# A gentle introduction to Cuntz semigroups
# Part 2

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# Reminder: Murray-von Neumann vs Cuntz semigroup

- Murray-von Neumann semigroup $V(A)$ encodes 'size' of projections in $A \otimes \mathbb{K}$.
- Cuntz semigroup $\mathrm{Cu}(A)$ encodes 'size' of positive elements in $A \otimes \mathbb{K}$.

| $A$ | $V(A)$ | $\mathrm{Cu}(A)$ |
|---|---|---|
| $\mathbb{C}$ | $\mathbb{N}$ | $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ |
| $C([0,1])$ | $\mathbb{N}$ | $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ |
| $\mathcal{Z}$ or $C^*_{\mathrm{red}}(\mathbb{F}_\infty)$ | $\mathbb{N}$ | $\mathbb{N} \sqcup (0, \infty]$ |

- $\mathrm{Cu}(A)$ encodes more information than $V(A)$ – for example, $\mathrm{Cu}(A)$ always encodes the ideal lattice and tracial simplex
- $\mathrm{Cu}(A)$ is more difficult to compute than $V(A)$ – for example, $V(\cdot)$ is homotopy invariant, while $\mathrm{Cu}(\cdot)$ is not

# Existence of traces I

- Projection $p$ is **finite** if $p \sim_{\mathrm{MvN}} q \leq p$ implies $q = p$.
- $A$ is **stably finite** if every projection in $A \otimes \mathbb{K}$ is finite.
- **Trace** is state $\tau\colon A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$ for all $a, b$.

## Lemma 1

*Let $A$ be simple, unital. If $A$ admits trace, then $A$ is stably finite.*

## Proof.

Let $\tau$ trace on $A$. Then $\tau$ is **faithful**: If $a \in A_+$ satisfies $\tau(a) = 0$, then $a = 0$. Let $p, q \in \mathrm{Proj}(A)$ with $p \sim_{\mathrm{MvN}} q \leq p$. Pick $v \in A$ such that $p = vv^*$ and $q = v^*v \leq p$. Then

$$\tau(p - v^*v) = \tau(p) - \tau(v^*v) = \tau(p) - \tau(vv^*) = 0.$$

Since $p - v^*v \in A_+$, get $p - v^*v = 0$. Thus, $q = p$.
Same argument applies in $M_n(A)$, since $M_n(A) = A \otimes M_n(\mathbb{C})$ is simple, unital and admits trace $\tau \otimes \mathrm{tr}_n$. □

# Existence of traces II

**Lemma**

*Let $A$ be simple, unital. If $A$ admits trace, then $A$ is stably finite.*

- Converse? Remains open. (One of the biggest open problems in C*-algebras.)
- $A$ is **exact** if $A \otimes -$ preserves short exact sequences: If $0 \to I \to B \to B/I \to 0$ is exact, then so is

$$0 \to A \otimes I \to A \otimes B \to A \otimes (B/I) \to 0.$$

- Most 'everyday life' $C^*$-algebras are exact: Every nuclear $C^*$-algebra is exact. Exactness passes to subalgebras, quotients and inductive limits.

**Theorem (Cuntz 1978, Blackadar-Handelman 1982, Haagerup 1991/2014)**

*Let $A$ be simple, unital, exact. Then $A$ admits a trace iff $A$ is stably finite.*

# Existence of traces III

**Theorem**

*Let $A$ be simple, unital, exact. Then $A$ admits a trace iff $A$ is stably finite.*

**Proof. (Sketch of $\Leftarrow$).**

- Cuntz 1978: $A$ admits a **dimension function**: Additive, order-preserving, normalized map $d\colon \mathrm{Cu}(A) \to [0,\infty]$. Consider $M := \{n[1_A] : n \in \mathbb{N}\} \subseteq \mathrm{Cu}(A)$ and define

$$d_0\colon M \to [0,\infty], \quad d_0(n[1_A]) := n.$$

  (Well-defined since $A$ stably finite.) Use order-theoretic Hahn-Banach to extend $d_0$ to $d\colon \mathrm{Cu}(A) \to [0,\infty]$.
- Blackadar-Handelman 1982:
  Dimension functions are induced by quasitraces.
- Haagerup 1991/2014:
  Quasitraces on exact $C^*$-algebras are traces. □

