The generator rank of C*-algebras

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The generator problem for von Neumann algebras

Problem 14 of Kadison's 1967 list of "Problems on von Neumann algebras"

Is every von Neumann algebra $M \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ singly generated, that is, is there $x \in \mathcal{B}(\ell^2(\mathbb{N}))$ such that $M = \{x, x^*\}''$?

Verified for the following classes of von Neumann algebras:

- abelian (von Neumann 1931)
- type I (Pearcy 1962)
- properly infinite (Wogen 1969)

$\rightsquigarrow$ reduced to type II_1 case

Theorem (Willig 1974)

Every separably-acting vNA is singly-generated if (and only if) every separably-acting II_1 factor is.

- It is unknown if the free group factors $W^*(\mathbb{F}_k)$ are singly-generated for $k \geq 3$.

The generator problem for C^*-algebras I

Definition

Given a C^*-algebra A, let $\mathrm{gen}(A)$ denote the minimal number of **self-adjoint** generators.

- $\mathrm{gen}(A) \leq n$ iff $A = C^*(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A_{\mathrm{sa}}$.
- If $a, b \in A_{\mathrm{sa}}$, then $C^*(a, b) = C^*(a + ib)$. Thus, A is singly-generated if and only if $\mathrm{gen}(A) \leq 2$.

Proposition

Let X be a compact, metrizable space. Then $\mathrm{gen}(C(X)) \leq n$ if and only if $X \hookrightarrow \mathbb{R}^n$.

Proof.

If $X \hookrightarrow \mathbb{R}^n$, then $\iota\colon X \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and $C(X)$ is generated by $\pi_j \circ \iota\colon X \to \mathbb{R}$ for coordinate projections $\pi_j\colon \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, n$. Conversely, if $C(X)$ is generated by $f_1, \ldots, f_n\colon X \to \mathbb{R}$, then $X \to \mathbb{R}^n$, $x \mapsto (f_1(x), \ldots, f_n(x))$ is injective. □

The generator problem for C^*-algebras II

The generator problem

Given a C^*-algebra A, determine $\mathrm{gen}(A)$. In particular, which C^*-algebras are singly generated?

Facts:

- If $I \subseteq A$ is an ideal, then $\mathrm{gen}(A/I) \leq \mathrm{gen}(A)$.
- $\mathrm{gen}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{gen}(A)-1}{n^2} \right\rceil + 1$.
  Thus, if A is generated by at most n^2+1 self-adjoint elements, then $A \otimes M_n$ is generated by two self-adjoints.

Examples:

- $C^*(\mathbb{F}_2)$ is not singly-generated, since $C^*(\mathbb{F}_2) \twoheadrightarrow C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$, and $\mathbb{T}^2 \not\hookrightarrow \mathbb{R}^2$.
- $C^*(\mathbb{F}_2) \otimes M_2$ is singly generated, since $\mathrm{gen}(C^*(\mathbb{F}_2)) \leq 4$.

The generator problem for C^*-algebras III

Slogan: The more noncommutative, the fewer generators.

Question

Is every separable C^*-algebra without finite-dimensional representations singly generated?
Is every separable, simple C^*-algebra singly-generated?

Verified for the following classes of C^*-algebras:

- separable, properly infinite (Kirchberg)
- separable, UHF-stable algebras (Olsen, Zame 1976)
- separable, $\mathcal{Z}$-stable algebras (T, Winter 2014)

Main open problem:

- It is unknown if the reduced free group C^*-algebras $C^*_{\mathrm{red}}(\mathbb{F}_k)$ are singly generated for $k \geq 2$.

The generator problem for C^*-algebras IV

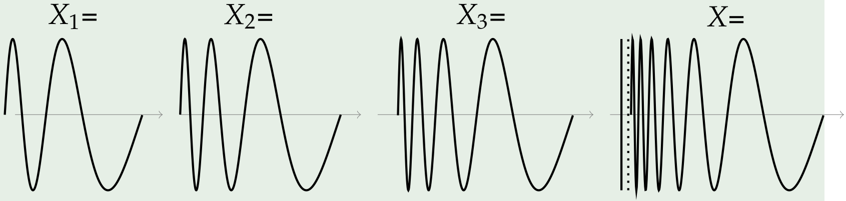
Difficulty: Invariant $\mathrm{gen}(\_)$ lacks nice permanence properties.

Example: $\mathrm{gen}(\_)$ may increase when passing to ideals

Set $A = C([0,1])$, $I = C_0((0,1))$. Then $\mathrm{gen}(A) = 1$, $\mathrm{gen}(I) = 2$.

