THE CONNECTION BETWEEN SHAPE THEORY AND E-THEORY.

HANNES THIEL

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This is a short note for a talk that I gave at the university of Copenhagen. It is a brief survey on noncommutative shape theory, E-theory and the connection between the two theories.

The central notion is that of a homotopy symmetric C^*-algebra, as defined by Dadarlat and Loring, [DL94]. They show that for stable, homotopy symmetric C^*-algebras the two theories agree, which allows to use K-theoretic computations to obtain shape equivalence.

By a morphism between C^*-algebras we mean a $*$-homomorphism. We use the symbol "$\simeq$" to denote homotopy equivalence.

Contents

1. Noncommutative shape theory

We consider shape theory for separable C^*-algebras as developed by Blackadar in [Bla85], to which we also refer the reader for details and proofs.

Definition 1.1. *A morphism $\varphi\colon A \to B$ is called **semiprojective**, if for any C^*-algebra C and increasing sequence of ideals $J_1 \lhd J_2 \lhd \ldots \lhd C$ and for any morphism $\sigma\colon B \to C/\overline{\bigcup_k J_k}$ there exists some k and a morphism $\psi\colon A \to C/J_k$ such that $\pi \circ \psi = \sigma \circ \varphi$, where $\pi\colon C/J_k \to C/\overline{\bigcup_k J_k}$ is the quotient morphism. The lifting is indicated in the following diagram:*

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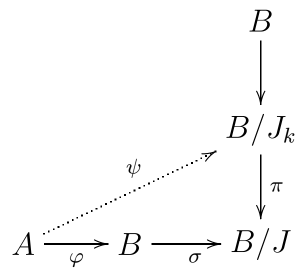
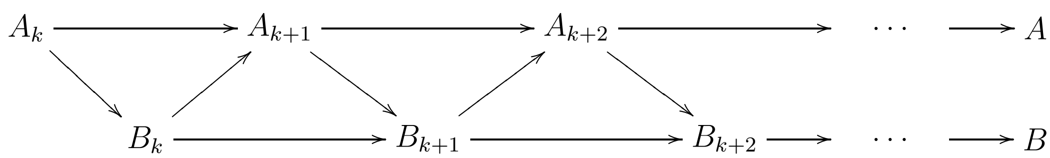


*A C^*-algebra A is called **semiprojective** if the identity map $\mathrm{id}_A : A \to A$ is semiprojective.*

Theorem 1.2 (Blackadar)**.** *Every C^*-algebra A has the form $A \cong \varinjlim_k A_k$ with semiprojective connecting maps $A_k \to A_{k+1}$. Such a system is called a **shape system** for A.*

Definition 1.3. *Two C^*-algebras A, B are called **shape equivalent**, denoted $A \sim_{Sh} B$, if one can find shape systems for A and B with intertwinings that make the following diagram commute up to homotopy:*



2. Asymptotic morphisms

For details and proofs in this section the reader is referred to Blackadar, [Bla98, Chapter 25], and Dadarlat, [Dad94].

Definition 2.1. *An **asymptotic morphism** from A to B is a family (φ_t) of mappings $\varphi_t \colon A \to B$ indexed over $t \in [0, \infty)$ such that:*

(i) *the map $t \mapsto \varphi_t(a)$ is continuous (for each $a \in A$)*
(ii) *(φ_t) is asymptotically a $*$-homomorphism (the identities for a $*$-homomorphism hold as a limit for $t \to \infty$):*
$\|\varphi_t(a + \lambda b) - [\varphi_t(a) + \lambda\varphi_t(b)]\| \to 0$ *(for all $a, b \in A$, $\lambda \in \mathbb{C}$)*
$\|\varphi_t(a^*) - \varphi_t(a)^*\| \to 0$ *(for all $a \in A$)*
$\|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| \to 0$ *(for all $a, b \in A$)*

Let $\mathrm{AMor}(A, B)$ denote the set of all asymptotic morphisms from A to B.

Definition 2.2. *We define two equivalence relations on $\mathrm{AMor}(A, B)$:*

(i) ***strict equivalence**: $(\varphi_t) \cong (\psi_t) \quad :\Leftrightarrow \|\varphi_t(a) - \psi_t(a)\| \to 0$ for all $a \in A$*
(ii) ***homotopy**: $(\varphi_t) \simeq (\psi_t) \quad :\Leftrightarrow \exists(\omega_t) \in \mathrm{AMor}(A, C([0,1], B)$ s.t. $\omega_t(a)(0) = \varphi_t(a)$ and $\omega_t(a)(1) = \psi_t(a)$ for all $a \in A$*

Definition 2.3. $[[A, B]] := \mathrm{AMor}(A, b)_{/\simeq}$.

