

The generator rank of C^* -algebras

Hannes Thiel

TU Dresden

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The generator problem for von Neumann algebras

Problem 14 of Kadison's 1967 list of "Problems on von Neumann algebras"

Is every von Neumann algebra $M \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ singly generated, that is, is there $x \in \mathcal{B}(\ell^2(\mathbb{N}))$ such that $M = \{x, x^*\}$?

Verified for the following classes of von Neumann algebras:

- abelian (von Neumann 1931)
- type I (Percy 1962)
- properly infinite (Wogen 1969)

↪ reduced to type II_1 case

Theorem (Willig 1974)

Every separably-acting vNA is singly-generated if (and only if) every separably-acting II_1 factor is.

- It is unknown if the free group factors $W^*(\mathbb{F}_k)$ are singly-generated for $k \geq 3$.

The generator problem for C^* -algebras I

Definition

Given a C^* -algebra A , let $\text{gen}(A)$ denote the minimal number of **self-adjoint** generators.

- $\text{gen}(A) \leq n$ iff $A = C^*(a_1, \dots, a_n)$ for some $a_1, \dots, a_n \in A_{\text{sa}}$.
- If $a, b \in A_{\text{sa}}$, then $C^*(a, b) = C^*(a + ib)$. Thus, A is singly-generated if and only if $\text{gen}(A) \leq 2$.

Proposition

Let X be a compact, metrizable space. Then $\text{gen}(C(X)) \leq n$ if and only if $X \hookrightarrow \mathbb{R}^n$.

Proof.

If $X \hookrightarrow \mathbb{R}^n$, then $\iota: X \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and $C(X)$ is generated by $\pi_j \circ \iota: X \rightarrow \mathbb{R}$ for coordinate projections $\pi_j: \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, \dots, n$. Conversely, if $C(X)$ is generated by $f_1, \dots, f_n: X \rightarrow \mathbb{R}$, then $X \rightarrow \mathbb{R}^n$, $x \mapsto (f_1(x), \dots, f_n(x))$ is injective. □

The generator problem for C^* -algebras II

The generator problem

Given a C^* -algebra A , determine $\text{gen}(A)$. In particular, which C^* -algebras are singly generated?

Facts:

■ If $I \subseteq A$ is an ideal, then $\text{gen}(A/I) \leq \text{gen}(A)$.

■ $\text{gen}(A \otimes M_n) \leq \left\lceil \frac{\text{gen}(A)-1}{n^2} \right\rceil + 1$.

Thus, if A is generated by at most $n^2 + 1$ self-adjoint elements, then $A \otimes M_n$ is generated by two self-adjoints.

Examples:

■ $C^*(\mathbb{F}_2)$ is not singly-generated, since $C^*(\mathbb{F}_2) \rightarrow C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$, and $\mathbb{T}^2 \not\cong \mathbb{R}^2$.

■ $C^*(\mathbb{F}_2) \otimes M_2$ is singly generated, since $\text{gen}(C^*(\mathbb{F}_2)) \leq 4$.

The generator problem for C^* -algebras III

Slogan: The more noncommutative, the fewer generators.

Question

Is every separable C^* -algebra without finite-dimensional representations singly generated?

Is every separable, simple C^* -algebra singly-generated?

Verified for the following classes of C^* -algebras:

- separable, properly infinite (Kirchberg)
- separable, UHF-stable algebras (Olsen, Zame 1976)
- separable, \mathcal{Z} -stable algebras (T, Winter 2014)

Main open problem:

- It is unknown if the reduced free group C^* -algebras $C_{\lambda}^*(\mathbb{F}_k)$ are singly generated for $k \geq 2$.