# Structure and classification of simple C*-algebras I

- Elliott program: Classify simple, nuclear $C^*$-algebras using $K$-theoretic and tracial data $\mathrm{Ell}(-)$.
- Simple $C^*$-algebras come in two main flavors: Stably finite (all projections in $A \otimes \mathbb{K}$ are finite) and **purely infinite**: Every projection in $A$ is infinite and there are 'many projections' (every hereditary subalgebra contains a nonzero projection)
  (Analogous to $\mathrm{II}_1$ factors, and $\mathrm{III}$ factors.)

**Theorem (Kirchberg-Phillips 2000)**

*If $A$ and $B$ are also purely infinite, then:*

$$A \cong B \quad \Leftrightarrow \quad A \sim_{\mathrm{KK}} B.$$

*If $A$ and $B$ also satisfy the Universal Coefficient Theorem (UCT), then $A \sim_{\mathrm{KK}} B$ iff $K_*(A) \cong K_*(B)$. In this case:*

$$A \cong B \quad \Leftrightarrow \quad K_*(A) \cong K_*(B) \quad \Leftrightarrow \quad \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

# Structure and classification of simple C*-algebras II

- Elliott program: Classify simple, nuclear $C^*$-algebras using $K$-theoretic and tracial data $\mathrm{Ell}(-)$.
- Main flavors: Stably finite and purely infinite.
- Kirchberg-Phillips (2000) handled purely infinite case.

**Theorem (Toms 2008)**

*There exist simple, nuclear, stably finite $A$ and $B$ with $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, yet $A \not\cong B$. (Since $\mathrm{Cu}(A) \not\cong \mathrm{Cu}(B)$.)*

Need more regularity to classify with $\mathrm{Ell}(-)$.

**Definition 2**

*$A$ is $\mathcal{Z}$**-stable** if $A \cong \mathcal{Z} \otimes A$. ($C^*$-analog of being McDuff)*

- Have $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$, and so $\mathcal{Z} \otimes A$ is always $\mathcal{Z}$-stable:

$$\mathcal{Z} \otimes A \cong \big(\mathcal{Z} \otimes \mathcal{Z}\big) \otimes A \cong \mathcal{Z} \otimes \big(\mathcal{Z} \otimes A\big).$$

- Have $\mathrm{Ell}(\mathcal{Z}) \cong \mathrm{Ell}(\mathbb{C})$, and so $\mathrm{Ell}(A) \cong \mathrm{Ell}(\mathcal{Z} \otimes A)$.

# Structure and classification of simple C*-algebras III

### Definition

*$A$ is $\mathcal{Z}$**-stable** if $A \cong \mathcal{Z} \otimes A$.*

### Theorem (many hands 1976-2019)

*Let $A$ and $B$ simple, nuclear, $\mathcal{Z}$-stable satisfying UCT. Then:*

$$A \cong B \quad \Leftrightarrow \quad \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

- This completes the Elliott program under the additional assumption of $\mathcal{Z}$-stability and UCT.

### Question 3 (The UCT problem)

Do all nuclear $C^*$-algebras satisfy the UCT?

# Structure and classification of simple C*-algebras IV

**Theorem (many hands 1976-2019)**

*Let $A$ and $B$ simple, nuclear, $\mathcal{Z}$-stable satisfying UCT. Then:*

$$A \cong B \quad \Leftrightarrow \quad \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

- This completes the Elliott program under the additional assumption of $\mathcal{Z}$-stability and UCT.

**Theorem (Kirchberg)**

*A simple, $\mathcal{Z}$-stable $C^*$-algebra is either stably finite or purely infinite.*

**Theorem (Rørdam 2003)**

*There exists a simple, unital, nuclear $C^*$-algebra satisfying UCT that contains a finite and an infinite projection.*
*(It is neither stably finite, nor purely infinite. It is not $\mathcal{Z}$-stable.)*

# The Toms-Winter conjecture I

**Toms-Winter conjecture 2009**

For simple, nuclear $A$, TFAE:

1. $A$ has finite nuclear dimension.
2. $A$ is $\mathcal{Z}$-stable.
3. $\mathrm{Cu}(A)$ is **almost unperforated**:
   If $(n+1)[a] \leq n[b]$ then $[a] \leq [b]$.