Remark 2.4. Every morphism naturally induces an asymptotic morphism. If $\mathrm{Mor}(A, B)$ denotes the set of morphisms from A to B, and $[A, B] = \mathrm{Mor}(A, B)_{/\simeq}$, then the following diagram naturally commutes:

$$\begin{array}{ccc} \mathrm{Mor}(A,B) & \longrightarrow & \mathrm{AMor}(A,B) \\ \downarrow & & \downarrow \\ [A,B] & \longrightarrow & [[A,B]] \end{array}$$

Note that the lower horizontal map is in general neither injective nor surjective, but see 2.9.

2.5. If $(\varphi_t) \in \mathrm{AMor}(A, B)$, then one can show that

$$\limsup_t \|\varphi_t(A)\| \leq \|a\|$$

It follows, that for each $a \in A$ the mapping $t \mapsto \varphi_t(a)$ is a well-defined element in

$$B_\infty := C_b([0,\infty), B)/C_0([0,\infty), B)$$

This induces a map $\mathrm{AMor}(A, B) \to \mathrm{Mor}(A, B_\infty)$, and two asymptotic morphisms induce the same map if and only if they are strictly equivalent. Thus, we obtain a natural bijection:

$$\mathrm{AMor}(A,B)_{/\cong} \quad \longleftrightarrow \quad \mathrm{Mor}(A, B_\infty)$$

2.6 (Tensor products)**.** Asymptotic morphisms $(\varphi_t) \in \mathrm{AMor}(A, C)$ and $(\psi_t) \in \mathrm{AMor}(B, D)$ induce an asymptotic morphism $(\varphi_t) \otimes (\psi_t) \in \mathrm{AMor}(A \otimes_{\max} B, C \otimes_{\max} D)$. This tensor product of asymptotic morphism respects homotopy, so we get induced tensor products:

$$[[A,C]] \times [[B,D]] \quad \longrightarrow \quad [[A \otimes_{\max} B, C \otimes_{\max} D]]$$

In particular, we get a suspension map $[[A, B]] \to [[SA, SB]]$ sending $[(\varphi_t)] \in [[A, B]]$ to $[\mathrm{id}_S \otimes (\varphi_t)]$, where $SA = C_0(\mathbb{R}) \otimes A$.

2.7 (Composition)**.** Given asymptotic morphisms $(\varphi_t) \in \mathrm{AMor}(A, B)$ and $(\psi_t) \in \mathrm{AMor}(B, C)$, the obvious composition $(\psi_t \circ \varphi_t)$ is not necessarily an asymptotic morphism from A to C. But one can find $(\varphi'_t) \simeq (\varphi_t)$ and $(\psi'_t) \simeq (\psi_t)$ so that $(\psi'_t \circ \varphi'_t) \in \mathrm{AMor}(A, C)$. This induces a composition

$$[[A,B]] \times [[B,C]] \quad \longrightarrow \quad [[A,C]]$$

This composition is associative, it commutes with tensor products, and the following natural diagram commutes:

$$\begin{array}{ccc} [A,B] \times [B,C] & \longrightarrow & [A,C] \\ \downarrow & & \downarrow \\ [[A,B]] \times [[B,C]] & \longrightarrow & [[A,C]] \end{array}$$

Definition 2.8. *The category $\mathcal{A}$ has as objects separable C^*-algebras and as morphisms between A and B the elements of $[[A, B]]$, with composition defined as in 2.7.*

Proposition 2.9 (Blackadar)**.** *If A is semiprojective, then the natural map $[A, B] \to [[A, B]]$ is bijective.*

Theorem 2.10 (Dadarlat)**.** *The following are equivalent for two C^*-algebras A and B:*

(i) $A \sim_{Sh} B$

(ii) $A \sim_{\mathcal{A}} B$, *i.e., there exist* $(\varphi_t) \in \mathrm{AMor}(A,B)$ *and* $(\psi_t) \in \mathrm{AMor}(B,A)$ *s.t.* $(\psi_t) \circ (\varphi_t) \simeq \mathrm{id}_A$ *and* $(\varphi_t) \circ (\psi_t) \simeq \mathrm{id}_B$

3. E-THEORY

For details on this section we refer to Blackadar, [Bla98, Chapter 25].