The generator problem for C^* -algebras IV

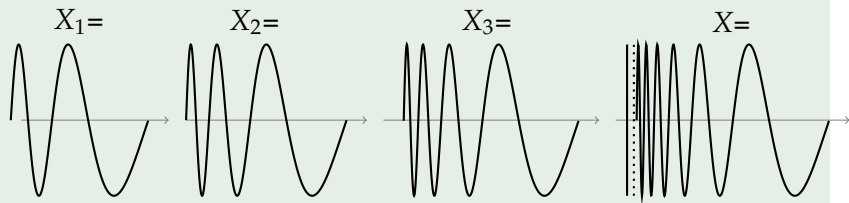
Difficulty: Invariant $\text{gen}(-)$ lacks nice permanence properties.

Example: $\text{gen}(-)$ may increase when passing to ideals

Set $A = C([0, 1])$, $I = C_0((0, 1))$. Then $\text{gen}(A) = 1$, $\text{gen}(I) = 2$.

Example: $\text{gen}(-)$ may increase when passing to inductive limits

Topologists sine-curve $X \subseteq \mathbb{R}^2$. Have $X \cong \varprojlim X_k$ with each $X_k \cong [0, 1]$. Then $C(X) \cong \varinjlim C(X_k)$ and $\text{gen}(C(X_k)) = 1$ for each k , but $\text{gen}(C(X)) = 2$.



The generator rank I

Idea: Instead of generating self-adjoint tuples, consider **stable** generating self-adjoint tuples.

Notation

$$\text{Gen}_n(A)_{\text{sa}} := \{(a_1, \dots, a_n) \in A_{\text{sa}}^n \mid A = C^*(a_1, \dots, a_n)\}.$$

Definition

If A is unital, then the **generator rank** $\text{gr}(A)$ is the minimal $n \in \mathbb{N}$ such that $\text{Gen}_{n+1}(A)_{\text{sa}}$ is dense in A_{sa}^{n+1} , or $\text{gr}(A) = \infty$ if no such n exists. If A is nonunital, set $\text{gr}(A) := \text{gr}(\tilde{A})$.

Notice the index shift!

Example

We have $\text{gr}(\mathbb{C}) = 0$, since $\text{Gen}_1(\mathbb{C})_{\text{sa}} = \mathbb{R} \setminus \{0\}$ is dense in $\mathbb{C}_{\text{sa}} = \mathbb{R}$.

The generator rank II

The index shift follows the usual convention in (noncommutative) dimension theory:

- The **covering dimension** of X satisfies $\dim(X) \leq n$ if for every finite, open cover of X there exists a finite, open refinement \mathcal{V} which can be coloured with $n + 1$ colours: $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1 \sqcup \dots \sqcup \mathcal{V}_n$ such that the elements in each \mathcal{V}_k are pairwise disjoint.
- The **nuclear dimension** of A satisfies $\dim_{\text{nuc}}(A) \leq n$ if $\text{id}: A \rightarrow A$ can be approximately factored by cp-maps $\psi: A \rightarrow F$ and $\varphi: F \rightarrow A$ with F finite-dimensional and φ a sum of $n + 1$ orthogonality-preserving maps.
- The **real rank** of unital A satisfies $\text{rr}(A) \leq n$ if

$$\left\{ (a_0, \dots, a_n) \in A_{\text{sa}}^{n+1} : \sum_k a_k^2 \text{ invertible} \right\} \subseteq A_{\text{sa}}^{n+1}$$

is dense.

- We have $\text{rr}(A) \leq \text{gr}(A)$.

The generator rank III

Recall:

$$\text{Gen}_n(A)_{\text{sa}} := \{(a_1, \dots, a_n) \in A_{\text{sa}}^n \mid A = C^*(a_1, \dots, a_n)\},$$

$$\text{gen}(A) \leq n \Leftrightarrow \text{Gen}_n(A)_{\text{sa}} \neq \emptyset$$

$$\text{gr}(A) \leq n \Leftrightarrow \text{Gen}_{n+1}(A)_{\text{sa}} \subseteq A_{\text{sa}}^{n+1} \text{ dense}$$

- Thus $\text{gen}(A) \leq \text{gr}(A) + 1$.
- $\text{gen}(A) \leq 2$ means A is singly-generated. $\text{gr}(A) \leq 1$ means that $\{x \in A : A = C^*(x)\}$ is a dense G_δ -subset of A . (A generic element of A is a generator.)