- Rørdam 2004: (2)⇒(3)
- Winter 2012: (1)⇒(2)
- Castillejos-Evington-Tikuisis-White-Winter 2021: (2)⇒(1)

The implication (3)⇒(2) remains open. Partial results:

- Matui-Sato 2012: (3)⇒(2) holds if $\partial_e T(A)$ finite.
- Kirchberg-Rørdam, Sato, Toms-White-Winter: (3)⇒(2) holds if $\partial_e T(A)$ compact and finite covering dimension.
- T 2020: (3)⇒(2) holds if $A$ is ASH-algebra (strong form of nuclearity) and has stable rank one.

# The Toms-Winter conjecture II

**Theorem 4 (T 2020)**

*Toms-Winter conj. holds for ASH-algebras of stable rank one.*

**Corollary 5**

*The Toms-Winter conjecture holds for minimal crossed products $C(X) \rtimes \mathbb{Z}$.*

**Proof.**

- Orbit breaking subalgebra $B$ of $A = C(X) \rtimes \mathbb{Z}$ is ASH.
- Phillips 2014: $\mathrm{Cu}(A)$ is almost unperforated iff $\mathrm{Cu}(B)$ is.
- Archey-Buck-Phillips 2018: $A$ is $\mathcal{Z}$-stable iff $B$ is.
- Alboiu-Lutley 2022: $B$ has stable rank one.

$\mathrm{Cu}(A)$ almost unperforated $\Rightarrow$ $\mathrm{Cu}(B)$ almost unperforated $\Rightarrow$ $B$ is $\mathcal{Z}$-stable $\Rightarrow$ $A$ is $\mathcal{Z}$-stable □

# The Toms-Winter conjecture III

**Toms-Winter conjecture 2009: For simple, nuclear $A$, TFAE:**

2. $A$ is $\mathcal{Z}$-stable.
3. $\mathrm{Cu}(A)$ is almost unperforated:
   If $(n+1)[a] \leq n[b]$ then $[a] \leq [b]$.

**Proposition 6**

*$\mathrm{Cu}(A) \cong \mathrm{Cu}(\mathcal{Z} \otimes A)$ iff $\mathrm{Cu}(A)$ is almost unperforated and* ***almost divisible****: For $[a]$ and $n$ there is $[b]$ with $n[b] \leq [a] \leq (n+1)[b]$.*

For simple, nuclear $A$, consider:

(2) $A$ is $\mathcal{Z}$-stable: $A \cong \mathcal{Z} \otimes A$.

(3a) $\mathrm{Cu}(A) \cong \mathrm{Cu}(\mathcal{Z} \otimes A)$.

(3b) $\mathrm{Cu}(A)$ is almost unperforated and almost divisible.

(3c) $\mathrm{Cu}(A)$ is almost unperforated.

- known: (2)$\Rightarrow$(3a)$\Leftrightarrow$(3b)$\Rightarrow$(3c)

# The Toms-Winter conjecture IV

Modulo ‘strong nuclearity’, Toms-Winter conjecture reduces to:

Does almost unperforated imply almost divisible for $\mathrm{Cu}(A)$?

**Theorem 7 (Antoine-Perera-Robert-Thiel 2022)**

*If $A$ has stable rank one, then $\mathrm{Cu}(A)$ has **Riesz interpolation**: If $[a_j] \leq [c_k]$ for $j, k = 1, 2$, there is $[b]$ such that $[a_j] \leq [b] \leq [c_k]$.*

**Corollary 8**

*If $A$ is separable, stable rank one, then $\mathrm{Cu}(A)$ is semilattice: For $[c_1], [c_2] \in \mathrm{Cu}(A)$ exists infimum $[c_1] \wedge [c_2]$.*

**Proof.**

Consider the set of lower bounds $L := \{[a] : [a] \leq [c_1], [c_2]\}$. Riesz interpolation shows $L$ is upward directed. Separability gives that $L$ has supremum. □

Thank you!

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