# Domains arising in operator algebras

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# What are operator algebras? I

## Definition

*A **C*-algebra** is a norm-closed $*$-invariant subalgebra $A \subseteq \mathcal{B}(H)$ of bounded operators on some Hilbert space $H$.*

## Example (Commutative $C^*$-algebras)

$C(X) = \{f \colon X \to \mathbb{C} : f \text{ continuous}\}$ for $X$ compact, Hausdorff, with pointwise addition, multiplication and involution, and norm

$$\|f\|_\infty = \sup\big\{|f(x)| : x \in X\big\}.$$

Acting as multiplication operators: $C(X) \subseteq \mathcal{B}(L^2(X, \mu))$.
Every unital, commutative $C^*$-algebra is of this form.

# What are operator algebras? II

- C*-algebras = norm-closed $*$-subalgebra of $\mathcal{B}(H)$
- Unital, commutative $C^*$-algebras = $C(X)$ for some compact, Hausdorff space $X$

## Contravariant equivalence of categories

$$\{\text{compact } T_2 \text{ spaces}\} \leftrightarrow \{\text{unital, commutative } C^*\text{-algebras}\}$$
$$\{\text{continuous maps}\} \leftrightarrow \{*\text{-homomorphisms}\}$$

- C*-algebra '=' functions on noncommutative topol. space
- Models noncommuting observables in quantum physics.

## Example (Reduced group C*-algebra)

Let $G$ be a discrete group. Left regular representation: $g \in G$ acts by unitary $u_g$ on $\ell^2(G)$, shifting indices $u_g\colon (\xi_h)_h \mapsto (\xi_{gh})_h$.

$$C^*_{\mathrm{red}}(G) := \overline{\operatorname{span}\{u_g : g \in G\}}^{\|\cdot\|} \subseteq \mathcal{B}(\ell^2(G))$$

# Comparison of projections (Murray-von Neumann) I

Throughout $A$ is a $C^*$-algebra.

## Definition

*Projection = self-adjoint idempotent ($p \in A$ with $p = p^2 = p^*$)*
***Murray-von Neumann (sub)equivalence** of projections in $A$:*

$$p \sim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = vv^*, q = v^*v, \ \ \textit{some } v \in A.$$
$$p \precsim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = rqr^*, \ \ \textit{some } r \in A.$$

***Murray-von Neumann semigroup**:*

$$V(A) := \mathrm{Proj}(A \otimes \mathcal{K})/_{\sim_{\mathrm{MvN}}}, \quad [p] + [q] := [p \oplus q].$$

- $\mathcal{K}$ denotes compact operators on $\ell^2(\mathbb{N})$, and $A \otimes \mathcal{K}$ is the **stabilization** of $A$, the completion of $\bigcup_n M_n(A)$.
- Idea: Equivalent projections have the same 'size' relative to $A$. $V(A)$ encodes 'sizes' of projections.
- If $A$ is unital, then $K_0(A)$ = Grothendieck group of $V(A)$.

# Comparison of projections (Murray-von Neumann) II

$$p \sim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = vv^*, q = v^*v, \ \text{some } v \in A.$$
$$p \precsim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = rqr^*, \ \text{some } r \in A.$$
$$V(A) := \mathrm{Proj}(A \otimes \mathbb{K})/_{\sim_{\mathrm{MvN}}}, \quad [p] + [q] := [p \oplus q].$$

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## Example 1 ($\mathbb{C}$)

Consider projection $p$ in $\mathbb{C} \otimes \mathcal{K} = \mathcal{K} = \mathcal{K}(H)$. The **rank** of $p$ is

$$\mathrm{rk}(p) := \dim_{\mathbb{C}} p(H) \in \{0, 1, 2, \ldots\} =: \mathbb{N}.$$

For $p, q \in \mathrm{Proj}(\mathcal{K})$, have

$$p \sim_{\mathrm{MvN}} q \quad \Leftrightarrow \quad \mathrm{rk}(p) = \mathrm{rk}(q)$$
$$p \precsim_{\mathrm{MvN}} q \quad \Leftrightarrow \quad \mathrm{rk}(p) \leq \mathrm{rk}(q).$$

Moreover, for every $n \in \mathbb{N}$ there is a projection of rank $n$. Thus

$$V(\mathbb{C}) = \mathrm{Proj}(\mathcal{K})/_{\sim_{\mathrm{MvN}}} \cong \mathbb{N}, \quad \text{and} \quad K_0(\mathbb{C}) = \mathrm{Gr}(\mathbb{N}) \cong \mathbb{Z}.$$