Example: $\text{gen}(A)$ can be (much) smaller than $\text{gr}(A)$.

For $A = C([0, 1]^k) \otimes M_d$, we have:

$$\text{gen}(A) = \left\lceil \frac{k-1}{d^2} \right\rceil + 1, \quad \text{gr}(A) = \left\lceil \frac{k+1}{2d-2} \right\rceil.$$

For $B = \bigoplus_{d=1}^{\infty} C([0, 1]^{d^2}) \otimes M_d$, have $\text{gen}(B) \leq 2$ but $\text{gr}(B) = \infty$.

The generator rank IV

Theorem (T 2020 - permanence properties)

Let $I \subseteq A$ be an ideal. Then

$$\max \{ \text{gr}(I), \text{gr}(A/I) \} \leq \text{gr}(A) \leq \text{gr}(I) + \text{gr}(A/I) + 1.$$

Further, if $A = \varinjlim_{\lambda} A_{\lambda}$ is an inductive limit, then

$$\text{gr}(A) \leq \liminf_{\lambda} \text{gr}(A_{\lambda}).$$

Corollary

Let A be an AF-algebra. Then $\text{gr}(A) \leq 1$, and so a generic element of A is a generator. In particular, A is singly generated.

Proof.

Show that $\text{gr}(M_n(\mathbb{C})) = 1$: Given self-adjoint $a, b \in M_n(\mathbb{C})$, conjugate by unitary to assume a is diagonal. Approximate a by diagonal matrix with pairwise different, nonzero eigenvalues. Approximate b by matrix with all entries nonzero. □

The commutative case I

Theorem (T 2020)

Let X be a compact, metrizable space. Then

$$\text{gr}(C(X)) = \dim(X \times X).$$

Proof.

Identify $C(X)_{\text{sa}}^{n+1}$ with $C(X, \mathbb{R}^{n+1})$. By Stone-Weierstrass, $f \in C(X, \mathbb{R}^{n+1})$ generates $C(X)$ if and only if (a) $f(x) \neq 0$ for all $x \in X$; and (b) $f(x) \neq f(y)$ for $x \neq y$. Thus, $\text{gr}(C(X)) \leq n$ iff:

- (a) $C(X, \mathbb{R}^{n+1} \setminus \{0\}) \subseteq C(X, \mathbb{R}^{n+1})$ is dense;
- (b) $E(X, \mathbb{R}^{n+1} \setminus \{0\}) \subseteq C(X, \mathbb{R}^{n+1} \setminus \{0\})$ is dense.

We have (a) iff $\dim(X) \leq n$ (classical); and (b) iff $\dim(X \times X) \leq n$ (using cohomological dimension and unstable intersection theory - Dranišnikov, Levin, Repovš, Ščepin, West). Result follows since $\dim(X) \leq \dim(X \times X)$. \square

The commutative case II

Theorem (T 2020)

Let X be a compact, metrizable space. Then

$$\text{gr}(C(X)) = \dim(X \times X).$$

Example

Consider $X = [0, 1]$. Then $C([0, 1])$ is generated by one self-adjoint, for example $x \mapsto 1 + x$. But

$$\text{gr}(C([0, 1])) = \dim([0, 1] \times [0, 1]) = 2,$$

that is, generating elements in $C([0, 1], \mathbb{R}^n)$ are dense for $n = 3$; but not for $n = 1$ ($x \mapsto -1 + 2x$ cannot be approximated by invertible functions) and neither for $n = 2$ (there exists $g: [0, 1] \rightarrow \mathbb{R}^2$ that cannot be approximated by embeddings).

Note: $\dim(X \times X)$ is either $2 \dim(X)$ or $2 \dim(X) - 1$ (that latter occurs by examples of Boltyanskiĭ from 1949).

