Traces and the Cuntz semigroup of ultraproduct C*-algebras

Ramon Antoine
UAB

joint work with F. Perera, L. Robert and H. Thiel

Cuntz semigroup workshop
Münster, July 2021

Motivation and Aim

Motivation:

The following are *"arranged"* excerpts of selected papers in the literature

- "The *Cuntz semigroup* of a C*-algebra is introduced mainly as a *technical device* in order to prove the existence of dimension functions (and hence traces) on stably finite algebras."
- "*Traces*, together with K-theory, are among the most standard invariants in the study and classification of nuclear C*-algebras."
- "*Ultraproducts* have a well established presence in the field of Operator Algebras."

Aim:

We illustrate how to use Cuntz Semigroup techniques to answer a particular problem concerning traces in ultraproducts of C*-algebras.

Cuntz Semigroup and Traces

A **topological cone** is an abelian monoid equipped with a scalar $(0, \infty)$-multiplication, and with both operations jointly continuous.

Quasitraces on a C*-algebra A

$\mathrm{QT}(A) = \{$lsc 2-qtraces $\tau : A_+ \to [0, \infty]\}$
topology: $(\tau_j) \to \tau$ if $\forall a \in A_+, \forall \epsilon > 0$:

$$\limsup_j \tau_j((a-\epsilon)_+) \leq \tau(a) \leq \liminf_j \tau_j(a),$$

$QT(A)$ is a topological cone.

Functionals on a Cuntz semigroup S

$F(S) = \{\lambda \colon S \to [0, \infty];\ 0, \leq, \sup, +\}$
topology: $(\lambda_j) \to \lambda$ if $\forall x' \ll x \in S$:

$$\limsup_j \lambda_j(x') \leq \lambda(x) \leq \liminf_j \lambda_j(x),$$

$F(S)$ is a topological cone.

Theorem (Elliott,Robert,Santiago, 2011)

Let A be a C*-algebra. Then, $\mathrm{QT}(A)$, $\mathrm{F}(\mathrm{Cu}(A))$ are compact Hausdorff cones, and there is a homeomorphism

$$\mathrm{QT}(A) \xrightarrow{\cong} F(\mathrm{Cu}(A)); \quad \tau \mapsto d_\tau$$

where $d_\tau([a]) = \sup_n \tau(a^{1/n})$.

Traces on ultraproducts

- Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ be C*-algebras.

- $\prod_n A_n$ is the product C*-algebra (i.e. norm-bounded sequences).
- $c_{\mathcal{U}} := \{(a_n) \in \prod_n A_n : \lim_{\mathcal{U}} \|a_n\| = 0\}$, closed two-sided ideal.
- $\prod_{\mathcal{U}} A_n := \prod_n A_n / c_{\mathcal{U}}$ is the **ultraproduct C*-algebra** of A_n along $\mathcal{U}$. (**ultrapower** in case $A_n = A \; \forall n$)

- Let $\tau_n \in \mathrm{QT}(A_n) \; \forall n$.

- $\prod_n A_n \xrightarrow{\pi_j} A_j \xrightarrow{\tau_j} [0,\infty]$. $\bar{\tau}_j := \tau_j \pi_j \in \mathrm{QT}(\prod_n A_n)$ for all $j \in \mathbb{N}$.
- $\mathrm{QT}(\prod_n A_n)$ compact Hausdorff. $\implies$ $\bar{\tau}_{\mathcal{U}} := \lim_{j\to\mathcal{U}} \bar{\tau}_j \in \mathrm{QT}(\prod_n A_n)$
- Have $\bar{\tau}_{\mathcal{U}}((a_n)) = \sup_{\epsilon>0} \lim_{n\to\mathcal{U}} \tau_n((a_n-\epsilon)_+)$. $\bar{\tau}_{\mathcal{U}}(c_{\mathcal{U}}) = 0$.
- $\exists!\; \tau_{\mathcal{U}} \in \mathrm{QT}(\prod_{\mathcal{U}} A_n)$ such that $\tau_{\mathcal{U}} \pi_{\mathcal{U}} = \bar{\tau}_{\mathcal{U}}$.

- Denote the set of (quasi)traces obtained like this as $\prod_n \mathrm{QT}(A_n)/\mathcal{U}$.

Question

Under what conditions is $\prod_n \mathrm{QT}(A_n)/\mathcal{U}$ dense in $\mathrm{QT}(\prod_{\mathcal{U}} A_n)$?

Plan

How do we translate this problem to the language of Cuntz Semigroups?

$$\Pi_n \mathrm{QT}(A_n)/\mathcal{U} \subseteq \mathrm{QT}(\Pi_{\mathcal{U}} A_n) \qquad \text{dense?}$$

$$\Downarrow \qquad \mathrm{QT}() \simeq \mathrm{F}(\mathrm{Cu}())$$

$$\Pi_n \mathrm{F}(\mathrm{Cu}(A_n))/\mathcal{U} = K \subseteq F(\Pi_{\mathcal{U}} \mathrm{Cu}(A_n)) = F(\mathrm{Cu}(\Pi_{\mathcal{U}} A_n))$$

- Describe $\mathrm{Cu}(\prod_{\mathcal{U}} A_n)$ as an ultraproduct of $\mathrm{Cu}(A_n)$.
- Describe K, as in QT but with limit functionals. ✓
- Study density in cones of the form $\mathrm{F}(S)$, $S \in \mathrm{Cu}$. (in terms of properties of S).

Structure of the category Cu

Theorem (Antoine, Perera, Thiel)

The category Cu is a closed, symmetric monoidal, bicomplete category. The functor Cu, suitably interpreted, preserves products and ultraproducts.

