Diffuse traces and Haar unitaries

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C*-algebras

Definition

A **C*-algebra** is a norm-closed *-invariant subalgebra $A \subseteq \mathcal{B}(H)$ for some Hilbert space H.

Examples

- compact Hausdorff space X
  $\rightsquigarrow C(X) = \{f \colon X \to \mathbb{C} : f$ continuous$\}$
- discrete group $G \rightsquigarrow$ left-regular representation $G \curvearrowright \ell^2(G)$
  $\rightsquigarrow$ **reduced group C*-algebra**
  $C^*_{\mathrm{red}}(G) = \overline{\mathrm{span}}\{u_g : g \in G\} \subseteq \mathcal{B}(\ell^2(G))$
  - $C^*_{\mathrm{red}}(\mathbb{Z}) \cong C(\mathbb{T})$

Traces and tracial states

Definition

Let A be a C*-algebra. A **trace** of A is a positive, linear functional $\tau\colon A \to \mathbb{C}$ that is **tracial**: $\tau(ab) = \tau(ba)$ for all $a, b \in A$. A **tracial state** is a trace τ with $\|\tau\| = 1$.

Examples

- Riesz theorem: Traces on $C(X)$ correspond to positive Borel measures on X. Measure μ corresponds to

$$\tau_\mu\colon C(X) \to \mathbb{C}, \quad \tau_\mu(f) = \int_X f(x)\,d\mu(x), \quad \text{for } f \in C(X).$$

  Tracial states on $C(X)$ $\leftrightsquigarrow$ probability measures on X
- Canonical tracial state

$$\tau_G\colon C^*_{\mathrm{red}}(G) \to \mathbb{C}, \quad \tau_G\Big(\sum_{g\in G} c_g u_g\Big) = c_1, \quad \text{for } c_g \in \mathbb{C}.$$

Haar unitaries

Definition

Let $\tau\colon A \to \mathbb{C}$ be a tracial state on a unital C*-algebra A.
A unitary $u \in A$ is a **Haar unitary** for τ if $\tau(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Exercise

A unitary $u \in A$ is a Haar unitary for τ if and only if $\mathrm{sp}(u) = \mathbb{T}$ (that is, $C^*(u) = C(\mathbb{T})$) and the trace $\tau|_{C(\mathbb{T})}$ corresponds to the normalized Lebesgue measure λ on $\mathbb{T}$.

Example

$\tau_G\colon C^*_{\mathrm{red}}(G) \to \mathbb{C}$ satisfies $\tau_G(u_g) = \begin{cases} 1, & g = 1 \\ 0, & g \neq 1 \end{cases}$

For $g \in G$ and $k \in \mathbb{Z}$ have $u_g^k = u_{g^k}$ and so $\tau_G(u_g^k) = 0$ iff $g^k \neq 1$.
Thus: $u_g \in C^*_{\mathrm{red}}(G)$ is a Haar unitary for τ_G if and only if g has infinite order.

Applications of Haar unitaries (1)

Recall that unital subalgebras $B, C \subseteq A$ are **free** with respect to τ if $\tau(a_1 a_2 \cdots a_n) = 0$ whenever $\tau(a_j) =$ for all j and either $a_1, a_3, \ldots \in B$ and $a_2, a_4, \ldots \in C$ or vice versa.

Proposition

Let $p, q \in A$ projections with $\tau(p) < \tau(q)$. Then $p \precsim q$, if there is Haar unitary $u \in A$ such that $C^*(p, q, 1)$ and $C^*(u)$ are free.

Proof.

