The rank problem for C*-algebras

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Introduction

The **rank** of a projection p with respect to a trace τ is $\tau(p)$.

In a II_1 factor with its unique tracial state τ, we have:

- *Comparison:* $\tau(p) \leq \tau(q)$ if and only if $p \precsim q$;
- *Range:* For every $t \in [0, 1]$, there exists p with $\tau(p) = t$.

What are analogs for simple, tracial C^*-algebras? Difficulties:

- there may be many tracial states, and one has to consider them all simultaneously;
- there may be few projections, and one has to consider positive elements.

Then, the analogs are:

- *Comparison:* Strict comparison of positive elements;
- *Range:* **The rank problem.**

The rank problem (1)

Let A be a C^*-algebra with nonempty tracial state space $T(A)$.

The **rank** of $a \in A_+$ with respect to $\tau \in T(A)$ is

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}) = \bar{\tau}(\mathrm{supp}(a))$$

where $\bar{\tau}\colon A^{**} \to \mathbb{C}$ is the unique extension to a normal tracial state and $\mathrm{supp}(a)$ is the support projection of a.

Definition

The **rank** of $a \in A_+$ is the function $\mathrm{rk}(a)\colon T(A) \to [0,1]$ given by

$$\mathrm{rk}(a)(\tau) := d_\tau(a).$$

The rank problem

Describe $\{\mathrm{rk}(a) : a \in A_+\}$.

The rank problem (2)

Let A be a unital, simple, separable, non-elementary C^*-algebra with nonempty tracial state space $T(A)$.

The rank problem: Describe $\{\mathrm{rk}(a) : a \in A_+\}$.

- $T(A)$ is a compact, convex set;
- For each $a \in A_+$, the function $\mathrm{rk}(a)\colon T(A) \to [0,1]$ is lower-semicontinuous and affine.
- If $a \neq 0$, then $\mathrm{rk}(a)$ is strictly positive.

Goal: Show that for every $f \in \mathrm{LAff}(T(A), (0,1])$ there exists $a \in A_+$ with $\mathrm{rk}(a) = f$.

Example: If $T(A) = \{\tau\}$, then we want that for every $t \in (0,1]$ there exists $a \in A_+$ with $d_\tau(a) = t$.

Elliott classification program

Theorem (Many hands 1976-2017)

Unital, simple, nuclear, separable C^*-algebras that are $\mathcal{Z}$-stable and satisfy the UCT are classified by their Elliott invariant.

The Toms-Winter conjecture

For A simple, nuclear, non-elementary, TFAE:

1. A is $\mathcal{Z}$-stable: $A \cong \mathcal{Z} \otimes A$;
2. A has strict comparison of positive elements.

Known: '(1)$\Rightarrow$(2)', and in many cases '(1)$\Leftarrow$(2)'.

Theorem (Winter 2012)

'(1)$\Leftarrow$(2)' holds if A has locally finite nuclear dimension and for every $f \in \mathrm{LAff}(T(A), (0, 1])$ there exists $a \in A_+$ with $\mathrm{rk}(a) = f$.

Thus: Solving the rank problem leads to verification of Toms-Winter conjecture

Known results

Let A be a unital, simple, separable, non-elementary C^*-algebra with nonempty tracial state space $T(A)$.

Goal: Show that for every $f \in \mathrm{LAff}(T(A), (0,1])$ there exists $a \in A_+$ with $\mathrm{rk}(a) = f$.

Positive solutions:

- If $T(A) = \{\tau\}$. More generally, if the extreme boundary $\partial_e T(A)$ is finite.
- If A is $\mathcal{Z}$-stable.
  (Brown-Perera-Toms 2008, Elliott-Robert-Santiago 2011)
- If A has strict comparison of positive elements and $T(A)$ is a Bauer simplex ($\partial_e T(A)$ is closed) and $\partial_e T(A)$ has finite covering dimension. (Dadarlat-Toms 2010)
- If A has stable rank one.
  (T 2020, Antoine-Perera-Robert-T)

Realizing functions as ranks (1)

- Functions in $\mathrm{LAff}(T(A))$ are determined by their values on $\partial_e T(A)$.
- If $T(A)$ is a Bauer simplex, then

$$\mathrm{LAff}(T(A)) \cong \mathrm{Lsc}(\partial_e T(A)).$$

- If $T(A)$ is not Bauer, then it is difficult to describe which functions on $\partial_e T(A)$ extend to lower-semicontinuous, affine functions on $T(A)$. (Dirichlet problem)
- Given $\tau \in \partial_e T(A)$ and $t \in (0,1]$, there exists $\chi_{\tau,t} \in \mathrm{LAff}(T(A),(0,1])$, called the **chisel** at τ with value t, such that for $\tau' \in \partial_e T(A)$ we have

$$\chi_{\tau,t}(\tau') = \begin{cases} t, & \text{if } \tau' = \tau \\ 1, & \text{if } \tau' \neq \tau \end{cases}$$

Realizing functions as ranks (2)

- The **chisel** at τ with value t is $\chi_{\tau,t} \in \mathrm{LAff}(T(A),(0,1])$ such that for $\tau' \in \partial_e T(A)$ we have
$$\chi_{\tau,t}(\tau') = \begin{cases} t, & \text{if } \tau' = \tau \\ 1, & \text{if } \tau' \neq \tau \end{cases}$$
- Given $f \in \mathrm{LAff}(T(A),(0,1])$, we have
$$f = \bigwedge_{\tau \in \partial_e T(A)} \chi_{\tau,f(\tau)}.$$
- A Stone-Weierstrass type result: A subset $R \subseteq \mathrm{LAff}(T(A),(0,1])$ is dense if:
  1. R contains every chisel.
  2. R is closed under infima: if $f, g \in R$, then $f \wedge g \in R$.
- For $R := \{\mathrm{rk}(a) \mid a \in A_+, a \neq 0\}$, (1) always holds (T 2020). When does (2) hold? If A has stable rank one, then $\mathrm{Cu}(A)$ is semilattice, and hence so is R. (Antoine-Perera-Robert-T)

Thank you!

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