# Comparison of projections (Murray-von Neumann) III

$$p \sim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = vv^*, q = v^*v, \ \text{some } v \in A.$$

$$p \precsim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = rqr^*, \ \text{some } r \in A.$$

$$V(A) := \mathrm{Proj}(A \otimes \mathbb{K})/_{\sim_{\mathrm{MvN}}}, \quad [p] + [q] := [p \oplus q].$$

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## Example 2 ($C([0,1])$)

Since $[0,1]$ is homotopy equivalent to $\{\mathrm{pt}\}$, have $C([0,1])$ homotopy equivalent to $C(\{\mathrm{pt}\}) = \mathbb{C}$. Since $V(\cdot)$ is homotopy invariant, we get

$$V\big(C([0,1])\big) \cong V(\mathbb{C}) \cong \mathbb{N}, \quad \text{and} \quad K_0\big(C([0,1])\big) \cong \mathbb{Z}.$$

## Example 3 (Reduced group $C^*$-algebra $C^*_{\mathrm{red}}(\mathbb{F}_n)$ of free group)

Pimsner-Voiculescu 1982: $K_0(C^*_{\mathrm{red}}(\mathbb{F}_n)) \cong \mathbb{Z}$

Dykema-Haagerup-Rørdam 1997: $V(C^*_{\mathrm{red}}(\mathbb{F}_n)) \cong \mathbb{N}$

# Comparison of positive elements (Cuntz) I

- Problem: Many interesting $C^*$-algebras contain few (if any) projections, and MvN semigroup contains no information.
- Idea (Cuntz): Comparison of (abundant) positive elements (= self-adjoint elements with spectrum in $[0, \infty)$)

Recall: $p \precsim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad p = rqr^*, \;$ some $r \in A$.

First Attempt: $a \precsim_{\mathrm{Cu}} b \quad :\Leftrightarrow \quad a = rbr^*, \;$ some $r \in A$.

## Definition (Cuntz 1978)

***Cuntz (sub)equivalence*** *of positive elements in A:*

$$a \precsim_{\mathrm{Cu}} b \quad :\Leftrightarrow \quad \forall\, \varepsilon > 0 : \|a - rbr^*\| < \varepsilon, \text{ for some } r \text{ in } A.$$

$$\Leftrightarrow \quad a = \lim_n r_n b r_n^*, \text{ for some } (r_n)_n \text{ in } A.$$

$$a \sim_{\mathrm{Cu}} b \quad :\Leftrightarrow \quad a \precsim_{\mathrm{Cu}} b \precsim_{\mathrm{Cu}} a.$$

***Cuntz semigroup****:*

$$\mathrm{Cu}(A) := (A \otimes \mathcal{K})_{+/\sim_{\mathrm{Cu}}},$$

# Comparison of positive elements (Cuntz) II

## Definition (Cuntz 1978)

$$a \precsim_{\mathrm{Cu}} b \quad :\Leftrightarrow \quad a = \lim_n r_n b r_n^*, \textit{ for some } (r_n)_n \textit{ in } A.$$

$$a \sim_{\mathrm{Cu}} b \quad :\Leftrightarrow \quad a \precsim_{\mathrm{Cu}} b \precsim_{\mathrm{Cu}} a.$$

***Cuntz semigroup*** $\mathrm{Cu}(A) := (A \otimes \mathcal{K})_{+/\sim_{\mathrm{Cu}}}$, *equipped with addition and partial order:*

$$[a] + [b] := [a \oplus b], \quad [a] \leq [b] \;:\Leftrightarrow\; a \precsim_{\mathrm{Cu}} b.$$

## Theorem (Coward-Elliott-Ivanescu 2008)

$\mathrm{Cu}(A)$ *is* $\omega$*-domain and commutative monoid:*

(O1) *Every increasing sequence has a supremum. (*$\omega$*-dcpo)*

(O2) *Every* $x$ *is* $x = \sup_n x_n$ *with* $x_n \ll_\omega x_{n+1}$*. (*$\omega$*-continuous)*

(O3) *If* $x' \ll_\omega x$ *and* $y' \ll_\omega y$*, then* $x' + y' \ll_\omega x + y$*.*

(O4) *If* $x \ll_\omega y + z$*, then* $x \ll_\omega y' + z'$ *some* $y' \ll_\omega y$*,* $z' \ll_\omega z$*.*

# Examples of Cuntz semigroups I

## Theorem (Coward-Elliott-Ivanescu 2008)

$\mathrm{Cu}(A)$ *is* $\omega$*-domain semigroup. If A is separable, then* $\mathrm{Cu}(A)$ *is countably based and thus a (continuous) domain.*
$*$*-homomorphism* $A \to B$ *induces Scott continuous monoid morphism* $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$*, preserving the way-below relation!*