- If $u \in A$ is a Haar unitary, and $B \subseteq A$ unital subalgebra such that $C^*(u)$ and B are free, then B and uBu^* are free.
- If $\bar{p}, \bar{q} \in A$ are projections such that $C^*(\bar{p}, 1)$ and $C^*(\bar{q}, 1)$ are free, and $\tau(\bar{p}) < \tau(\bar{q})$, then $\bar{p} \precsim \bar{q}$.
- Consider $\bar{p} := p$ and $\bar{q} := uqu^*$. □

Applications of Haar unitaries (2)

Proposition

Let $p, q \in A$ projections with $\tau(p) < \tau(q)$. Then $p \precsim q$, if there is Haar unitary $u \in A$ such that $C^*(p, q, 1)$ and $C^*(u)$ are free.

Theorem (Dykema-Rørdam 2000)

Let (A_k, τ_k) such that each τ_k admits a Haar unitary for $k \in \mathbb{N}$. If projections p, q in the reduced free product of all (A_k, τ_k) satisfy $\tau(p) < \tau(q)$, then $p \precsim q$.

- Robert 2012: Works also for comparison of positive elements. Can compute Cuntz semigroup of $C^*_{\mathrm{red}}(\mathbb{F}_\infty)$, but not (yet) of $C^*_{\mathrm{red}}(\mathbb{F}_2)$.
- Popa 1995: For every II_1 factor M, there exists a Haar unitary u in the ultrapower M^ω such that $C^*(u)$ and M are free in M^ω.

The question

Question

When does a tracial state admit a Haar unitary?

The commutative case

Proposition (Dykema-Haagerup-Rørdam 1997)

$\tau\colon C(X) \to \mathbb{C}$ *admits a Haar unitary if and only if the associated measure* μ *on* X *is **diffuse**:* $\mu(\{x\}) = 0$ *for every* $x \in X$*.*

Proof. (different from DHR)

$\Rightarrow$: Let $u \in C(X)$ be a Haar unitary. The inclusion $C(\mathbb{T}) = C^*(u) \subseteq C(X)$ corresponds to a surjective map $h\colon X \to \mathbb{T}$ such that $h_*(\mu) = \lambda$. Let $x \in X$. Then

$$\mu(\{x\}) \leq \mu(h^{-1}(\{h(x)\})) = \lambda(\{h(x)\}) = 0.$$

$\Leftarrow$: Assume μ is diffuse. Sierpiński's theorem gives Borel sets $(E_t)_{t\in[0,1]}$ such that: (a) $\mu(E_t) = t$; and (b) $E_{t'} \subseteq E_t$ if $t' \leq t$.
Using regularity of the measure can find open sets $(U_t)_{t\in[0,1]}$ such that: (a) $\mu(U_t) = t$; and (b) $\overline{U_{t'}} \subseteq U_t$ if $t' < t$; and (c) $U_t = \bigcup\{U_{t'} : t' < t\}$.
$\rightsquigarrow$ Get $f\colon X \to [0,1]$ with $U_t = f^{-1}([0,t))$. Use $u := \exp(2\pi i f)$.

The main result

Theorem (T-2020)

Let (A, τ) be a unital C^*-algebra with a tracial state. TFAE:

1. *τ admits a Haar unitary;*
2. *there exists a (maximal) unital, abelian sub-C^*-algebra $C(X) \subseteq A$ such that τ induces a diffuse measure on X;*
3. *τ is **diffuse**: the unique extension to a normal, tracial state $A^{**} \to \mathbb{C}$ vanishes on every minimal projection in A^{**};*
4. *$\pi_\tau(A)''$ is a diffuse von Neumann algebra;*
5. *τ does not dominate a trace that factors through a finite-dimensional quotient of A.*

(1)⇒(2): consider $C(\mathbb{T}) \cong C^*(u) \subseteq A$
(3)⇔(4)⇔(5) not so difficult.

Sketch of the proof

1. τ admits a Haar unitary;
2. there exists a unital, abelian sub-C^*-algebra $C(X) \subseteq A$ such that τ induces a diffuse measure μ on X;
3. τ is diffuse;
5. τ does not dominate a trace that factors through a finite-dimensional quotient of A.

