Pure C*-algebras

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Logic and C*-Algebras
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Pureness and Z-stability 1

Definition 1 (Winter 2012)

A C-algebra is **pure** if it has*

strict comparison (very good comparison) and

almost divisibility (very good divisibility)

Definition 2

A C-algebra A is $\mathcal{Z}$**-stable** if it tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$, that is, $A \cong \mathcal{Z} \otimes A$.*

Theorem 3 (Rørdam 2004)

$$A \text{ is } \mathcal{Z}\text{-stable} \quad \Rightarrow \quad A \text{ is pure}$$

Example 4

VNAs without type I part are pure, but not $\mathcal{Z}$-stable.

Pureness and Z-stability 2

- Pureness = very good comparison + very good divisibility
- $\mathcal{Z}$-stability $\Rightarrow$ pureness

Theorem 5 (Antoine-Perera-T 2018)

Pureness means $\mathcal{Z}$-stability at the level of Cuntz semigroups:

$$A \text{ is pure} \quad \Leftrightarrow \quad \mathrm{Cu}(A) \cong \mathrm{Cu}(\mathcal{Z}) \otimes \mathrm{Cu}(A)$$

Question 6 (Problem LXXVI/IC of Schafhauser-Tikuisis-White)

$$A \text{ is pure} \quad \overset{?}{\Leftrightarrow} \quad \mathrm{Cu}(A) \cong \mathrm{Cu}(\mathcal{Z} \otimes A)$$

Remarks

'$\Leftarrow$' always holds. '$\Rightarrow$' holds if A is simple.
Does $\mathrm{Cu}(\mathcal{Z}) \otimes \mathrm{Cu}(A) = \mathrm{Cu}(\mathcal{Z} \otimes A)$ hold if A is pure?
(It can fail for non-pure C^*-algebras.)

Pureness and Z-stability 3

- pureness = $\mathcal{Z}$-stability at the level of Cuntz semigroups

Ultrapower $A_\mathcal{U}$ of A wrt. a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Theorem 7 (Perera-T-Vilalta 2025)

For separable A, $\mathcal{Z}$-stability means central pureness:

$$A \text{ is } \mathcal{Z}\text{-stable} \quad \Leftrightarrow \quad A' \cap A_\mathcal{U} \text{ is pure}$$

Definition 8

*A is **separably $\mathcal{Z}$-stable** if for every separable $B_0 \subseteq A$ there exists a separable, $\mathcal{Z}$-stable B with $B_0 \subseteq B \subseteq A$.*

Theorem 9 (Perera-T-Vilalta 2025)

A (nonseparable) C^*-algebra A is separably $\mathcal{Z}$-stable iff

$$B' \cap A_\mathcal{U} \text{ is pure for every separable } B \subseteq A_\mathcal{U}.$$

Pureness and Z-stability 4

- $\mathcal{Z}$-stability $\Rightarrow$ pureness

Conjecture 10

For a separable, nuclear C^*-algebra A:

$$A \text{ is } \mathcal{Z}\text{-stable} \quad \Leftrightarrow \quad A \text{ is pure}$$

Remarks

For simple C^*-algebras part of the Toms-Winter conjecture, and known if:

- *A has locally finite nuclear dimension (Winter 2012)*
- *$T(A)$ has finite-dimensional compact extreme boundary (Sato '12, Kirchberg-Rørdam '14, Toms-White-Winter '15)*

Also known if A has locally finite nuclear dimension, no simple purely infinite ideal-quotients, and $\mathrm{Prim}(A)$ is zero-dimensional (Robert-Tikuisis 2017)

Purely infinite C^*-algebras

Definition 11 (Kirchberg-Rørdam 2000)

*A C^*-algebra is **purely infinite** if it admits no characters and Cuntz equivalence is determined by the primitive ideal space.*