Example: $\mathrm{gen}(\_)$ may increase when passing to inductive limits

Topologists sine-curve $X \subseteq \mathbb{R}^2$. Have $X \cong \varprojlim X_k$ with each $X_k \cong [0,1]$. Then $C(X) \cong \varinjlim_k C(X_k)$ and $\mathrm{gen}(C(X_k)) = 1$ for each k, but $\mathrm{gen}(C(X)) = 2$.



The generator rank I

Idea: Instead of generating self-adjoint tuples, consider **stable** generating self-adjoint tuples.

Notation

$$\mathrm{Gen}_n(A)_{\mathrm{sa}} := \big\{(a_1,\ldots,a_n) \in A_{\mathrm{sa}}^n \mid A = C^*(a_1,\ldots,a_n)\big\}.$$

Definition

If A is unital, then the **generator rank** $\mathrm{gr}(A)$ is the minimal $n \in \mathbb{N}$ such that $\mathrm{Gen}_{n+1}(A)_{\mathrm{sa}}$ is dense in A_{sa}^{n+1}, or $\mathrm{gr}(A) = \infty$ if no such n exists. If A is nonunital, set $\mathrm{gr}(A) := \mathrm{gr}(\widetilde{A})$.

Notice the index shift!

Example

We have $\mathrm{gr}(\mathbb{C}) = 0$, since $\mathrm{Gen}_1(\mathbb{C})_{\mathrm{sa}} = \mathbb{R} \setminus \{0\}$ is dense in $\mathbb{C}_{\mathrm{sa}}^1 = \mathbb{R}$.

The generator rank II

The index shift follows the usual convention in (noncommutative) dimension theory:

- The **covering dimension** of X satisfies $\dim(X) \leq n$ if for every finite, open cover of X there exists a finite, open refinement $\mathcal{V}$ which can be coloured with $n+1$ colours: $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1 \sqcup \ldots \sqcup \mathcal{V}_n$ such that the elements in each $\mathcal{V}_k$ are pairwise disjoint.
- The **nuclear dimension** of A satisfies $\dim_{\mathrm{nuc}}(A) \leq n$ if $\mathrm{id}\colon A \to A$ can be approximately factored by cp-maps $\psi\colon A \to F$ and $\varphi\colon F \to A$ with F finite-dimensional and φ a sum of $n+1$ orthogonality-preserving maps.
- The **real rank** of unital A satisfies $\mathrm{rr}(A) \leq n$ if

$$\left\{(a_0, \ldots, a_n) \in A_{\mathrm{sa}}^{n+1} : \sum_k a_k^2 \text{ invertible } \right\} \subseteq A_{\mathrm{sa}}^{n+1}$$

  is dense.
- We have $\mathrm{rr}(A) \leq \mathrm{gr}(A)$.

The generator rank III

Recall:

$$\mathrm{Gen}_n(A)_{\mathrm{sa}} := \{(a_1,\dots,a_n) \in A_{\mathrm{sa}}^n \mid A = C^*(a_1,\dots,a_n)\},$$
$$\mathrm{gen}(A) \leq n \Leftrightarrow \mathrm{Gen}_n(A)_{\mathrm{sa}} \neq \emptyset$$
$$\mathrm{gr}(A) \leq n \Leftrightarrow \mathrm{Gen}_{n+1}(A)_{\mathrm{sa}} \subseteq A_{\mathrm{sa}}^{n+1} \text{ dense}$$

- Thus $\mathrm{gen}(A) \leq \mathrm{gr}(A) + 1$.
- $\mathrm{gen}(A) \leq 2$ means A is singly-generated. $\mathrm{gr}(A) \leq 1$ means that $\{x \in A : A = C^*(x)\}$ is a dense G_δ-subset of A. (A generic element of A is a generator.)

Example: $\mathrm{gen}(A)$ can be (much) smaller than $\mathrm{gr}(A)$.

For $A = C([0,1]^k) \otimes M_d$, we have:

$$\mathrm{gen}(A) = \left\lceil \frac{k-1}{d^2} \right\rceil + 1, \qquad \mathrm{gr}(A) = \left\lceil \frac{k+1}{2d-2} \right\rceil.$$

For $B = \bigoplus_{d=1}^{\infty} C([0,1]^{d^2}) \otimes M_d$, have $\mathrm{gen}(B) \leq 2$ but $\mathrm{gr}(B) = \infty$.

The generator rank IV

Theorem (T 2020 - permanence properties)

Let $I \subseteq A$ be an ideal. Then

$$\max\left\{ \mathrm{gr}(I), \mathrm{gr}(A/I) \right\} \leq \mathrm{gr}(A) \leq \mathrm{gr}(I) + \mathrm{gr}(A/I) + 1.$$

Further, if $A = \varinjlim_\lambda A_\lambda$ is an inductive limit, then

$$\mathrm{gr}(A) \leq \liminf_\lambda \mathrm{gr}(A_\lambda).$$

AF-algebras

Theorem (T 2020)

Let A be an AF-algebra. Then $\mathrm{gr}(A) \leq 1$, and so a generic element of A is a generator. In particular, A is singly generated.