## Example 1 ($\mathbb{C}$, revisited)

The **rank** of a positive element $a$ in $\mathbb{C} \otimes \mathcal{K} = \mathcal{K} = \mathcal{K}(H)$ is:

$$\mathrm{rk}(a) := \dim_{\mathbb{C}} a(H) \in \{0, 1, 2, \dots, \infty\} =: \overline{\mathbb{N}}.$$

For $a, b \in \mathcal{K}_+$, have

$$a \sim_{\mathrm{Cu}} b \Leftrightarrow \mathrm{rk}(a) = \mathrm{rk}(b), \quad a \precsim_{\mathrm{Cu}} b \Leftrightarrow \mathrm{rk}(a) \leq \mathrm{rk}(b).$$

Moreover, every $n \in \overline{\mathbb{N}}$ is realized by positive element. Thus

$$\mathrm{Cu}(\mathbb{C}) \cong \overline{\mathbb{N}}.$$

# Examples of Cuntz semigroups II

$$a \precsim_{\mathrm{Cu}} b \;:\Leftrightarrow\; a = \lim_n r_n b r_n^*, \text{for some } (r_n)_n \text{ in } A.$$

$$a \sim_{\mathrm{Cu}} b \;:\Leftrightarrow\; a \precsim_{\mathrm{Cu}} b \precsim_{\mathrm{Cu}} a, \quad \mathrm{Cu}(A) := (A \otimes \mathcal{K})_+/_{\sim_{\mathrm{Cu}}}.$$

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## Example 2 ($C([0,1])$, revisited)

The Cuntz semigroup is not homotopy invariant.
For $f, g \in C([0,1])_+$, have

$$f \precsim_{\mathrm{Cu}} g \quad \Leftrightarrow \quad \{t \in [0,1] : f(t) \neq 0\} \subseteq \{t \in [0,1] : g(t) \neq 0\}.$$

We get

$$C([0,1])_+/_{\sim_{\mathrm{Cu}}} \quad \cong \quad \mathcal{O}([0,1]) \quad = \quad \mathrm{Lsc}\big([0,1], \{0,1\}\big),$$

and using $C([0,1]) \otimes \mathcal{K} \cong C([0,1], \mathcal{K})$ we deduce

$$\mathrm{Cu}\big(C([0,1])\big) \quad \cong \quad C([0,1], \mathcal{K})_+/_{\sim_{\mathrm{Cu}}} \quad \cong \quad \mathrm{Lsc}\big([0,1], \overline{\mathbb{N}}\big),$$

while $V\big(C([0,1])\big) \cong \mathbb{N}$.

# Examples of Cuntz semigroups III

## Example 3 (Reduced group $C^*$-algebra $C^*_{\mathrm{red}}(\mathbb{F}_n)$, revisited)

Amrutan, Gao, Elayavalli, Patchell (Invent. Math. 2025):
$\mathrm{Cu}(C^*_{\mathrm{red}}(\mathbb{F}_n)) \cong \mathbb{N} \sqcup (0, \infty]$

## Theorem (Antoine-Perera-T 2018, 2020)

*The category of $\omega$-domain semigroups is bicomplete, and the functor $\mathrm{Cu}(\cdot)$ preserves inductive limits and products.*

## Example 4 (Uniformly hyperfinite (UHF) algebras)

Consider the inductive limit

$$M_{2^\infty} := \varinjlim \Big( M_2(\mathbb{C}) \xrightarrow{x \mapsto x \otimes 1} M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \to \ldots \bigotimes^n M_2(\mathbb{C}) \ldots \Big)$$

Since $\bigotimes_n M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$, and $\mathrm{Cu}(\cdot)$ preserves $\varinjlim$, get

$$\mathrm{Cu}(M_{2^\infty}) \cong \varinjlim \Big( \overline{\mathbb{N}} \xrightarrow{x \mapsto 2x} \overline{\mathbb{N}} \to \ldots \Big) \cong \mathbb{N}\Big[\frac{1}{2}\Big] \sqcup (0, \infty].$$

# Examples of Cuntz semigroups IV

## Example 4 (Uniformly hyperfinite (UHF) algebras, continued)

Similarly, can consider UHF algebra for supernatural number $\mathbf{p} = 2^{n_1}3^{n_2}5^{n_3}7^{n_4}11^{n_5}\cdots$ with $n_1, n_2, \ldots \in \overline{\mathbb{N}}$, which has Cuntz semigroup $\mathbb{N}[\frac{1}{\mathbf{p}}] \sqcup (0, \infty]$.
For $n_1 = n_2 = \ldots = \infty$, get universal UHF algebra $Q$ with

$$\mathrm{Cu}(Q) \cong \mathbb{Q}_+ \sqcup (0, \infty].$$

The Cuntz classes of positive elements in $Q$ (and not $Q \otimes \mathcal{K}$) correspond to

$$Q_+/_{\sim_{\mathrm{Cu}}} \cong \big(\mathbb{Q} \cap [0, 1]\big) \sqcup (0, 1].$$

