The generator rank of C*-algebras

Hannes Thiel

TU Dresden

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The generator problem for C^*-algebras I

Definition

Given a C^*-algebra A, let $\mathrm{gen}(A)$ denote the minimal number of **self-adjoint** generators.

- $\mathrm{gen}(A) \leq n$ iff $A = C^*(a_1, \dots, a_n)$ for some $a_1, \dots, a_n \in A_{\mathrm{sa}}$.
- If $a, b \in A_{\mathrm{sa}}$, then $C^*(a, b) = C^*(a + ib)$. Thus, A is singly-generated if and only if $\mathrm{gen}(A) \leq 2$.

The generator problem

Given a C^*-algebra A, determine $\mathrm{gen}(A)$. In particular, which C^*-algebras are singly generated?

Example

Let X be a compact, metrizable space. Then $\mathrm{gen}(C(X)) \leq n$ if and only if $X \hookrightarrow \mathbb{R}^n$. Thus: $C(X)$ singly-generated $\Leftrightarrow$ X planar.

The generator problem for C^*-algebras II

The generator problem: Determine $\mathrm{gen}(A)$.

Facts:

- If $I \subseteq A$ is an ideal, then $\mathrm{gen}(A/I) \leq \mathrm{gen}(A)$.
- $\mathrm{gen}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{gen}(A)-1}{n^2} \right\rceil + 1$.

Examples:

- $C^*(\mathbb{F}_2)$ is not singly-generated, since $C^*(\mathbb{F}_2) \twoheadrightarrow C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$, and $\mathbb{T}^2 \not\hookrightarrow \mathbb{R}^2$.
- $C^*(\mathbb{F}_2) \otimes M_2$ is singly generated, since $\mathrm{gen}(C^*(\mathbb{F}_2)) \leq 4$.

Slogan: The more noncommutative, the fewer generators.

Question

Is every separable C^*-algebra without finite-dimensional representations singly generated?
Is every separable, simple C^*-algebra singly-generated?

- Main open problem: What is $\mathrm{gen}(C^*_{\mathrm{red}}(\mathbb{F}_k))$, for $k \geq 2$?

The generator problem for C^*-algebras III

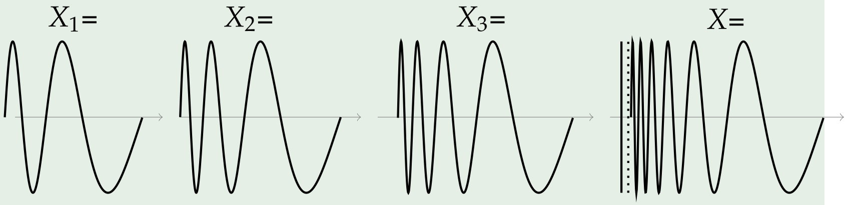
Difficulty: Invariant $\mathrm{gen}(\_)$ lacks nice permanence properties.

Example: $\mathrm{gen}(\_)$ may increase when passing to ideals

Set $A = C([0,1])$, $I = C_0((0,1))$. Then $\mathrm{gen}(A) = 1$, $\mathrm{gen}(I) = 2$.

Example: $\mathrm{gen}(\_)$ may increase when passing to inductive limits

Topologists sine-curve $X \subseteq \mathbb{R}^2$. Have $X \cong \varprojlim X_k$ with each $X_k \cong [0,1]$. Then $C(X) \cong \varinjlim_k C(X_k)$ and $\mathrm{gen}(C(X_k)) = 1$ for each k, but $\mathrm{gen}(C(X)) = 2$.



The generator rank I

Idea: Instead of generating self-adjoint tuples, consider **stable** generating self-adjoint tuples.

Definition

Set

$$\mathrm{Gen}_n(A)_{\mathrm{sa}} := \left\{(a_1,\dots,a_n)\in A_{\mathrm{sa}}^n \mid A = C^*(a_1,\dots,a_n)\right\}.$$

If A is unital, then the **generator rank** $\mathrm{gr}(A)$ is the minimal $n\in\mathbb{N}$ such that $\mathrm{Gen}_{n+1}(A)_{\mathrm{sa}}$ is dense in A_{sa}^{n+1}, or $\mathrm{gr}(A)=\infty$ if no such n exists. If A is nonunital, set $\mathrm{gr}(A) := \mathrm{gr}(\widetilde{A})$.

The index shift follows the usual convention in (noncommutative) dimension theory.