Proof.

Reduce to show that $\mathrm{gr}(M_n(\mathbb{C})) = 1$. Given self-adjoint $a, b \in M_n(\mathbb{C})$, find unitary u such that uau^* is diagonal:

$$uau^* = \begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}, \quad ubu^* = \begin{pmatrix} * & * & \cdots & * \\ * & * & & * \\ \vdots & & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}.$$

Approximate uau^* by diagonal matrix c with pairwise different, nonzero eigenvalues. Approximate ubu^* by matrix d with all entries nonzero. Then $C^*(u^*cu, u^*du) = C^*(c, d) = M_d$ and $u^*cu \approx a$ and $u^*du \approx b$. □

The commutative case I

Theorem (T 2020)

Let X be 2nd-countable, locally compact, Hausdorff. Then

$$\mathrm{gr}(C_0(X)) = \dim(X \times X).$$

Proof.

Reduce to X compact. Identify $C(X)_{\mathrm{sa}}^{n+1}$ with $C(X, \mathbb{R}^{n+1})$. Then $f \in C(X, \mathbb{R}^{n+1})$ generates $C(X)$ iff (a) $f(x) \neq 0$ for all $x \in X$; and (b) $f(x) \neq f(y)$ for $x \neq y$. Thus, $\mathrm{gr}(C(X)) \leq n$ iff:

(a) $C(X, \mathbb{R}^{n+1} \setminus \{0\}) \subseteq C(X, \mathbb{R}^{n+1})$ is dense;

(b) $E(X, \mathbb{R}^{n+1} \setminus \{0\}) \subseteq C(X, \mathbb{R}^{n+1} \setminus \{0\})$ is dense.

We have (a) iff $\dim(X) \leq n$ (classical); and (b) iff $\dim(X \times X) \leq n$ (using cohomological dimension and unstable intersection theory - Dranišnikov, Levin, Repovš, Ščepin, West). Result follows since $\dim(X) \leq \dim(X \times X)$. □

The commutative case II

Theorem (T 2020)

Let X be 2nd-countable, locally compact, Hausdorff. Then

$$\mathrm{gr}(C_0(X)) = \dim(X \times X).$$

Note: $\dim(X \times X)$ is either $2\dim(X)$ or $2\dim(X) - 1$ (that latter occurs by examples of Boltyanskiĭ from 1949).

Example

Consider $X = [0,1]$. Then $C([0,1])$ is generated by one self-adjoint, for example $x \mapsto 1 + x$. But

$$\mathrm{gr}(C([0,1])) = \dim([0,1] \times [0,1]) = 2,$$

that is, generating elements in $C([0,1], \mathbb{R}^n)$ are dense for $n = 3$; but not for $n = 1$ ($x \mapsto -1 + 2x$ cannot be approximated by invertible functions) and neither for $n = 2$ (there exists $g\colon [0,1] \to \mathbb{R}^2$ that cannot be approximated by embeddings).

The homogeneous case I

- A C^*-algebra A is d**-homogeneous** if every irreducible representation of A has dimension d.
- Typical example: $C_0(X) \otimes M_d$.
- Let A be a d-homogeneous C^*-algebra. Then the primitive ideal space $\mathrm{Prim}(A)$ is locally compact and Hausdorff, and A is the algebra of sections (vanishing at infinity) of a locally trivial M_d-bundle over $\mathrm{Prim}(A)$.

Theorem (T 2020)

Let A be a separable, d-homogeneous C^*-algebra. Set $X := \mathrm{Prim}(A)$. If $d = 1$, then $\mathrm{gr}(A) = \dim(X \times X)$. If $d \geq 2$, then:

$$\mathrm{gr}(A) = \left\lceil \frac{\dim(X) + 1}{2d - 2} \right\rceil .$$

The homogeneous case II

Let A be a separable, d-homogeneous C^*-algebra with $d \geq 2$. Set $X := \mathrm{Prim}(A)$. Then: $\mathrm{gr}(A) = \left\lceil \frac{\dim(X)+1}{2d-2} \right\rceil$.

Proof.