# Examples of Cuntz semigroups V

## Example 5 ($\mathrm{II}_1$-factor)

Let $N$ be a $\mathrm{II}_1$-factor, with its unique tracial state $\tau$. For $p, q \in \mathrm{Proj}(N)$, Murray-von Neumann showed:

$$p \sim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad \tau(p) = \tau(q),$$
$$p \precsim_{\mathrm{MvN}} q \quad :\Leftrightarrow \quad \tau(p) \leq \tau(q).$$

Moreover, for every $t \in [0,1]$, there is $p$ with $\tau(p) = t$. Thus

$$\mathrm{Proj}(N)_{/\sim_{MvN}} \cong [0,1].$$

The same holds in each $M_n(N)$, and we get

$$\mathrm{Proj}(M_n(N))_{/\sim_{MvN}} \cong [0,n],$$

and then

$$V(N) \cong [0,\infty), \quad \text{and} \quad \mathrm{Cu}(N) \cong [0,\infty) \sqcup (0,\infty].$$

# Murray-von Neumann vs Cuntz semigroup

Examples:

| $A$ | $V(A)$ | $\mathrm{Cu}(A)$ |
|---|---|---|
| $\mathbb{C}$ | $\mathbb{N}$ | $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ |
| $C([0,1])$ | $\mathbb{N}$ | $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ |
| $C^*_{\mathrm{red}}(\mathbb{F}_n)$ | $\mathbb{N}$ | $\mathbb{N} \sqcup (0, \infty]$ |
| $M_{2^\infty}$ | $\mathbb{N}[\frac{1}{2}]$ | $\mathbb{N}[\frac{1}{2}] \sqcup (0, \infty]$ |
| $Q$ | $\mathbb{Q}_+$ | $\mathbb{Q}_+ \sqcup (0, \infty]$ |
| $\mathrm{II}_1$-factor | $[0, \infty)$ | $[0, \infty) \sqcup (0, \infty]$ |

- $\mathrm{Cu}(A)$ encodes more information than $V(A)$ – for example, $\mathrm{Cu}(A)$ always encodes the ideal lattice and tracial simplex
- $\mathrm{Cu}(A)$ is more difficult to compute than $V(A)$ – for example, $V(\cdot)$ is homotopy invariant, while $\mathrm{Cu}(\cdot)$ is not

# Semilattice structure of Cuntz semigroup 1

**Definition**

*A unital $C^*$-algebra $A$ has **stable rank one** if the set of invertible elements in $A$ is norm-dense.*

Examples: $\mathbb{C}$, $C([0,1])$, $C^*_{\mathrm{red}}(\mathbb{F}_n)$, UHF-algebras, $\mathrm{II}_1$-factors.

**Theorem (Antoine-Perera-Robert-T, Duke Math. J. 2022)**

*If $A$ is a separable $C^*$-algebra of stable rank one, then $\mathrm{Cu}(A)$ is a semilattice. Moreover,*

$$x \wedge \bigvee{}^{\uparrow} y_n = \bigvee{}^{\uparrow}(x \wedge y_n) \qquad (\textit{continuous preframe})$$

$$x + (y \wedge z) = (x+y) \wedge (x+z) \qquad (\textit{inf-semilattice ordered})$$

Application for C*-algebras of stable rank one:

- Solution of the Global Glimm Problem.
- Verification of the Blackadar-Handelman Conjecture.
- Solution of the Rank Problem.

# Semilattice structure of Cuntz semigroup 2

**Theorem (Antoine-Perera-Robert-T, Duke Math. J. 2022)**

*If $A$ is separable with stable rank one, then $\mathrm{Cu}(A)$ is semilattice.*

**Proof.**

1. For $C^*$-algebras of stable rank (possibly nonseparable), we verify that $\mathrm{Cu}(A)$ has the Riesz interpolation property:
Given $x_1, x_2$ and $y_1, y_2$ in $\mathrm{Cu}(A)$ satisfying

$$x_j \leq y_k \qquad \text{(for } j = 1, 2 \text{ and } k = 1, 2\text{)}$$

there exists $z \in \mathrm{Cu}(A)$ such that

$$x_j \leq z \leq y_k \qquad \text{(for } j = 1, 2 \text{ and } k = 1, 2\text{)}$$

2. Given $y_1, y_2 \in \mathrm{Cu}(A)$, the set of lower bounds

$$L := \left\{ x \in \mathrm{Cu}(A) : x \leq y_1, y_2 \right\}$$

is directed. Then $y_1 \wedge y_2 = \sup^{\uparrow} L$ exists as $\mathrm{Cu}(A)$ is dcpo. □

# Thank you!


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