Theorem 12 (Kirchberg-Rørdam 2000)

$$A \text{ is } \mathcal{O}_\infty\text{-stable} \quad \Rightarrow \quad A \text{ is purely infinite}$$

Theorem 13 (Antoine-Perera-T 2018)

Pure infiniteness means $\mathcal{O}_\infty$-stability at Cuntz semigroup level:

$$A \text{ is purely infinite} \quad \Leftrightarrow \quad \mathrm{Cu}(A) \cong \mathrm{Cu}(\mathcal{O}_\infty) \otimes \mathrm{Cu}(A)$$

Question 14 (Kirchberg-Rørdam 2002)

For a separable, nuclear C^*-algebra A:

$$A \text{ is } \mathcal{O}_\infty\text{-stable} \quad \overset{?}{\Leftrightarrow} \quad A \text{ is purely infinite}$$

Pureness of simple C^*-algebras 1

- Pure infiniteness = strong form of infiniteness

Definition 15 (Rieffel 1983)

A unital C^*-algebra has ***stable rank one*** *if A^{-1} is norm-dense. (A strong form of stable finiteness.)*

Theorem 16 (Dichotomy - Lin 2024)

A simple, pure C^*-algebra is either of stable rank one (type II) or purely infinite (type III).

Remark

There exist simple C^*-algebras that are neither of stable rank one nor purely infinite. (Neither stably finite nor properly infinite). (Rørdam 2003)

Pureness of simple C^*-algebras 2

Every (quasi)trace $\tau \colon A \to \mathbb{C}$ induces a dimension function:

$$d_\tau \colon A_+ \to [0, \infty], \quad d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n}).$$

The rank of $a \in A_+$ is

$$\mathrm{rk}(a) \colon \mathrm{QT}(A) \to [0, 1], \quad \tau \mapsto d_\tau(a).$$

The Rank Problem - XXIII/IC of Schafhauser-Tikuisis-White

Which functions arise as $\mathrm{rk}(a)$ for $a \in A_+$? In particular, do all lower-semicontinuous, affine functions $\mathrm{QT}(A) \to (0, 1]$ arise?

Proposition 17

A simple C^*-algebra is pure iff it has very good comparison and all lower-semicontinuous, affine functions $\mathrm{QT}(A) \to (0, 1]$ arise as ranks.

Pureness of simple C^*-algebras 3

- A simple C^*-algebra is pure iff it has very good comparison and all ranks arise.

Theorem 18 (T 2020)

Solution to Rank Problem for simple C^*-algebra of stable rank one: All lower-semicontinuous, affine functions $\mathrm{QT}(A) \to (0, 1]$ arise as ranks.

Corollary 19

A simple C^*-algebra with very good comparison and stable rank one is pure. (An 'automatic divisibility' result.)

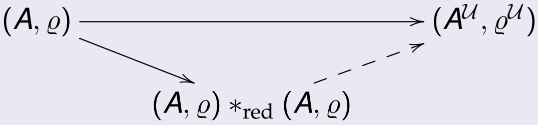
Question 20

Is every simple C^*-algebra with very good comparison either elementary (type I) or pure (type II or type III)?

Pureness of simple C^*-algebras 4

Definition 21 (Robert 2025)

A C^*-algebra A with GNS-faithful state ϱ is ***selfless*** *if the first factor embedding $(A, \varrho) \to (A, \varrho) *_{\mathrm{red}} (A, \varrho)$ is existential, that is,*



Theorem 22 (Robert 2025; Gould for non-faithful state)

Every selfless C^*-algebra is simple and pure. In the finite case, it has stable rank one and a unique tracial state (namely ϱ).

