Are C*-algebras determined by their linear and orthogonality structure?

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Introduction 1

C*-algebras A and B are $*$-isomorphic, denoted $A \cong B$, if there exists a $*$-isomorphism:
a linear, multiplicative, $*$-preserving bijection $A \to B$.

Question

How can we deduce such a $*$-isomorphism?

- From the $*$-linear structure? No: The map

$$M_2(\mathbb{C}) \to \mathbb{C}^4, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b + c, i(b - c), d).$$

is bijective, linear and $*$-preserving, but $M_2(\mathbb{C}) \ncong \mathbb{C}^4$.

Introduction 2

Question: How can we deduce a $*$-isomorphism?

- From the $*$-ring structure? No: Given a C^*-algebra A, consider the opposite C^*-algebra A^{op}. Then the map

$$A \to A^{\mathrm{op}}, \quad a \mapsto a^*,$$

is bijective, multiplicative and $*$-preserving, but there exist examples by Phillips (2001) such that $A \ncong A^{\mathrm{op}}$.

- From the algebra structure? Yes:

Theorem (Gardner 1965)

Let $\varphi\colon A \to B$ be an isomorphism (linear, multiplicative bijection). Then $\varphi = \alpha \circ \pi$ for some $*$-isomorphism $\pi\colon A \to B$ and some 'weakly inner' automorphism $\alpha\colon B \to B$, given by $\alpha(b) = cbc^{-1}$, for some invertible $c \in B^{**}$.
In particular, φ is automatically continuous and $A \cong B$.

The main question

Does it suffice if instead of the whole multiplicative structure we record when two elements are orthogonal?

Definition 1

A map $\varphi\colon A \to B$ is said to ***preserve zero-products*** *if $ab = 0$ implies $\varphi(a)\varphi(b) = 0$, for all $a, b \in A$.*

Question 2

Let $\varphi\colon A \to B$ be a linear bijection such that φ and φ^{-1} preserve zero-products.
(That is, $ab = 0$ if and only if $\varphi(a)\varphi(b) = 0$, for all $a, b \in A$.)
Is φ automatically continuous and $A \cong B$? What can we say about the structure of φ?

Zero-product preserving maps (1)

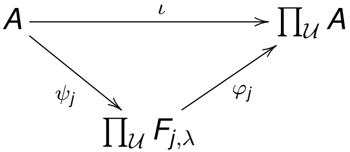
Zero-product preserving maps are much easier to construct than $*$-homomorphisms.

Winter-Zacharias: A completely positive map $\varphi\colon A \to B$ is **order-zero** if $ab = 0$ implies $\varphi(a)\varphi(b) = 0$ for $a, b \in A_+$

Theorem 3 (Gardella-T)

A linear map between C^*-algebras is completely positive, order-zero iff it is positive and preserves zero-products.

Robert (2011): A has nuclear dimension $\leq n$ iff there exist finite-dimensional $F_{j,\lambda}$, an ultrafilter $\mathcal{U}$ on $\{j\}$, and cpc. order-zero maps ψ_j, φ_j for $j = 0, \ldots, n$ such that $\iota = \sum_j \varphi_j \circ \psi_j$:



Zero-product preserving maps (2)

Question 2

Let $\varphi\colon A \to B$ be a linear bijection such that φ and φ^{-1} preserve zero-products.
Is φ automatically continuous and $A \cong B$? What can we say about the structure of φ?

Theorem (Alaminos-Bresar-Extremera-Villena 2009)

*Let $\varphi\colon A \to B$ be a **bounded** linear bijection that preserves zero-products. Then φ is a **weighted isomorphism**, that is, there exists a central, invertible multiplier $h \in Z(M(B))^{-1}$ and an isomorphism $\psi\colon A \to B$ such that $\varphi(a) = h\psi(a)$ for all $a \in A$. In particular, $A \cong B$.*

Thus, Question 2 is asking for automatic continuity. Results showing automatic continuity are difficult and highly nontrivial.

