Domains arising in operator algebras

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What are operator algebras?

Definition

*A **C*-algebra** is a Banach algebra with an involution such that*

$$\|a^* a\| = \|a\|^2 \qquad\qquad (\textit{for all } a \in A)$$

We consider only unital, separable C*-algebras.

Theorem (Gelfand 1940s)

A C-algebra A is commutative if and only if $A \cong C(X)$, where*

$$C(X) = \{ f \colon X \to \mathbb{C} \mid f \text{ continuous} \}$$

with $\|\cdot\|_\infty$-norm, for some compact, metrizable space X.

- C*-algebra '=' functions on noncommutative topol. space
- Models noncommuting observables in quantum physics.
- C*-algebras associated to groups or dynamical systems.

What are Hilbert modules?

Definition

Given C-algebra A, a **Hilbert A-module** is an A-module E together with a generalized inner product* $\langle \cdot, \cdot \rangle \colon E \times E \to A$ *such that E is complete for* $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$.

Examples

- Hilbert $\mathbb{C}$-module = Hilbert space
- $A^{\oplus n}$ w.r.t. $\big\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \big\rangle = \sum_k a_k^* b_k$
- every finitely generated, projective A-module
- $\overline{\bigoplus_{\mathbb{N}} A} = \big\{ (a_n)_{n \in \mathbb{N}} : a_n \in A, \sum_n a_n^* a_n \text{ converges in } A \big\}$
- $\overline{\bigoplus_{\mathbb{N}} \mathbb{C}} = \ell^2(\mathbb{N})$

The domain semigroup of Hilbert modules 1

Given a C^*-algebra A, we consider:

$$\mathcal{H}(A) := \{\text{countably generated, Hilbert } A\text{-modules}\}/_{\cong}$$

- $\mathcal{H}(A)$ is abelian monoid with $[E] + [F] := [E \oplus F]$.
- $\mathcal{H}(A)$ is pre-ordered with $[E] \leq [F]$ if $E \hookrightarrow F$ as closed submodule.

Theorem (Coward-Elliott-Ivanescu, Crelle 2008)

Under the mild assumption of 'stable rank one' (ensuring that $\mathcal{H}(A)$ is partially ordered), $\mathcal{H}(A)$ is a ***domain semigroup****:*

- *Every directed set in $\mathcal{H}(A)$ has a supremum (dcpo).*
- *Every $[E]$ is the supremum of $\{[E'] \in \mathcal{H}(A) : [E'] \ll [E]\}$.*
- *If $[E'] \ll [E]$ and $[F'] \ll [F]$, then $[E' \oplus F'] \ll [E \oplus F]$.*
- *If $C, D \subseteq \mathcal{H}(A)$ directed, then $\sup(C + D) = \sup C + \sup D$.*

The domain semigroup of Hilbert modules 2

Given a C^*-algebra A, we consider:

$$\mathcal{H}(A) := \{\text{countably generated, Hilbert } A\text{-modules}\}/_{\cong}$$

Similar to Murray-von Neumann semigroup:

$$V(A) := \{\text{finitely generated, projective } A\text{-modules}\}/_{\cong}$$

The Grothendieck completion of $V(A)$ is $K_0(A)$.

Example ($A = \mathbb{C}$)

- We have $V(\mathbb{C}) \cong \mathbb{N} = \{0, 1, 2, \ldots\}$.
  Every finitely generated, projective $\mathbb{C}$-modules E is free, and so $E \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$.
- We have $\mathcal{H}(\mathbb{C}) \cong \overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$.
  Hilbert $\mathbb{C}$-module = Hilbert space
  Countably generated Hilbert $\mathbb{C}$-modules = separable Hilbert spaces: $\mathbb{C}^n$ for $n \in \mathbb{N}$ and also $\ell^2(\mathbb{N})$

The domain semigroup of Hilbert modules 3

Examples:

A	$V(A)$	$\mathcal{H}(A)$
$\mathbb{C}$	$\mathbb{N}$	$\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$
$C([0,1])$	$\mathbb{N}$	$\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$
$C^*_{\mathrm{red}}(\mathbb{F}_\infty)$	$\mathbb{N}$	$\mathbb{N} \sqcup (0, \infty]$

- $\mathcal{H}(A)$ encodes more information than $V(A)$ – for example, $\mathcal{H}(A)$ always encodes the ideal lattice and tracial simplex
- $\mathcal{H}(A)$ is more difficult to compute than $V(A)$ – for example, $V(\cdot)$ is homotopy invariant, while $\mathcal{H}(\cdot)$ is not
- Understanding $\mathcal{H}(A)$ is crucial for the structure theory of C*-algebras

Semilattice structure of Hilbert modules 1

Theorem (Coward-Elliott-Ivanescu, Crelle 2008)

*If A has stable rank one, then $\mathcal{H}(A)$ is a **domain semigroup**.*

Theorem (Antoine-Perera-Robert-T, Duke 2022)

*If A has stable rank one, then $\mathcal{H}(A)$ is a **semilattice**:*

- *Given $[E], [F] \in \mathcal{H}(A)$, the infimum $[E] \wedge [F] \in \mathcal{H}(A)$ exists.*

Application for C*-algebras of stable rank one:

- Solution of the Global Glimm Problem:
  Highly noncommutative C*-algebras can be untwisted.
- Verification of the Blackadar-Handelman Conjecture:
  Dimension functions form a Choquet simplex.
- Solution of the Rank Problem:
  Every lower-semicontinuous, affine, strictly positive function arises as the rank of an operator.

Semilattice structure of Hilbert modules 2

Theorem (Antoine-Perera-Robert-T, Duke 2022)

If A has stable rank one, then $\mathcal{H}(A)$ is a ***semilattice****.*

Proof.

1. We verify the Riesz interpolation property: Given countably generated Hilbert modules E_1, E_2 and F_1, F_2 satisfying

$$E_j \hookrightarrow F_k \qquad \text{(for } j = 1, 2 \text{ and } k = 1, 2)$$

there exists a countably generated Hilbert module G such that

$$E_j \hookrightarrow G \hookrightarrow F_k \qquad \text{(for } j = 1, 2 \text{ and } k = 1, 2)$$

2. Given $[F_1], [F_2] \in \mathcal{H}(A)$, the set of lower bounds

$$L := \{[E] \in \mathcal{H}(A) : [E] \leq [F_1], [F_2]\}$$

is directed. Then $[E] \wedge [F] = \sup L$ exists as $\mathcal{H}(A)$ is dcpo. □

Thank you!

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