Theorem 23 (Amrutam-Gao-Kunna. Elayavalli-Patchell 2025)

$C^*_\lambda(\mathbb{F}_2)$ is selfless, and hence pure (but not $\mathcal{Z}$-stable)

A dimension reduction result 1

Definition 24 (Winter 2012)

*Let $m, n \geq 0$. A C^*-algebra A is **(m,n)-pure** if:*

- *A has m-**comparison**: If $x <_s y_0, \ldots, y_m$ in $\mathrm{Cu}(A)$, then $x \leq y_0 + \ldots + y_m$. (Where $x <_s y$ if $(k+1)x \leq ky$ some k.)*
- *A is n-**divisible**: For $x' \ll x$ in $\mathrm{Cu}(A)$ and $k \geq 2$, there exists $y \in \mathrm{Cu}(A)$ such that $ky \leq x$ and $x' \leq (n+1)(k+1)y$.*

***strict comparison** (very good comparison) = 0-comparison*
***almost divisibility** (very good divisibility) = 0-divisibility*
Pure means $(0,0)$-pure.

Example 25

$C(X)$ has m-comparison for $m = \dim(X)$. (Robert 2011)

A dimension reduction result 2

Theorem 26 (Antoine-Perera-T-Vilalta 2024)

$$A \text{ is } (m,n)\text{-pure for some } m,n \quad \Rightarrow \quad A \text{ is pure}$$

Proof.

- *m*-comparison ⇒ **controlled comparison** (=no silly traces = limit traces are dense in $T(A_{\mathcal{U}})$)
- *n*-divisibility ⇒ **functional divisibility**
- controlled comparison + functional divisibility ⇒ pure (decent comparison + decent divisibility ⇒ very good comparison + very good divisibility) □

Remarks

m-comparison does not imply 0*-comparison.*
Presumably, n-divisibility does not imply 0*-divisibility.*

Extensions of pure C^*-algebras 1

Theorem 27 (Perera-T-Vilalta 2025)

In short exact sequence, A is pure iff I and B are pure:

$$0 \to I \to A \to B \to 0.$$

Proof.

'$\Leftarrow$': Verify algebraically:

$$I \text{ has } m_1\text{-comp., } B \text{ has } m_2\text{-comp. } \Rightarrow A\ (m_1 + m_2 + 1)\text{-comp.}$$

$$I \text{ is } n_1\text{-div., } B \text{ is } n_2\text{-div } \Rightarrow A \text{ is } \max\{2n_1 + 1, 2n_2 + 1\}\text{-div.}$$

Thus, A is $(1,1)$-pure. Then apply dimension reduction. □

Question 28

If I and B have 0-comparison, then A has 1-comparison. Does A have 0-comparison? (Similarly, for 0-divisibility.)

Extensions of pure C^*-algebras 2

- Pureness passes to extensions.

Corollary 29

Let A be separable, unital, simple, non-elementary C^*-algebra with strict comparison by a unique tracial state. Then the stable multiplier algebra $M(A \otimes \mathbb{K})$ is pure.

Proof.

Consider short exact sequence

$$0 \to A \otimes \mathbb{K} \to M(A \otimes \mathbb{K}) \to M(A \otimes \mathbb{K})/(A \otimes \mathbb{K}) \to 0.$$

The stable corona algebra $M(A \otimes \mathbb{K})/(A \otimes \mathbb{K})$ is purely infinite (Kaftal-Ng-Zhang 2019), and hence pure. □

Examples

$M(\mathcal{Z} \otimes \mathbb{K})$ and $M(C^*_\lambda(\mathbb{F}_2) \otimes \mathbb{K})$ are pure.

Extensions of pure C^*-algebras 3

Question 30

Does pureness pass to multiplier algebras of σ-unital C^*-algebras?

Yes, for simple C^*-algebras. (Work-in-progress with Ng.)

Theorem 31 (Farah-Szabo 2024)

Separable $\mathcal{Z}$-stability passes to multiplier algebras of σ-unital C^*-algebras.

Example

$M(\mathcal{Z} \otimes \mathbb{K})$ is not only pure, but even separably $\mathcal{Z}$-stable.

Thank you!

And Happy Birthday, Ilijas.

References

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