_1, t_2 \in G_{\mathrm{tor}}$ with $t = t_1 + t_2$, $mt_1 = 0$ and $g + t_2 \geq 0$. Set $\mathbf{t}_1 := (t_1, 0)$ and $\mathbf{t}_2 := (t_2, t')$, which fulfill the desired properties.

Conversely, assume K is weakly unperforated. Let $g \in G^+, t \in G_{\mathrm{tor}}$ and $ng + mt \in G^+$ for some $n, m \in \mathbb{Z}$, $n \geq 1$. Set $\mathbf{g} := (g, 0)$, $\mathbf{t} := (t, 0)$. Then $n\mathbf{g} + m\mathbf{t} \in K^+$ and we get $\mathbf{t}_1, \mathbf{t}_2 \in K_{\mathrm{tor}}$ with $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$, $m\mathbf{t}_1 = 0$, and $\mathbf{g} + \mathbf{t}_1 \in K^+$. Then t_1, t_2 are the desired elements. □

D. C*-algebras

In this section, we collect some facts from the theory of C*-algebras that we need elsewhere in this paper.

We begin by noting, that it is ambiguous to speak of "finite-dimensional" C*-algebras, since we are introducing several different dimension theories for C*-algebras. Usually, if one says a C*-algebra A is "finite-dimensional", then one means the vector-space dimension. To avoid confusion we will call this property **vs-finite-dimensional**. Thus, a C*-algebra A is vs-finite-dimensional if and only if it is a finite direct sum of matrix algebras, i.e. $A = M_{n_1} \oplus \ldots \oplus M_{n_k}$.

D.1: The spectrum and primitive ideal space

For any C*-algebra A we will denote the spectrum (equivalence classes of irreducible representations) by $\widehat{A}$, and the primitive ideal space by $\mathrm{Prim}(A)$. We will also for $k \in \mathbb{N}$ denote by $\widehat{A}_k$ (resp. $\widehat{A}_{\leq k}$) the part of the spectrum corresponding to irreducible representations on a Hilbert-space of dimension k (resp. $\leq k$).

The set $\widehat{A}_{\leq k}$ is always closed in $\widehat{A}$, and we denote by $A_{\leq k}$ the corresponding quotient of A. Next, $\widehat{A}_k$ is open in $\widehat{A}_{\leq k}$, and we denote by A_k the corresponding ideal of $A_{\leq k}$. Note that $A_{\leq k}$ is a k-subhomogeneous C*-algebra (see 3.1), and A_k is k-homogeneous (see 2.1). In particular $\widehat{A}_k$ is Hausdorff.

D.2: Hereditary sub-C*-algebras

There is a tight connection between the primitive ideal space of a C*-algebra and a hereditary sub-C*-algebra of it. This will be used in 1.17. Precisely, we have the following:

Let A be a C*-algebra, and B be a hereditary sub-C*-algebra of A. For each irreducible representation $\pi : A \to B(H_\pi)$ let H'_π denote the essential subspace of $\pi(B)$, i.e. $H'_\pi = \pi(B)H_\pi$. Denote by π' the representation of B on H'_π which is irreducible if H'_π is nonzero. Define maps as follows:

$$\begin{aligned} \Phi : \widehat{A} &\to \widehat{B} \cup \{0\} & \qquad \Psi : \mathrm{Prim}(A) &\to \mathrm{Prim}(B) \cup \{B\} \\ \pi &\mapsto \pi' & I &\mapsto I \cap B \end{aligned}$$

This gives the following commutative diagram:

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{\Phi} & \widehat{B} \cup \{0\} \\ \downarrow{\scriptstyle \ker} & & \downarrow{\scriptstyle \ker} \\ \mathrm{Prim}(A) & \xrightarrow{\Psi} & \mathrm{Prim}(B) \cup \{B\} \end{array}$$

Then:

1.) Φ and Ψ are continuous and open
2.) The image of Φ contains $\widehat{B}$ (it might not contain 0)
3.) The image of Ψ contains $\mathrm{Prim}(B)$ (it might not contain B)
4.) Ψ is a homeomorphism when restricted to $\Psi^{-1}(\mathrm{Prim}(B))$ (i.e. on the part that is not mapped to B)

List of Symbols

A^{-1}	The invertible elements in a unital C*-algebra A.
A_{sa}	The self-adjoint elements in the C*-algebra A.
H^*	Čech-Cohomoloy.
$S(G)$	The space of states on an ordered group G with order unit.
$\mathrm{Aff}_0(C)$	The continuous affine maps $C \to \mathbb{R}$ on a topological convex cone C that vanish at zero, with the strict ordering.
$\mathbb{N}$	The natural numbers $\{0, 1, 2, \ldots\}$.
$\mathrm{Pos}(G)$	The order-homomorphisms $G \to \mathbb{R}$ for a (pre-)ordered group G. This space is given the topology of pointwise convergence.
$\mathrm{Prim}(A)$	The primitive ideal space of A.
$\mathrm{Prim}(A)_k$	The part of the primitive ideal space of A, the corresponds to irreducible representations of dimension k (see D.1).
$\mathbb{T}$	The circle, i.e. $\mathbb{T} = \{\lambda \in \mathbb{C} \, : \, \lvert\lambda\rvert = 1$.
$\lceil x \rceil$	The least integer n such that $x \leq n$.
$\lfloor x \rfloor$	The biggest integer n such that $n \leq x$.
$\lhd$	$I \lhd A$ means I is a (two-sided, closed) ideal in the C*-algebra A.
$\mathcal{Z}$	The Jiang-Su algebra.
rr	The decomposition rank (see 1.23).
rr	The real rank (see 1.5).
rr	The stable rank (see 1.5).
rr	The topological dimension (see 1.15).
$\simeq$	Homotopy equivalence.
$\underline{\lim}$	The limes inferior.
$\widehat{A}$	The spectrum of the C*-algebra A (irreducible representations up to unitary equivalence).
$\widehat{A}_k$	The spectrum of the C*-algebra A, that corresponds to irreducible representations of dimesio k (see D.1).
$\widetilde{A}$	The smallest unitalization of a C*-algebra A. If A is unital, then $\widetilde{A} = A$, otherwise $\widetilde{A} = A \oplus \mathbb{C}$.
$f \gg 0$	The strict order for a function f. It means that either $f = 0$ (i.e. $f(x)$ for all points), or $f(x) > 0$ for all x.

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Erkärung der Urheberschaft

Ich versichere hiermit, dass die vorstehende Diplomarbeit mit dem Titel "One-dimensional C*-algebras and their K-theory" von mir selbstständig verfasst wurde und alle Quellen und Hilfsmittel angegeben sind.

Ort, Datum

Unterschrift
(Hannes Thiel)