ARE C*-ALGEBRAS DETERMINED BY THEIR LINEAR AND ORTHOGONALITY STRUCTURE?

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Abstract. It is well-known that every C*-algebra is determined by its linear and multiplicative structure: Two C*-algebras are *-isomorphic if and only if they admit a multiplicative, linear bijection.

We study if instead of the whole multiplicative structure it suffices to record when two elements have zero product. While it is not clear if every C*-algebra is determined this way, we obtain many positive results. In particular, two unital, simple C*-algebras are *-isomorphic if and only if they admit a linear bijection that preserves zero products.

These are notes for a talk on 14. March 2022 at the conference 'Noncommutativity in the north' in Gothenburg. It is about joint work with Eusebio Gardella to appear in forthcoming work [GT22].

One says that two C*-algebras A and B are *-isomorphic, denoted $A \cong B$, if there exists a *-isomorphism $A \to B$, that is, a bijective, linear, multiplicative, *-preserving map $A \to B$.

Question 1. How can we deduce such a *-isomorphism?

• From the *-linear structure? No: The map

$$M_2(\mathbb{C}) \to \mathbb{C}^4, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b+c, i(b-c), d).$$

is bijective, linear and *-preserving, but $M_2(\mathbb{C}) \ncong \mathbb{C}^4$.

• From the *-ring structure? No: Given a C*-algebra A, consider the opposite C*-algebra A^{op}. Then the map

$$A \to A^{\mathrm{op}}, \quad a \mapsto a^*,$$

is bijective, multiplicative and *-preserving, but there exist examples ([Phi01]) such that $A \ncong A^{\mathrm{op}}$.

• From the algebra structure? Yes:

Theorem 2 (Gardner 1965, [Gar65])**.** *Let $\varphi\colon A \to B$ be an isomorphism between C^*-algebras, that is, φ is bijective, linear and multiplicative. Then $\varphi = \alpha \circ \pi$ for some *-isomorphism $\pi\colon A \to B$ and some 'weakly inner' automorphism $\alpha\colon B \to B$, given by $\alpha(b) = cbc^{-1}$, for some invertible operator $c \in B^{**}$. In particular, φ is automatically continuous.*

Can the result of Gardner be strengthened? Does it suffice if instead of the whole multiplicative structure we record when two elements are orthogonal in the sense that they have zero-product? We formalize this:

Definition 3. A map $\varphi\colon A \to B$ between C^*-algebras is said to *preserve zero-products* if $xy = 0$ implies $\varphi(x)\varphi(y) = 0$, for all $x, y \in A$.

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Question 4. Let $\varphi\colon A \to B$ be a bijective, linear map between C^*-algebras such that φ and φ^{-1} preserve zero-products. (That is, we have $xy = 0$ if and only if $\varphi(x)\varphi(y) = 0$, for all $x, y \in A$.)

Do we obtain $A \cong B$? What can we say about the structure of φ? In particular, is φ automatically continuous?

Theorem 5 (Alaminos-Bresar-Extremera-Villena 2009, [ABEV09])**.** *Let $\varphi\colon A \to B$ be a* bounded*, bijective, linear, zero-product preserving map between C^*-algebras. Then φ is a* weighted isomorphism*, that is, there exists a central, invertible multiplier $h \in Z(M(B))^{-1}$ and an isomorphism $\psi\colon A \to B$ such that $\varphi(a) = h\psi(a)$ for all $a \in A$. In particular, $A \cong B$.*

Theorem 5 shows that Question 4 is really a question about automatic continuity. Results showing automatic continuity are difficult and highly nontrivial.

Example 6 (Jarosz 1990, [Jar90]). Let X and Y be compact, Hausdorff spaces, and let $\varphi\colon C(X) \to C(Y)$ be a bijective, linear map preserving zero-products. Then φ is automatically continuous, and consequently $C(X) \cong C(Y)$, and thus X and Y are homeomorphic.

Proposition 7. *Let $\varphi\colon A \to B$ be a bijective, linear map between C^*-algebras. Then the following are equivalent:*

(1) φ is a weighted isomorphism;
(2) $\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$ for all $a, b, c \in A$.

Proof. The implication '(1)⇒(2)' is clear. The idea for the implication '(2)⇒(1)' is to define $\alpha\colon B \to B$ by $\alpha(\varphi(x)\varphi(y)) = \varphi(xy)$ for $x, y \in A$. One shows that α is a well-defined, central, invertible multiplier on B and that $\alpha \circ \varphi\colon A \to B$ is an isomorphism. □

Our approach to Question 4 is to verify the formula in Proposition 7(2). For the purpose of this talk, we use the following definition. (The actual definition is slightly different.)

Definition 8 (Gardella-T)**.** A C^*-algebra A is said to be *zero-product balanced* if

$$ab \otimes c - a \otimes bc \in \operatorname{span}\{u \otimes v \in A \otimes A : uv = 0\},$$

for all $a, b, c \in A$, where $A \otimes A$ denotes the algebraic tensor product.