Zero-product preserving maps (3)

Example (Jarosz 1990)

Let $\varphi\colon C(X) \to C(Y)$ be a linear bijection that preserves zero-products. Then φ is automatically continuous, and consequently $C(X) \cong C(Y)$, and $X \cong Y$.

Proposition 4

Let $\varphi\colon A \to B$ be a linear bijection. TFAE:

1. *φ is a weighted isomorphism.*
2. *$\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$ for all $a, b, c \in A$.*

Proof.

'(1)⇒(2)' is clear. For '(2)⇒(1)', define $\alpha\colon B \to B$ by $\alpha(\varphi(a)\varphi(b)) = \varphi(ab)$ for $a, b \in A$. One shows that α is a well-defined, central, invertible multiplier on B and that $\alpha \circ \varphi\colon A \to B$ is an isomorphism. □

Zero-product balanced C*-algebras (1)

Definition 5 (Gardella-T)

*A is **zero-product balanced** if*

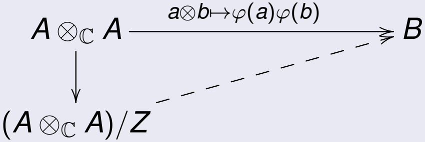
$$ab \otimes c - a \otimes bc \in Z := \operatorname{span}\{u \otimes v \in A \otimes_{\mathbb{C}} A : uv = 0\},$$

for all $a, b, c \in A$.

Lemma 6

Let $\varphi\colon A \to B$ be linear, zero-product preserving, and A zero-product balanced. Then $\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$ for all a, b, c.

Proof.



Zero-product balanced C*-algebras (2)

The next result is an immediate consequence of Proposition 4 and Lemma 6:

Theorem 7

Let $\varphi\colon A \to B$ be a linear bijectiion that preserves zero-products. Assume that A is zero-product balanced. Then φ is a weighted isomorphism and $A \cong B$.

Example 8

Let X be a compact, Hausdorff space with $|X| = \infty$. Choose $f \in C(X)$ with $|f(X)| = \infty$. Then

$$1 \otimes f - f \otimes 1 \notin Z := \operatorname{span}\big\{u \otimes v \in A \otimes_{\mathbb{C}} A : uv = 0\big\}.$$

Thus, $C(X)$ is not zero-product balanced.

Zero-product balanced C*-algebras (3)

Proposition 9

Assume that A is generated (as an algebra) by its idempotents. Then A is zero-product balanced.

Proof.

Let $a, c \in A$, and let $e = e^2 \in A$. Then

$$ae \otimes c - ae \otimes ec = ae \otimes (1 - e)c \in Z, \qquad \text{and}$$
$$ae \otimes ec - a \otimes ec = a(e - 1) \otimes ec \in Z.$$

Hence, $ae \otimes c - a \otimes ec \in Z$.
Thus, $ab \otimes c - a \otimes bc \in Z$ whenever b is an idempotent, and then whenever b is a product of idempotents, and then whenever b is a linear combination of products of idempotents, and thus (by assumption) for every b. □

Zero-product balanced C*-algebras (4)

Proposition 9

Assume that A is generated (as an algebra) by its idempotents. Then A is zero-product balanced.

Example 10

Let A be a unital C^*-algebra, and let $n \geq 2$. Then $M_n(A)$ is generated by idempotents and hence is zero-product balanced. Indeed, given $a \in A$, we have

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and}$$

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus, for any B we have $M_n(A) \cong B$ if and only if there exists a bijective, linear, zero-product preserving map $M_n(A) \to B$.

Zero-product balanced C*-algebras (5)

Theorem 11 (Gardella-T)

Every unital C^*-algebra without characters (one-dimensional, irreducible representations) is zero-product balanced.
For such A and any B we have $A \cong B$ iff there exists a linear, zero-product preserving bijection $A \to B$.