(2)⇒(5): Let $\pi\colon A \to M_n(\mathbb{C})$ and $c > 0$ such that $\tau \geq c \cdot \mathrm{tr}_n \circ \pi$. There are $x_1, \ldots, x_n \in X$ such that $\pi|_{C(X)}\colon C(X) \to M_n(\mathbb{C})$ is unitarily conjugate to $f \mapsto \mathrm{diag}(f(x_1), \ldots, f(x_n))$. Then

$$\mu(\{x_1, \ldots, x_n\}) \geq c > 0. \quad ↯$$

(3)⇒(1): An **open projection** is $p \in A^{**}$ that is the weak*-limit of an increasing net in A_+.
Using that τ is diffuse, we construct open projections $(p_t)_{t\in[0,1]}$ such that: (a) $\tau(p_t) = t$; and (b) $\overline{p_{t'}} \leq p_t$ if $t' < t$; and (c) $p_t = \sup\{p_{t'} : t' < t\}$.
$\rightsquigarrow$ contractive $a \in A_+$ with $p_t = \mathbb{1}_{[0,t)}(a)$. Use $u := \exp(2\pi i a)$.

Direct consequences

Theorem (T-2020)

Let (A, τ) be a unital C^*-algebra with a tracial state. TFAE:

1. *τ admits a Haar unitary;*
5. *τ does not dominate a trace that factors through a finite-dimensional quotient of A.*

Corollary

A unital C^*-algebra has no finite-dimensional representations if and only if every of its tracial states admits a Haar unitary.

Corollary

Every tracial state on an infinite-dimensional, simple, unital C^*-algebra admits a Haar unitary.

Comparison with von Neumann algebras

Proposition

Let M be a diffuse von Neumann algebra, and $\tau : M \to \mathbb{C}$ a normal trace. Then every masa of M contains a Haar unitary.

The analog is not true for (diffuse) traces on C^*-algebras:

Example

Let $\mathcal{T} \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ Toeplitz algebra. Masa $\ell^\infty(\mathbb{N}) \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ leads to masa $B := \ell^\infty(\mathbb{N}) \cap \mathcal{T}$ in $\mathcal{T}$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{T} & \xrightarrow{\pi} & C(\mathbb{T}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & c_0(\mathbb{N}) & \longrightarrow & B & \longrightarrow & \mathbb{C} & \longrightarrow & 0
\end{array}$$

Diffuse trace τ_0 on $C(\mathbb{T})$ induces diffuse trace $\tau := \tau_0 \circ \pi$ on $\mathcal{T}$. But B contains no Haar unitary for τ: Given $u \in B$, have $\pi(u) = z \in \mathbb{T} \subseteq \mathbb{C}$ and then $\tau(u) = z \neq 0$. ↯

Application: Group C*-algebras (1)

Let G be a discrete group.

Proposition

G is infinite if and only if $\tau_G \colon C^*_{\mathrm{red}}(G) \to \mathbb{C}$ admits a Haar unitary.

Proof.

$$
\begin{aligned}
G \text{ infinite } &\Leftrightarrow W^*(G) = \pi_{\tau_G}(G)'' \text{ diffuse (Dykema 1993)} \\
&\Leftrightarrow \tau_G \text{ diffuse} \\
&\Leftrightarrow \tau_G \text{ admits Haar unitary}
\end{aligned}
$$

□

Example

Let G be an infinite torsion group (e.g. $\mathbb{Q}/\mathbb{Z}$). Then τ_G admits a Haar unitary, but none of the unitaries u_g for $g \in G$ is Haar.

Application: Group C*-algebras (2)

Proposition

G is infinite if and only if τ_G admits a Haar unitary.

Proposition

G is nonamenable if and only if every tracial state of $C^*_{\mathrm{red}}(G)$ admits a Haar unitary.

Proof.

