A gentle introduction to Cuntz semigroups
Part 1

Hannes Thiel

Chalmers University of Technology and University of Gothenburg

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Comparison of projections (Murray-von Neumann) I

Throughout A is a C^*-algebra.

Definition 1

Murray-von Neumann (sub)equivalence *of projections in* A*:*

$$p \sim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = vv^*, q = v^*v, \ \textit{some } v \in A.$$
$$p \precsim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p \sim_{\mathrm{MvN}} p' \leq q, \ \textit{some } p'.$$

Murray-von Neumann semigroup*:*

$$V(A) := \mathrm{Proj}(A \otimes \mathbb{K})_{/\sim_{MvN}}, \quad [p] + [q] := [p \oplus q].$$

- $\mathbb{K}$ denotes compact operators on $\ell^2(\mathbb{N})$, and $A \otimes \mathbb{K}$ is the **stabilization** of A, the completion of $\bigcup_n M_n(A)$.
- Idea: Equivalent projections have the same 'size' relative to A. $V(A)$ encodes 'sizes' of projections.
- If A is unital, then $K_0(A)$ = Grothendieck group of $V(A)$.

Comparison of projections (Murray-von Neumann) II

$$p \sim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = vv^*, q = v^*v, \text{ some } v \in A.$$
$$V(A) := \mathrm{Proj}(A \otimes \mathbb{K})_{/\sim_{MvN}}, \quad [p] + [q] := [p \oplus q].$$

Example 2 ($\mathbb{C}$)

Consider projection p in $\mathbb{C} \otimes \mathbb{K} = \mathbb{K} = \mathbb{K}(H)$. The **rank** of p is

$$\mathrm{rk}(p) := \dim_{\mathbb{C}} p(H) \in \{0, 1, 2, \ldots\} =: \mathbb{N}.$$

For $p, q \in \mathrm{Proj}(\mathbb{K})$, have

$$p \sim_{\mathrm{MvN}} q \quad \Leftrightarrow \quad \mathrm{rk}(p) = \mathrm{rk}(q)$$
$$p \precsim_{\mathrm{MvN}} q \quad \Leftrightarrow \quad \mathrm{rk}(p) \leq \mathrm{rk}(q).$$

Moreover, for every $n \in \mathbb{N}$ there is a projection of rank n. Thus

$$V(\mathbb{C}) = \mathrm{Proj}(\mathbb{K})_{/\sim_{MvN}} \cong \mathbb{N}, \quad \text{and} \quad K_0(\mathbb{C}) = \mathrm{Gr}(\mathbb{N}) \cong \mathbb{Z}.$$

Comparison of projections (Murray-von Neumann) III

$$p \sim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = vv^*, q = v^*v, \text{ some } v \in A.$$

$$V(A) := \mathrm{Proj}(A \otimes \mathbb{K})_{/\sim_{MvN}}, \quad [p] + [q] := [p \oplus q].$$

Example 3 ($\mathbb{B}$)

$\mathbb{B}$ denotes bounded opertors on $H = \ell^2(\mathbb{N})$. For $p \in \mathrm{Proj}(\mathbb{B})$:

$$\mathrm{rk}(p) := \dim_{\mathbb{C}} p(H) \in \{0, 1, 2, \ldots, \infty\} =: \overline{\mathbb{N}}.$$

Have $p \in \mathbb{K}$ iff $\mathrm{rk}(p) < \infty$, and $p \sim_{\mathrm{MvN}} 1$ iff $\mathrm{rk}(p) = \infty$. Similarly, projections in $\mathbb{B} \otimes \mathbb{K}$ with infinite rank are MvN equivalent. Thus

$$V(\mathbb{B}) \cong \overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}.$$

For all $x, y \in \overline{\mathbb{N}}$, have $x + \infty = \infty = y + \infty$, and so

$$K_0(\mathbb{B}) = \mathrm{Gr}(\overline{\mathbb{N}}) = 0.$$

Comparison of projections (Murray-von Neumann) IV

Example 4 (II_1-factor)

Let N be a II_1-factor, with its unique tracial state τ. For $p, q \in \mathrm{Proj}(N)$, Murray-von Neumann showed:

$$\begin{aligned} p \sim_{\mathrm{MvN}} q \quad &:\Leftrightarrow \quad \tau(p) = \tau(q), \\ p \precsim_{\mathrm{MvN}} q \quad &:\Leftrightarrow \quad \tau(p) \leq \tau(q). \end{aligned}$$

Moreover, for every $t \in [0, 1]$, there is p with $\tau(p) = t$. Thus

$$\mathrm{Proj}(N)_{/\sim_{MvN}} \cong [0, 1].$$

The same holds in each $M_n(N)$, and we get

$$\mathrm{Proj}(M_n(N))_{/\sim_{MvN}} \cong [0, n].$$

and finally

$$V(N) \cong [0, \infty), \quad \text{and} \quad K_0(N) = \mathrm{Gr}([0, \infty)) = \mathbb{R}.$$

Comparison of projections: Applications

- Type classification of von Neumann algebras
- Classification of AF-algebras: $A \cong B$ iff $V(A) \cong V(B)$.
- Classification of finitely-generated, projective modules: Given $p \in \mathrm{Proj}(M_n(A))$, the module $E_p := p(A^{\oplus n})$ is f.g. projective. Every f.g. projective module arises this way, and $E_p \cong E_q$ iff $p \sim_{\mathrm{MvN}} q$. Thus:

$$V(A) := \mathrm{Proj}(A \otimes \mathbb{K})_{/\sim_{MvN}} \cong \{\text{f.g. projective } A\text{-modules}\}_{/\cong}.$$

Example 5 ($C^*_{\mathrm{red}}(\mathbb{F}_n)$)

Reduced group C^*-algebra $C^*_{\mathrm{red}}(\mathbb{F}_n)$ of free group.