3.1. Problem: $[[A,B]]$ has not additive structure.

3.2 (Idea 1)**.** Define an addition on $[[A, B\otimes\mathbb{K}]]$ as in K-theory: Given $(\varphi_t),(\psi_t) \in \mathrm{AMor}(A, B\otimes \mathbb{K})$, consider the asymptotic morphism $(\varphi_t)\oplus(\psi_t)$ from A to $(B\otimes\mathbb{K})\oplus(B\otimes\mathbb{K}) \subset B\otimes\mathbb{K}\otimes M_2 \cong B\otimes\mathbb{K}$. Different choices of isomorphisms $B\otimes\mathbb{K}\otimes M_2 \cong B\otimes\mathbb{K}$ are homotopic, so we get a well-defined structure of $[[A, B\otimes\mathbb{K}]]$ as an abelian *semigroup*.

3.3 (Idea 2)**.** Define an addition on $[[A, SB]]$, similar to the definition of the fundamental group in algebraic topology: An asymptotic morphism (φ_t) from A to $SB = C_0(\mathbb{R}, B)$ is homotopic to an asymptotic morphism (φ'_t) from A to $C_0((-\infty,0,B) \subset SB$. Similarly, any $(\psi_t) \in \mathrm{AMor}(A, SB)$ is homotopic to (ψ'_t) from A to $C_0((0,\infty), B) \subset SB$. Note that $\varphi'_t + \psi'_t$ is an asymptotic morphism, which allows to define $[(\varphi_t)] + [(\psi_t)] := [(\varphi'_t + \psi_t)]$. This gives $[[A, SB]$ the structure of a (not necessarily abelian) group.

3.4. On $[[A, SB\otimes\mathbb{K}]]$ the two additions agree and give the structure of an abelian group.

Definition 3.5. $E(A,B) := [[SA, SB\otimes\mathbb{K}]]$

Remark 3.6. For any C^*-algebras A, B there is a natural isomorphism $[[A, B\otimes\mathbb{K}]] \xrightarrow{\cong} [[A\otimes\mathbb{K}, B\otimes\mathbb{K}]]$. This makes it possibly to use the composition defined in 2.7 to obtain a natural composition between $E(A,B) = [[SA, SB\otimes\mathbb{K}]]$ and $E(B,C) \cong [[SA\otimes\mathbb{K}, SB\otimes\mathbb{K}]]$.

Definition 3.7. *The additive category* E *has as objects separable* C^**-algebras and as set of morphisms between* A *and* B *the abelian group* $E(A,B)$*.*

4. HOMOTOPY SYMMETRIC C^*-ALGEBRAS

For details on this section we refer to Dadarlat and Loring, [DL94], [Dad93].

Definition 4.1. *A* C^**-algebra* A *is* ***homotopy symmetric*** *if the map* $\mathrm{id}_A\colon A \to A\otimes\mathbb{K}$, $a \mapsto a\otimes e_{11}$ *has an additive inverse in* $[[A, A\otimes\mathbb{K}]]$*. This means exactly that* $[[A, A\otimes\mathbb{K}]]$ *is a group.*

Let H *be the set of all* C^**-algebras that are homotopy symmetric.*

Theorem 4.2 (Dadarlat, Loring)**.** *Let* $A \in H$*, and* B *any* C^**-algebra. Then the suspension map is an isomorphism, i.e.,*

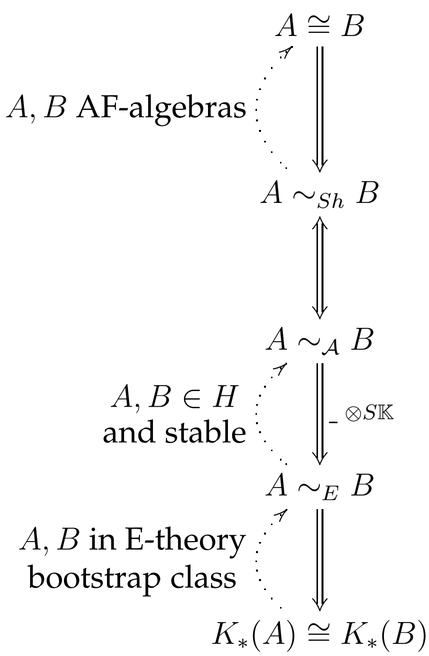
$$[[A, B\otimes\mathbb{K}]] \xrightarrow{\cong} [[SA, SB\otimes\mathbb{K}]] = E(A,B)$$

4.3 (Closure properties of H)**.** The following hold:

1) $A \in H$, B any C^*-algebra $\Rightarrow A \otimes_{\max} B \in H$
2) $A \cong \varinjlim A_k$, $A_k \in H$ $\Rightarrow A \in H$
3) $A_k \in H$ $\Rightarrow \oplus A_k \in H$

Theorem 4.4 (Dadarlat)**.** *Let* A *and* B *be two stable* C^**-algebras in the E-theory bootstrap class. Assume* $A, B \in H$*. Then* $A \sim_{Sh} B$ *if and only if* $K_*(A) \cong K_*(B)$*.*