The next result shows that the formula in Proposition 7(2) holds for all (not necessarily bijective) zero-product preserving maps if the domain is zero-product balanced.

Lemma 9. *Let $\varphi\colon A \to B$ be a linear, zero-product preserving map between C^*-algebras. Assume that A is zero-product balanced. Then $\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$ for all $a, b, c \in A$.*

The next result is an immediate consequence of Proposition 7 and Lemma 9:

Theorem 10. *Let $\varphi\colon A \to B$ be a bijective, linear, zero-product preserving map between C^*-algebras. Assume that A is zero-product balanced. Then φ is a weighted isomorphism and $A \cong B$.*

Example 11. If X is a compact, Hausdorff space with infinitely many points, then $C(X)$ is not zero-product determined: there exists a (discontinuous) linear map $\varphi\colon C(X) \to \mathbb{C}$ that preserves zero-products, but such that $\varphi(a)\varphi(1) \neq \varphi(1)\varphi(a)$ for some $a \in C(X)$. (This is based on Proposition 2.12 in [Bre16].)

This should be compared with a result of Dales from 1979, [Dal79], which shows that for X a compact, Hausdorff space with infinitely many points, there exists (assuming the continuum hypothesis) a discontinuous homomorphism from $C(X)$ to a Banach algebra.

Proposition 12. *Assume that a C^*-algebra A is generated (as an algebra) by its idempotents. Then A is zero-product determined.*

Proof. For simplicity, we only verify the conclusion of Lemma 9. Let $\varphi\colon A \to B$ be a linear, zero-product preserving map to another C^*-algebra. Let $a, c \in A$ and let $e \in A$ be an idempotent. Then $[ae][(1-e)c] = 0$, and therefore $\varphi(ae)\varphi((1-e)c) = 0$. Adding $\varphi(ae)\varphi(ec)$, we obtain

$$\varphi(ae)\varphi(c) = \varphi(ae)\varphi(ec).$$

An analogous argument shows that $\varphi(ae)\varphi(ec) = \varphi(a)\varphi(ec)$, and thus

$$\varphi(ae)\varphi(c) = \varphi(ae)\varphi(ec) = \varphi(a)\varphi(ec).$$

If e_1 and e_2 are idempotents, then

$$\varphi(ae_1e_2)\varphi(c) = \varphi(ae_1)\varphi(e_2c) = \varphi(a)\varphi(e_1e_2c).$$

Inductively, we get $\varphi(ae_1e_2\cdots e_n)\varphi(c) = \varphi(a)\varphi(e_1e_2\cdots e_nc)$ for every idempotents $e_1, \ldots, e_n$. Using linearity and the assumption that A is generated by its idempotents, we deduce that $\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$ for all $a, b, c \in A$. □

Example 13. Let A be a unital C^*-algebra, and let $n \geq 2$. Then $M_n(A)$ is generated by idempotents and hence is zero-product balanced. (See, for example, Proposition 2.18 in [Bre16].)

Thus, for any other C^*-algebra B we have $M_n(A) \cong B$ if and only if there exists a bijective, linear, zero-product preserving map $M_n(A) \to B$.

Theorem 14 (Gardella-T)**.** *Let A be a unital C^*-algebra without one-dimensional irreducible representations. Then A is zero-product balanced.*

Corollary 15. *Let A be a unital C^*-algebra without one-dimensional irreducible representations. Then, for any other C^*-algebra B we have $A \cong B$ if and only if there exists a bijective, linear, zero-product preserving map $A \to B$.*

Remarks 16. (1) Theorem 14 applies in particular to every unital, simple C^*-algebra. There are important examples (like the Jiang-Su algebra $\mathcal{Z}$) of such C^*-algebras that contain no idempotents different from 0 and 1.

(2) The idea to our proof of Theorem 14 is to show that A is generated by 'special' square-zero elements that can be 'transferred' like idempotents in the proof of Proposition 12.

(3) Theorem 14 also applies to non-unital C^*-algebras assuming that $M(A)$ has no one-dimensional irreducible representations. This raises the next question.

Question 17. Let A be a non-unital C^*-algebra. When does $M(A)$ have no one-dimensional irreducible representations?

A necessary condition for a positive answer to Question 17 is that A itself has no one-dimensional irreducible representations. But this condition is not sufficient: Robert-Rørdam, [RR13], show that there exists a separable (non-simple) C^*-algebra A that has no one-dimensional irreducible representations (not even any finite-dimensional irreducible representations), but such that $M(A)$ has a one-dimensional irreducible representation.

Based on results of Sakai and Pedersen, on can show that there exists a simple (non-separable) C^*-algebra A such that $M(A)/A \cong \mathbb{C}$; see Example 4.15 in [TV21].

Does there exist a (non-unital) simple *and* separable C^*-algebra A such that $M(A)$ has a one-dimensional irreducible representation?

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