Example 12

Consider the **dimension-drop algebra**

$$Z_{2,3} := \big\{ f \in C([0,1], M_2 \otimes M_3) : f(0) \in M_2 \otimes 1, f(1) \in 1 \otimes M_3 \big\}.$$

Then $Z_{2,3}$ has no idempotents except $0, 1$. Nevertheless, by Theorem 11, $Z_{2,3}$ is zero-product balanced.

Zero-product balanced C*-algebras (6)

Theorem 11 (Gardella-T)

Every unital C^*-algebra without characters is zero-product balanced.

Corollary 13

Simple, unital C^*-algebras A and B are $*$-isomorphic iff they admit a linear, zero-product preserving bijection $A \to B$.

- The idea to our proof of Theorem 11 is to show that A is generated by 'special' square-zero elements that can be 'transferred' like idempotents in the proof of Proposition 9.
- Theorem 11 also applies to non-unital C^*-algebras assuming that $M(A)$ has no characters.

Question 14

When does $M(A)$ have no characters?

Multiplier algebras without characters (1)

Question14

When does $M(A)$ have no characters?

Necessarily, A has no characters. But this is not sufficient:

- Robert-Rørdam (2013): There exist simple, unital C^*-algebras $A_1, A_2, \dots$ such that $\prod_n A_n$ has a character. Then $\bigoplus A_n$ is separable and has no characters, but $M(\bigoplus_n A_n) = \prod_n A_n$.
- Sakai (1971), Pederson (1972): Let M be a II_1 factor, $\varphi\colon M \to \mathbb{C}$ a pure state, and set

$$A := \{a \in M : \varphi(a^*a) = \varphi(aa^*) = 0\}.$$

  Then A is simple (nonseparable), but $M(A)/A \cong \mathbb{C}$.
- We do not know if there exists simple **and** separable A such that $M(A)$ has a character.

Multiplier algebras without characters (2)

Question14

When does $M(A)$ have no characters?

Example 15

Consider

$$A = \left\{ (x^{(n)})_n \in C_b(\mathbb{N}, M_3) : x^{(n)} \text{ converges to } \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

For the ideal $I = C_0(\mathbb{N}, M_3)$, we have $A/I \cong M_2$, and so A has only irreducible representations of dimension 2 and 3.
Thus, A is a subhomogeneous AF-algebra. We have $M(A) \subseteq C_b(\mathbb{N}, M_3)$, consisting of $(x^{(n)})_n$ such that

$$\begin{pmatrix} x_{11}^{(n)} & x_{12}^{(n)} & x_{13}^{(n)} \\ x_{21}^{(n)} & x_{22}^{(n)} & x_{23}^{(n)} \\ x_{31}^{(n)} & x_{32}^{(n)} & \end{pmatrix} \to \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \end{pmatrix}$$

It follows that $M(A)/A \cong C_b(\mathbb{N})/C_0(\mathbb{N})$.

Multiplier algebras without characters (3)

Question14

When does $M(A)$ have no characters?

C^*-algebras whose multiplier algebras have no characters:

- σ-unital, weakly purely infinite C^*-algebras: the multiplier algebra is weakly purely infinite. (Kirchberg-Rørdam 2002).
- Minimal tensor products $B \otimes C$ where $M(B)$ has no characters: $M(B) \otimes M(C)$ embeds unitally into $M(B \otimes C)$. (Akemann-Pedersen-Tomiyama 1973)
- Stable C^*-algebras, $\mathcal{Z}$-stable C^*-algebras.
- If A admits essential $*$-homomorphism $B \to A$, and $M(B)$ has no characters, then neither does $M(A)$, since we have unital $M(B) \to M(A)$.
- Simple, separable C^*-algebras A of real rank zero: There is simple AF-algebra B (which is $\mathcal{Z}$-stable) and essential $*$-homomorphism $B \to A$ (Perera-Rørdam 2004).

Thank you!

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