The subhomogeneous case II

- A C^* -algebra A is d -**(sub)homogeneous** if every irreducible representation of A has dimension (at most) d .
- Typical example: $C(X) \otimes M_d$ is d -homogeneous.
- Subalgebras of $C(X) \otimes M_d$ are d -subhomogeneous.
- Let A be a unital, d -homogeneous C^* -algebra. Then the primitive ideal space $\text{Prim}(A)$ is compact and Hausdorff, and A is the section algebra of a locally trivial M_d -bundle over $\text{Prim}(A)$.
- Let A be a unital, d -subhomogeneous C^* -algebra. Then there is a unique chain of ideals

$$\{0\} = I_{d+1} \subseteq I_d \subseteq I_{d-1} \subseteq \dots \subseteq I_1 = A$$

such that I_k/I_{k+1} is k -homogeneous.

(I_k corresponds to the ‘part’ of A where all irreducible representations have dimension $\geq k$)

The subhomogeneous case II

Theorem (T 2020)

Let A be a unital, separable d -subhomogeneous C^* -algebra. For each $k \leq d$, set $X_k := \text{Prim}_k(A)$, the subset of the primitive ideal space of A corresponding to k -dimensional irreducible representations. Then:

$$\text{gr}(A) = \max \left\{ \dim(X_1 \times X_1), \sup_{k \geq 2} \left\lceil \frac{\dim(X_k) + 1}{2k - 2} \right\rceil \right\}.$$

- Note: If A is subhomogeneous with $\text{gr}(A) < \infty$, then $\text{gr}(A \otimes M_n) = 1$ for n sufficiently large.
- If A is subhomogeneous, then $A = \varinjlim_k A_k$ for subhomogeneous A_k with $\text{gr}(A_k) < \infty$.
- UHF-algebra $M_{p^\infty} = \bigotimes_{\mathbb{N}} M_p \cong \varinjlim_n M_{p^n}$.
- Thus: If A is subhomogeneous, then $\text{gr}(A \otimes \text{UHF}) = 1$.

The approximately subhomogeneous case

- A C^* -algebra is **approximately subhomogeneous** (ASH) if it is an inductive limit of subhomogeneous C^* -algebras.
- Includes many interesting classes of nuclear C^* -algebras: AF-algebras, irrational rotation algebras, (certain) crossed products.
- The Jiang-Su algebra \mathcal{Z} is an interpolation between M_{p^∞} for all primes p .
- A C^* -algebra A is **\mathcal{Z} -stable** if $A \cong \mathcal{Z} \otimes A$. (The C^* -analog of being McDuff.)

Corollary

Let A be a separable, \mathcal{Z} -stable ASH-algebra. Then $\text{gr}(A) = 1$, and so a generic element of A is a generator.

The classifiable case

- Simple, nuclear A is **classifiable** if it is unital, separable, \mathcal{Z} -stable and satisfies the Universal Coefficient Theorem.
- By recent breakthrough in the Elliott classification program, simple, nuclear, classifiable C^* -algebras are isomorphic iff their Elliott invariants (K -theoretic and tracial data) are.

Theorem (T 2020)

Let A be a simple, nuclear, classifiable C^ -algebra. Then $\text{gr}(A) = 1$, and so a generic element of A is a generator.*

Proof.

A is either stably finite or purely infinite. If purely infinite, then A is a Kirchberg algebra and so has real rank zero (handled with different methods). If stably finite, then A is ASH. \square

- Villadsen: There exist simple, separable, unital, nuclear (ASH) algebras A with $\text{gr}(A) = \infty$.

Thank you.

Reference:

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- 3 Thiel. Generators in \mathcal{Z} -stable C^* -algebras of real rank zero. arXiv:2006.08404, 17p.
- 4 Thiel, Winter. The generator problem for \mathcal{Z} -stable C^* -algebras. Trans. Amer. Math. Soc. 366 (2014), 2327-2343.