G nonamenable

$\Leftrightarrow$ $C^*_{\mathrm{red}}(G)$ has no finite-dimensional representations

$\Leftrightarrow$ every tracial state on $C^*_{\mathrm{red}}(G)$ admits Haar unitary

□

Application: Free products (1)

We consider unital C^*-algebras with faithful tracial states.

The **reduced free product** of (A, τ_A) and (B, τ_B) is the (unique) (C, τ_C) together with unital embeddings $A, B \subseteq C$ such that

1. τ_C restricts to τ_A on A, and to τ_B on B;
2. $C = C^*(A, B)$;
3. A and B are free with respect to τ_C.

We write $(C, \tau_C) = (A, \tau_A) *_{\text{red}} (B, \tau_B)$.

Motivating example:

$$(C^*_{\text{red}}(G), \tau_G) *_{\text{red}} (C^*_{\text{red}}(H), \tau_H) \cong (C^*_{\text{red}}(G * H), \tau_{G*H}).$$

Application: Free products (2)

Example

$$(C^*_{\mathrm{red}}(\mathbb{F}_2), \tau_{\mathbb{F}_2}) \cong (C^*_{\mathrm{red}}(\mathbb{Z} * \mathbb{Z}), \tau_{\mathbb{Z}*\mathbb{Z}}) \cong (C(\mathbb{T}), \mu_\lambda) *_{\mathrm{red}} (C(\mathbb{T}), \mu_\lambda).$$

$C^*_{\mathrm{red}}(\mathbb{F}_2)$ is simple (Powers 1975).
$C^*_{\mathrm{red}}(\mathbb{F}_2)$ has stable rank one (Dykema-Haagerup-Rørdam '97).

Definition (Rieffel 1983)

A unital C^*-algebra has **stable rank one** (SR1) if its invertible elements are dense.

Important in K-theory. Recall $K_0(A) = \mathrm{Gr}(V(A))$ for

$$V(A) = \{\text{finitely generated, projective } A\text{-modules}\}/\cong .$$

If A has SR1, then $V(A)$ is cancellative and then $V(A) \subseteq K_0(A)$.
Have: $K_0(C^*_{\mathrm{red}}(\mathbb{F}_2)) \cong \mathbb{Z}$. It follows that $V(C^*_{\mathrm{red}}(\mathbb{F}_2)) \cong \mathbb{N}$.
$\rightsquigarrow$ Every finitely generated, projective $C^*_{\mathrm{red}}(\mathbb{F}_2)$-module is free.

Application: Free products (3)

Theorem

Let (A, τ_A) and (B, τ_B) be unital, simple C^*-algebras with stable rank one. Then so is $(A, \tau_A) *_{\mathrm{red}} (B, \tau_B)$.

Proof.

If $A = \mathbb{C}$ or $B = \mathbb{C}$, then reduced free product is B or A. So assume $A, B \neq \mathbb{C}$.
Case 1: A and B finite-dimensional (Avitzour 1982, DHR 1997)
Case 2: A or B infinite-dimsional. Then τ_A or τ_B admits Haar unitary. Can apply result of Dykema 1999. □

This is the C^*-analog of: reduced free product of II_1-factors is again II_1-factor. (We consider infinite-dimensional, unital, simple C^*-algebras with stable rank one as C^*-analogs of II_1-factors.)

Questions

Question 1

When is there $u \in A$ with $\tau(u) = \tau(u^2) = \ldots = \tau(u^k) = 0$?
Does every τ admit such u if and only if A has no k-dimensional representations?
Already interesting for $k = 1$.

Question 2

If $\varphi\colon A \to \mathbb{C}$ is a state, the **centralizer** is

$$A_\varphi := \{a \in A : \varphi(ab) = \varphi(ba) \text{ for all } b \in A\}.$$

When does A_φ contain a Haar unitary?

Thank you.

Reference:
Thiel. Diffuse traces and Haar unitaries. arXiv:2009.06940, 25 pp.