Rigidity results for L^p-operator algebras

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L^p-operator algebras

Let $p \in [1, \infty)$.

Definition

An L^p**-operator algebra** is a Banach algebra that admits an isometric representation on some L^p-space.

Examples

- closed subalgebras of $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p)$
- $C(X) = \{f \colon X \to \mathbb{C} \mid f \text{ continuous}\} \subseteq \mathcal{B}(L^p(X, \mu))$
- Herz 1973: reduced group L^p-operator algebra $F^p_\lambda(G) = \overline{\operatorname{span}}\{\lambda_g \mid g \in G\} \subseteq \mathcal{B}(\ell^p(G))$
- Phillips 2012: reduced L^p-operator crossed product $F^p_\lambda(G, X) = \overline{\operatorname{span}}\{f\lambda_g \mid f \in C(X), g \in G\} \subseteq \mathcal{B}(\ell^p(G, L^p(\mu)))$ for dynamical system $G \curvearrowright X$
- Phillips 2012: L^p-Cuntz algebra $\mathcal{O}^p_2 \subseteq \mathcal{B}(\ell^p)$

Rigidity results

Theorem (Gardella-T 2018)

Let G and H be discrete groups, and $p \in [1,\infty) \setminus \{2\}$. Then:

$$F^p_\lambda(G) \cong F^p_\lambda(H) \quad \Leftrightarrow \quad G \cong H.$$

Fails for $p = 2$: $C^*_\lambda(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong C^*_\lambda(\mathbb{Z}_4)$.

Theorem (Choi-Gardella-T 2019)

Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free actions of discrete groups on compact Hausdorff spaces, and $p \in [1,\infty) \setminus \{2\}$. Then $F^p_\lambda(G, X) \cong F^p_\lambda(H, Y)$ if and only if $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent.

Also fails for $p = 2$: Let $F \curvearrowright C$ be a topologically free action with $C^*_\lambda(F, C) \cong \mathcal{O}_2$. Then for any minimal, topologically free action $G \curvearrowright X$, we have $C^*_\lambda(F \times G, C \times X) \cong \mathcal{O}_2$.

The C*-core

Theorem (CGT 2019)

Let A be a unital L^p-operator algebra. Then there exists a largest C*-subalgebra of A. (Call it the **C*-core** of A, denoted by $\mathrm{core}(A)$.) If $p \neq 2$, then $\mathrm{core}(A)$ is commutative.

Idea for $p \neq 2$: Let $A \subseteq \mathcal{B}(L^p(\mu))$. Let A_h denote the hermitian elements. Then $A_h = A \cap \mathcal{B}(L^p(\mu))_h = A \cap L^\infty_{\mathbb{R}}(\mu)$ is closed under multiplication. Hence, $\mathrm{core}(A) = A_h + iA_h$ is commutative C*-algebra.

Examples

- $\mathrm{core}(F^p_\lambda(G)) = \mathbb{C}$
- $\mathrm{core}(F^p_\lambda(G, X)) = C(X)$
- $\mathrm{core}(\mathcal{O}^p_2) = C(\text{Cantor space})$

The Weyl groupoid

Let A be a unital L^p-operator algebra with C*-core $C(X) \subseteq A$. We say that a partial homeomorphism $\alpha\colon U \to V$ (for open subsets $U, V \subseteq X$) is **realizable** if there exist $a, b \in A$ such that:

1. for $f \in C(X)_+$, we have $afb, bfa \in C(X)_+$;
2. $U = \{x \in X \mid ba(x) > 0\}$ and $V = \{x \in X \mid ab(x) > 0\}$;
3. we have $f(\alpha(x))ba(x) = (bfa)(x)$ for $f \in C_0(V)$ and $x \in U$;
4. we have $g(\alpha^{-1}(y))ab(y) = (agb)(y)$ for $g \in C_0(U)$ and $y \in V$.

Definition (CGT 2019)

The **Weyl groupoid** of A is the groupoid $\mathcal{G}_A$ of germs associated to the inverse semigroup of realizable partial homeomorphisms on X.

Theorem (CGT 2019)

Let $G \curvearrowright X$ be topologically free. Then the Weyl groupoid of $F^p_\lambda(G, X)$ is isomorphic to the transformation groupoid $G \ltimes X$.

Rigidity of reduced L^p-operator crossed products

Theorem (CGT 2019)

Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free actions of discrete groups on compact Hausdorff spaces, and $p \in [1, \infty) \setminus \{2\}$. Then TFAE:

1. $F^p_\lambda(G, X) \cong F^p_\lambda(H, Y)$;
2. $G \ltimes X \cong H \ltimes Y$;
3. $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent.

(1)⇒(2): If $F^p_\lambda(G, X) \cong F^p_\lambda(H, Y)$, then the Weyl groupoids of $F^p_\lambda(G, X)$ and $F^p_\lambda(H, Y)$ are isomorphic, hence $G \ltimes X \cong H \ltimes Y$.

(1)⇐(2): Surprisingly difficult.

(2)⇔(3) well-known

Application to L^p-Cuntz algebras (1)

Leavitt algebra = universal $\mathbb{C}$-algebra:

$$L_2 = \langle s_1, s_2, t_1, t_2 \mid t_1 s_1 = t_2 s_2 = s_1 t_1 + s_2 t_2 = 1,\ t_1 s_2 = t_2 s_1 = 0 \rangle.$$

Cuntz algebra $\mathcal{O}_2$ is the completion of L_2 for representations $\pi\colon L_2 \to \mathcal{B}(H)$ such that $\pi(s_j) = \pi(t_j)^*$.

Phillips: For $p \in [1, \infty)$, the L^p-Cuntz algebra $\mathcal{O}_2^p$ is the completion of L_2 for 'spatial' representations $\pi\colon L_2 \to \mathcal{B}(L^p(\mu))$.

Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

Question (Phillips): $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \cong \mathcal{O}_2^p$ for $p \neq 2$?

Application to L^p-Cuntz algebras (2)

Theorem (CGT 2019)

If $p \neq 2$, then $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \not\cong \mathcal{O}_2^p$.

Idea:

- There exists an amenable, topologically free action of $G := \mathbb{Z}_2 * \mathbb{Z}_3$ on the Cantor space X such that $\mathcal{O}_2^p \cong F_\lambda^p(G, X)$.
- Then $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \cong F_\lambda^p(G \times G, X \times X)$.
- Medynets-Sauer-Thom 2017: Two groups are bi-Lipschitz equivalent iff they admit continuously orbit equivalent actions on compact Hausdorff spaces.
- Assuming $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \cong \mathcal{O}_2^p$, we get
$$F_\lambda^p(G \times G, X \times X) \cong F_\lambda^p(G, X).$$
Then $G \times G \curvearrowright X \times X$ and $G \curvearrowright X$ are continuously orbit equivalent. Hence, $(\mathbb{Z}_2 * \mathbb{Z}_3) \times (\mathbb{Z}_2 * \mathbb{Z}_3)$ and $\mathbb{Z}_2 * \mathbb{Z}_3$ are bi-Lipschitz equivalent, a contradiction.

Thank you!

References

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