4.5 (Summary)**.** We get the following connection between the various equivalence relations on C^*-algebras:



isomorphism of C^*-algebras
- very hard to compute

shape theory - structure of the C^*-algebra
- hard to compute

E-theory = "suspended, stabilized shape theory"
- easier to compute

K-theory - easiest to compute

5. Examples

5.1 (Some homotopy symmetric C^*-algebras)**.** If X is a compact, connected, metric space, and $x \in X$ a fixed point, then set:

$$C_0(X_0) := C_0(X \setminus \{x\})$$

This algebra is homotopy symmetric and $K_*(C_0(X_0)) \cong \widetilde{K}^*(X) = K^*(X, \{x\})$.

The algebras

$$A_n := \{f\colon (0,1] \to M_n \ : \ f(1) \in \mathbb{C} \cdot 1_{M_n}\}$$

are homotopy symmetric and $K_*(A_n) \cong (0, \mathbb{Z}_{/n})$.

The algebra

$$q\mathbb{C} = \{f\colon (0,1] \to M_2 \ : \ f(1) \text{ diagonal } \}$$

is homotopy symmetric with $K_*(q\mathbb{C}) \cong (\mathbb{Z}, 0)$.

5.2 (Dadarlat)**.** If X, Y are two compact, connected, metric spaces, then:

$$C_0(X_0) \sim_{Sh} C_0(Y_0) \quad \Leftrightarrow \quad (X, x) \sim_{Sh} (Y, y)$$

This means that noncommutative shape theory agrees with classical shape theory for commutative C^*-algebras. If we stabilize, then the situation is completely changed:

$$C_0(X_0) \otimes \mathbb{K} \sim_{Sh} C_0(Y_0) \otimes \mathbb{K} \quad \Leftrightarrow \quad \widetilde{K}^*(X) \cong \widetilde{K}^*(Y)$$

Example 5.3. Calculating the K-thoery, we get:

$$C_0(\mathbb{R}^3)\otimes\mathbb{K}\sim_{Sh} C_0(\mathbb{R})\otimes\mathbb{K}$$

since $\mathbb{R}^3=S^3\setminus\{x\},\mathbb{R}=S^2\setminus\{y\}$ and $\widetilde{K}^*(S^3)\cong(\mathbb{Z},0)\cong\widetilde{K}^*(S^1)$.

Let $\mathbb{T}^2=S^1\times S^1$ be the two-dimensional torus, then

$$C_0(\mathbb{T}^2_0)\otimes\mathbb{K}\sim_{Sh} C_0(\mathbb{R}^2\sqcup\mathbb{R}\sqcup\mathbb{R})\otimes\mathbb{K}$$

since $K_*(C_(\mathbb{T}^2_0))\cong(\mathbb{Z}^2,\mathbb{Z}^2)\cong K_*(C_0(\mathbb{R}^2\sqcup\mathbb{R}\sqcup\mathbb{R}))$.

Let $S\mathcal{O}_{n+1}$ be the suspension of the Cuntz algebra $\mathcal{O}_{n+1}$, then

$$S\mathcal{O}_{n+1}\otimes\mathbb{K}\sim_{Sh} A_n\otimes\mathbb{K}$$

since $K_*(S\mathcal{O}_{n+1})\cong(0,\mathbb{Z}_{/n})\cong K_*(A_n)$.

Example 5.4 (K-theory with coefficients)**.** The C^*-algebras A_n are semiprojective, homotopy symmetric and K-nuclear (they are subhomogeneous). K-theory with ceofficients for a C^*-algebra B is defined as:

$$K_0(B;\mathbb{Z}_n):=KK(A_n,B)$$

We get the following:

$$\begin{aligned} K_0(B;\mathbb{Z}_n) &= KK(A_n,B) \\ &\cong E(A_b,B) && [\text{ since } A_n \text{ is } K\text{-nuclear }] \\ &\cong [[A_n,B\otimes\mathbb{K}]] && [\text{ since } A_n\in H\text{ }] \\ &\cong [A_n,B\otimes\mathbb{K}] && [\text{ since } A_n \text{ is semiprojective }] \end{aligned}$$

Example 5.5 (Shulman)**.** As shown by Shulman, [Shu10], the C^*-algebras $qA\otimes\mathbb{K}$ and $S^2A\otimes\mathbb{K}$ are shape equivalent. It follows that $[[qA,qA\otimes\mathbb{K}]]$ always is a group, so that $qA\in H$ for every C^*-algebra A.

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Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen Ø, Denmark

E-mail address: thiel@math.ku.dk