Pimsner-Voiculescu 1982: $K_0(C^*_{\mathrm{red}}(\mathbb{F}_n)) \cong \mathbb{Z}$
Dykema-Haagerup-Rørdam 1997: $V(C^*_{\mathrm{red}}(\mathbb{F}_n)) \cong \mathbb{N}$

Consequence: Every f.g. projective $C^*_{\mathrm{red}}(\mathbb{F}_n)$-module is free.

Comparison of positive elements (Cuntz) I

- Problem: Many interesting C^*-algebras contain only few (if any) projections.
- Kadison-Kaplansky conjecture:
  If G is torsion-free, then $\mathrm{Proj}(C^*_{\mathrm{red}}(G)) = \{0, 1\}$.
- Cuntz: Study comparision of positive elements.

Recall: $p \precsim_{\mathrm{MvN}} q \;:\Leftrightarrow\; p = vv^*, v^*v \leq q,$ some $v \in A$.

First attempt: For $a, b \in A_+$:

$$a \precsim b \quad :\Leftrightarrow \quad a = vv^*, v^*v \in \overline{bAb}, \text{ some } v \in A.$$

(Essentially $p_a \precsim_{\mathrm{MvN}} p_b$ in A^{**} for support projections p_a, p_b.)

Much better behaved:

$$a \precsim_{\mathrm{Cu}} b \quad :\Leftrightarrow \quad \forall\, \varepsilon > 0 : (a - \varepsilon)_+ = vv^*, v^*v \in \overline{bAb}, \text{ some } v.$$
$$\Leftrightarrow \quad a = \lim_n w_n b w_n^*, \text{ some } (w_n)_n.$$

- $(a - \varepsilon)_+$ is the ε**-cut-down** of a, given by functional calculus with $f(t) = \max\{0, t - \varepsilon\}$.

Comparison of positive elements (Cuntz) II

Definition 6 (Cuntz 1978, Coward-Elliott-Ivanescu 2008)

Cuntz (sub)equivalence *of positive elements in A:*

$$a \precsim_{\mathrm{Cu}} b \quad :\Leftrightarrow \quad \forall\, \varepsilon > 0 : (a-\varepsilon)_+ = vv^*, v^*v \in \overline{bAb}, \ \text{ some } v.$$

$$\Leftrightarrow \quad a = \lim_n w_n b w_n^*, \ \text{ some } (w_n)_n.$$

$$a \sim_{\mathrm{Cu}} b \quad :\Leftrightarrow \quad a \precsim_{\mathrm{Cu}} b \precsim_{\mathrm{Cu}} a.$$

Cuntz semigroup*:*

$$\mathrm{Cu}(A) := (A \otimes \mathbb{K})_{+/\sim_{\mathrm{Cu}}},$$

equipped with addition and partial order:

$$[a] + [b] := [a \oplus b], \quad [a] \leq [b] \ :\Leftrightarrow \ a \precsim_{\mathrm{Cu}} b.$$

Comparison of positive elements (Cuntz) III

Example 7 ($\mathbb{C}$)

Positive element a in $\mathbb{C} \otimes \mathbb{K} = \mathbb{K}$ is diagonalizable:

$$a = \sum_{n=1}^{\infty} \lambda_n e_n$$

with rank-one projections e_n, and decreasing sequence $\lambda_n \to 0$.

Case 1: If $\mathrm{sp}(a)$ finite, then $a \sim_{\mathrm{Cu}} p$ for finite-rank projection p.

Case 2: If $\mathrm{sp}(a)$ is infinite, then $(a - \varepsilon)_+ \sim_{\mathrm{Cu}} p$ for finite-rank projection p, with rank $\to \infty$ as $\varepsilon \to 0$. It follows that all positive elements with infinite spectrum are Cuntz equivalent. Thus:

$$\mathrm{Cu}(\mathbb{C}) = \mathbb{K}_{+/\sim_{Cu}} \cong \overline{\mathbb{N}}.$$

Murray-von Neumann vs Cuntz semigroup I

Examples:

A	$V(A)$	$\mathrm{Cu}(A)$
$\mathbb{C}$	$\mathbb{N}$	$\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$
$C([0,1])$	$\mathbb{N}$	$\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$
II_1-factor	$[0,\infty)$	$[0,\infty) \sqcup (0,\infty]$
$C^*_{\mathrm{red}}(\mathbb{F}_\infty)$	$\mathbb{N}$	$\mathbb{N} \sqcup (0,\infty]$
M_{2^∞}	$\mathbb{N}[\frac{1}{2}]$	$\mathbb{N}[\frac{1}{2}] \sqcup (0,\infty]$