The generator rank II

Recall:

$$\mathrm{Gen}_n(A)_{\mathrm{sa}} := \{(a_1,\ldots,a_n) \in A_{\mathrm{sa}}^n \mid A = C^*(a_1,\ldots,a_n)\},$$
$$\mathrm{gen}(A) \le n \Leftrightarrow \mathrm{Gen}_n(A)_{\mathrm{sa}} \ne \emptyset$$
$$\mathrm{gr}(A) \le n \Leftrightarrow \mathrm{Gen}_{n+1}(A)_{\mathrm{sa}} \subseteq A_{\mathrm{sa}}^{n+1} \text{ dense}$$

- Thus $\mathrm{gen}(A) \le \mathrm{gr}(A) + 1$.
- $\mathrm{gen}(A) \le 2$ means A is singly-generated.
- $\mathrm{gr}(A) \le 1$ means that $\{x \in A : A = C^*(x)\}$ is a dense G_δ-subset of A. (A generic element of A is a generator.)

Example: $\mathrm{gen}(A)$ can be (much) smaller than $\mathrm{gr}(A)$.

For $A = C([0,1]^k) \otimes M_d$, we have:

$$\mathrm{gen}(A) = \left\lceil \frac{k-1}{d^2} \right\rceil + 1, \qquad \mathrm{gr}(A) = \left\lceil \frac{k+1}{2d-2} \right\rceil.$$

For $B = \bigoplus_{d=1}^\infty C([0,1]^{d^2}) \otimes M_d$, have $\mathrm{gen}(B) \le 2$ but $\mathrm{gr}(B) = \infty$.

The generator rank III

Theorem (T 2020 - permanence properties)

Let $I \subseteq A$ *be an ideal. Then*

$$\max\left\{ \mathrm{gr}(I), \mathrm{gr}(A/I) \right\} \leq \mathrm{gr}(A) \leq \mathrm{gr}(I) + \mathrm{gr}(A/I) + 1.$$

Further, if $A = \varinjlim_\lambda A_\lambda$ *is an inductive limit, then*

$$\mathrm{gr}(A) \leq \liminf_\lambda \mathrm{gr}(A_\lambda).$$

Corollary

Let A *be an AF-algebra. Then* $\mathrm{gr}(A) \leq 1$*, and so a generic element of* A *is a generator. In particular,* A *is singly generated.*

Generator rank of commutative C^*-algebras

Theorem (T 2020)

Let X be a compact, metrizable space. Then

$$\mathrm{gr}(C(X)) = \dim(X \times X).$$

Example

Consider $X = [0,1]$. Then $C([0,1])$ is generated by one self-adjoint, for example $x \mapsto 1 + x$. But

$$\mathrm{gr}(C([0,1])) = \dim([0,1] \times [0,1]) = 2,$$

that is, generating elements in $C([0,1], \mathbb{R}^n)$ are dense for $n = 3$; but not for $n = 1$ ($x \mapsto -1 + 2x$ cannot be approximated by invertible functions) and neither for $n = 2$ (there exists $g\colon [0,1] \to \mathbb{R}^2$ that cannot be approximated by embeddings).

Note: $\dim(X \times X)$ is either $2\dim(X)$ or $2\dim(X) - 1$ (that latter occurs by examples of Boltyanskiĭ from 1949).

Generator rank of subhomogeneous C^*-algebras

Theorem (T 2020)

Let A be a unital, separable d-subhomogeneous C^*-algebra. For each $k \leq d$, set $X_k := \mathrm{Prim}_k(A)$, the subset of the primitive ideal space of A corresponding to k-dimensional irreducible representations. Then:

$$\mathrm{gr}(A) = \max\left\{ \dim(X_1 \times X_1), \sup_{k \geq 2} \left\lceil \frac{\dim(X_k) + 1}{2k - 2} \right\rceil \right\}.$$

- If A is subhomogeneous with $\mathrm{gr}(A) < \infty$, then $\mathrm{gr}(A \otimes M_n) = 1$ for n sufficiently large.
- A C^*-algebra A is $\mathcal{Z}$**-stable** if $A \cong \mathcal{Z} \otimes A$. (The C^*-analog of being McDuff.)

Corollary

Let A be a separable, $\mathcal{Z}$-stable ASH-algebra. Then $\mathrm{gr}(A) = 1$.

Generator rank of classifiable C^*-algebras

- Simple, nuclear A is **classifiable** if it is unital, separable, $\mathcal{Z}$-stable and satisfies the Universal Coefficient Theorem.
- By recent breakthrough in the Elliott classification program, simple, nuclear, classifiable C^*-algebras are isomorphic iff their Elliott invariants (K-theoretic and tracial data) are.

Theorem (T 2020)

Let A be a simple, nuclear, classifiable C^*-algebra. Then $\mathrm{gr}(A) = 1$, and so a generic element of A is a generator.

Proof.

A is either stably finite or purely infinite. If purely infinite, then A is a Kirchberg algebra and so has real rank zero (handled with different methods). If stably finite, then A is ASH. □

- Villadsen: There exist simple, separable, unital, nuclear (ASH) algebras A with $\mathrm{gr}(A) = \infty$.

Thank you.

References:

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