Reduce to $A = C(X, M_d)$. For $n \in \mathbb{N}$, set:

$$G_d^{n+1} := \mathrm{Gen}_{n+1}(M_d)_{\mathrm{sa}} \quad \subseteq \quad (M_d)_{\mathrm{sa}}^{n+1} \cong \mathbb{R}^{(n+1)d^2}.$$

Then $f \in C(X, (M_d)_{\mathrm{sa}}^{n+1}) = A_{\mathrm{sa}}^{n+1}$ generates $C(X, M_d)$ iff (a) $f(x) \in G_d^{n+1}$ for each $x \in X$; and (b) for $x \neq y$ in X there is no $\alpha \in \mathrm{Aut}(M_d)$ with $\alpha(f(x)) = f(y)$. Thus, $\mathrm{gr}(C(X, M_d)) \leq n$ iff:

(a) $C(X, G_d^{n+1}) \subseteq C(X, (M_d)_{\mathrm{sa}}^{n+1})$ is dense;

(b) $E(X, G_d^{n+1}/\mathrm{Aut}(M_d)) \subseteq C(X, G_d^{n+1}/\mathrm{Aut}(M_d))$ is dense.

Using Beggs-Evans, we have (a) iff

$$\dim(X) < \dim\left((M_d)_{\mathrm{sa}}^{n+1}\right) - \dim\left((M_d)_{\mathrm{sa}}^{n+1} \setminus G_d^{n+1}\right) = n(2d-2).$$

Have (b) iff $2\dim(X) < \dim\left(G_d^{n+1}/\mathrm{Aut}(M_d)\right)$ (Luukkainen). □

The subhomogeneous case I

- A C^*-algebra A is d**-subhomogeneous** if every irreducible representation of A has dimension at most d.
- Typical example: Subalgebra of $C(X) \otimes M_d$.
- Let A be a d-subhomogeneous C^*-algebra. Then there is a unique chain of ideals

  $$\{0\} = I_{d+1} \subseteq I_d \subseteq I_{d-1} \subseteq \ldots \subseteq I_1 = A$$

  such that I_k/I_{k+1} is k-homogeneous.
  (I_k corresponds to the 'part' of A where all irreducible representations have dimension $\geq k$)

The subhomogeneous case II

Theorem (T 2020)

Let A be a separable d-subhomogeneous C^*-algebra. For each $k \leq d$, set $X_k := \mathrm{Prim}_k(A)$, the subset of the primitive ideal space of A corresponding to k-dimensional irreducible representations. Then:

$$\mathrm{gr}(A) = \max\left\{ \dim(X_1 \times X_1), \sup_{k \geq 2} \left\lceil \frac{\dim(X_k) + 1}{2k - 2} \right\rceil \right\}.$$

- Note: If A is subhomogeneous with $\mathrm{gr}(A) < \infty$, then $\mathrm{gr}(A \otimes M_n) = 1$ for n sufficiently large.
- If A is subhomogeneous, then $A = \varinjlim_k A_k$ for subhomogeneous A_k with $\mathrm{gr}(A_k) < \infty$. (Ng-Winter)
- UHF-algebra $M_{p^\infty} = \bigotimes_{\mathbb{N}} M_p \cong \varinjlim_n M_{p^n}$.
- Thus: If A is subhomogeneous, then $\mathrm{gr}(A \otimes UHF) = 1$.

The approximately subhomogeneous case

- A C^*-algebra is **approximately subhomogeneous** (ASH) if it is an inductive limit of subhomogeneous C^*-algebras.
- Includes many interesting classes of nuclear C^*-algebras: AF-algebras, irrational rotation algebras, (certain) crossed products.
- The Jiang-Su algebra $\mathcal{Z}$ is an interpolation between M_{p^∞} for all primes p.
- A C^*-algebra A is **$\mathcal{Z}$-stable** if $A \cong \mathcal{Z} \otimes A$. (The C^*-analog of being McDuff.)

Corollary

Let A be a separable, $\mathcal{Z}$-stable ASH-algebra. Then $\mathrm{gr}(A) = 1$, and so a generic element of A is a generator.

The classifiable case

- Simple, nuclear A is **classifiable** if it is unital, separable, $\mathcal{Z}$-stable and satisfies the Universal Coefficient Theorem.
- By recent breakthrough in the Elliott classification program, simple, nuclear, classifiable C^*-algebras are isomorphic iff their Elliott invariants (K-theoretic and tracial data) are.

Theorem (T 2020)

Let A be a simple, nuclear, classifiable C^*-algebra. Then $\mathrm{gr}(A) = 1$, and so a generic element of A is a generator.

Proof.

A is either stably finite or purely infinite. If purely infinite, then A is a Kirchberg algebra and so has real rank zero (handled with different methods). If stably finite, then A is ASH. □

- Villadsen: There exist simple, separable, unital, nuclear (ASH) algebras A with $\mathrm{gr}(A) = \infty$.

Thank you.

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