- $\mathrm{Cu}(A)$ encodes more information than $V(A)$ – for example, $\mathrm{Cu}(A)$ always encodes the ideal lattice and tracial simplex
- $\mathrm{Cu}(A)$ is more difficult to compute than $V(A)$ – for example, $V(\cdot)$ is homotopy invariant, while $\mathrm{Cu}(\cdot)$ is not

Murray-von Neumann vs Cuntz semigroup II

- $V(\cdot)$ classifies finitely-generated, projective modules.

$$V(A) \cong \{\text{f.g. projective } A\text{-modules}\}_{/\cong}.$$

- If A has **stable rank one** ($A^{-1} \subseteq A$ is norm-dense), then $\mathrm{Cu}(A)$ classifies countably-generated Hilbert modules.

$$\mathrm{Cu}(A) \cong \{\text{c.g. Hilbert } A\text{-modules}\}_{/\cong}.$$

Example 8 ($C^*_{\mathrm{red}}(\mathbb{F}_\infty)$)

Dykema-Haagerup-Rørdam 1997: $C^*_{\mathrm{red}}(\mathbb{F}_\infty)$ stable rank one
Dykema-Rørdam '00, Robert '12: $\mathrm{Cu}(C^*_{\mathrm{red}}(\mathbb{F}_\infty)) \cong \mathbb{N} \sqcup (0, \infty]$.

Consequence: We know all c.g. Hilbert $C^*_{\mathrm{red}}(\mathbb{F}_\infty)$-modules.

Question 9

Is $\mathrm{Cu}(C^*_{\mathrm{red}}(\mathbb{F}_n)) \cong \mathbb{N} \sqcup (0, \infty]$ for $n \in \{2, 3, \ldots\}$?

Classification of morphisms (1)

B has **stable rank one** if $B^{-1} \subseteq B$ is dense.

Proposition

Let A be AF-algebra, B stable rank one. Then:

- (∃) *Every $V(A) \to V(B)$ is induced by $A \to B$.*
- (!) *$\varphi, \psi \colon A \to B$ approx. unitarily equivalent iff $V(\varphi) = V(\psi)$.*

Robert's class = inductive limits of one-dimensional noncommutative CW-complexes with trivial K_1-groups.
Examples: Interval algebra $C([0,1], M_n)$, dimension-drop algebra $\{f \in C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1, f(1) \in 1 \otimes M_q\}$.

Theorem (Robert 2012)

Let A in Robert's class, B stable rank one. Then:

- (∃) *Every $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$ is induced by $A \to B$.*
- (!) *$\varphi, \psi \colon A \to B$ approx. unitarily equivalent iff $\mathrm{Cu}(\varphi) = \mathrm{Cu}(\psi)$.*

Classification of morphisms (2)

- If A in Robert's class, B stable rank one, then morphisms $A \to B$ (up to a.u.) correspond to $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$.

Applications:

- Classification of Robert's class: $A \cong B$ iff $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$.
- Construction of the **Jiang-Su algebra** $\mathcal{Z}$ as the (unique) algebra in Robert's class with $\mathrm{Cu}(\mathcal{Z}) \cong \mathbb{N} \cup (0, \infty]$.
  ($\mathcal{Z}$ is unital, separable, simple, nuclear, projectionless, unique trace; the C^*-analog of the hyperfinite II_1-factor)
- Embedding results: $\mathcal{Z} \subseteq C^*_{\mathrm{red}}(\mathbb{F}_\infty)$
  (using $\mathrm{Cu}(\mathcal{Z}) \cong \mathrm{Cu}(C^*_{\mathrm{red}}(\mathbb{F}_\infty))$ and $\mathrm{sr}(C^*_{\mathrm{red}}(\mathbb{F}_\infty)) = 1$)

$\rightsquigarrow$ $C^*_{\mathrm{red}}(\mathbb{F}_\infty) \otimes C^*_{\mathrm{red}}(\mathbb{F}_\infty)$ is singly generated (T-Winter 2014)

Question 10

Does $\mathcal{Z}$ embed into every simple C^*-algebra with stable rank one?

Thank you!

References (1)

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Facts about Cuntz comparison of positive elements

- Let X locally compact, Hausdorff, and $f, g \in C_0(X)_+$. Then $f \precsim_{\mathrm{Cu}} g$ iff $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$.
- $f(a) \precsim_{\mathrm{Cu}} a$ for every continuous $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$.
- $f(a) \sim_{\mathrm{Cu}} a$ for every continuous $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ and $f(t) > 0$ for $t > 0$.
- $a \sim_{\mathrm{Cu}} a^t$ and $a \sim_{\mathrm{Cu}} ta$ for every $t > 0$.
- $xx^* \sim_{\mathrm{Cu}} x^*x$ for every $x \in A$.
- If $a \leq b$, then $a \precsim_{\mathrm{Cu}} b$.