- *Symmetric monoidal*: Cu has tensor products, with unit $(\mathbb{N} \sqcup \{\infty\})$, symmetry isomorphism.
- *Closed*: There is an adjoint to the tensor product (similar to, but not the, Hom functor).
- *Bicomplete*: Cu admits limits and colimits. In particular, products and coproducts (and ultraproducts).

Categorical ultraproducts: Given $(S_j)_{j\in J}$, and an ultrafilter $\mathcal{U}$, one defines

$$\prod_{\mathcal{U}} S_j = \lim_{F\in\mathcal{U}} \prod_F S_j.$$

Reflection, coreflection and ultraproducts in Cu

Limits and colimits are obtained through certain *reflector* and *coreflector* functors on simpler categories of ordered monoids.

- *Reflector* functor: equivalence classes of $\prec$-increasing sequences $(s_n)_n$ of elements.
- *Coreflector* functor: equivalence classes of *paths*:

C*-algebras

Given $a \in A_+$, we have

- $[a] = \sup_{\epsilon>0}[(a-\epsilon)_+]$ with
  $$[(a-\epsilon)_+] \ll [(a-\epsilon')_+] \text{ if } \epsilon' < \epsilon$$
- $[a] \leq [b]$ iff $\forall\epsilon\ \exists\delta$ s.t.
  $$(a-\epsilon)_+ \precsim (b-\delta)_+$$

Q semigroups

Given $(S, \prec)$ in Q, consider

- *paths* in S, $(s_\lambda)_{\lambda\in(-\infty,0)}$
  $$s_\lambda \prec s_{\lambda'} \text{ if } \lambda < \lambda'$$
- Set $(s_\lambda)_\lambda \precsim (t_\mu)_\mu$ if $\forall\lambda_0\ \exists\mu_0$ s.t.
  $$s_{\lambda_0} \prec t_{\mu_0}$$

Given $\mathcal{U}$ an ultrafilter on $\mathbb{N}$ and $S_n \in \mathrm{Cu}$,

$$\prod_{\mathcal{U}} S_n = \{(a_{n,t})_{n,t} \mid a_{n,t'} \ll a_{n,t} \in S_n,\ n\in\mathbb{N}, t' < t < 0\}_{/(\precsim,\succsim)}$$

where $(a_{n,t})_{n,t} \precsim (b_{n,t})_{n,t}$ if for all t there is t' such that $\{n\colon a_{n,t} \ll b_{n,t'}\} \in \mathcal{U}$

Theorem (APT)

If $(A_n)_{n\in\mathbb{N}}$ are stable C*-algebras, then $\mathrm{Cu}(\prod_n A_n) \cong \prod_n \mathrm{Cu}(A_n)$ and $\mathrm{Cu}(\prod_{\mathcal{U}} A_n) \cong \prod_{\mathcal{U}} \mathrm{Cu}(A_n)$.

What about non stable C*-algebras?
$\mathrm{Cu}(A) := W(A \otimes \mathcal{K})$ can not distinguish A from $A \otimes \mathcal{K}$, and yet

$$\prod_{\mathcal{U}} \mathbb{C} \simeq \mathbb{C}, \qquad \prod_{\mathcal{U}} \mathcal{K} \not\simeq \mathcal{K} \quad \text{e.g. not simple...}$$

$$\mathrm{Cu}(\mathbb{C}) = \mathrm{Cu}(\mathcal{K}) = \overline{\mathbb{N}}, \quad \prod_{\mathcal{U}} \overline{\mathbb{N}} = \overline{\mathbb{N}}^{\mathcal{U}} \supseteq \overline{\mathbb{N}} \quad \text{as an ideal.}$$

Scaled products

We consider Cu-semigroups S with a **scale** Σ, which is a sup-closed, downwards hereditary subset of S that generates S as an ideal:
for all $s' \ll s \in S$, exists d such that $s' \leq x \in \Sigma + \overset{d)}{\cdots} + \Sigma \qquad (\leftarrow \Sigma^{(d)})$.

The *scale* of A is $\Sigma_A = \{x \in \mathrm{Cu}(A) \colon \forall x' \ll x \, \exists a \in A_+ \text{ s.t. } x' \leq [a]\}$. $\quad \Sigma_A^{(d)} \simeq \Sigma_{M_d(A)}$.
We define the **scaled Cuntz semigroup** as

$$\mathrm{Cu}_{\mathrm{sc}}(A) = (\mathrm{Cu}(A), \Sigma_A).$$

Theorem (APT)

The category of scaled Cu-semigroups is bicomplete and we have:

$$\mathrm{Cu}_{\mathrm{sc}}(\textstyle\prod_n A_n) \cong \prod_n (\mathrm{Cu}(A_n), \Sigma_{A_n}) \text{ and } \mathrm{Cu}_{\mathrm{sc}}(\prod_{\mathcal{U}} A_n) \cong \prod_{\mathcal{U}} (\mathrm{Cu}(A_n), \Sigma_{A_n}).$$

Given $\mathcal{U}$ an ultrafilter on $\mathbb{N}$ and $(S_n, \Sigma_n) \in \mathrm{Cu}_{sc}$,

$$\prod_{\mathcal{U}} (S_n, \Sigma_n) = \{(a_{n,t})_{n,t} \mid a_{n,t'} \ll a_{n,t} \in \Sigma_n^{(d_t)},\ n \in \mathbb{N}, t' < t < 0\}_{/(\precsim, \succsim)}$$

where $(a_{n,t})_{n,t} \precsim (b_{n,t})_{n,t}$ if for all t there is t' such that $\{n \colon a_{n,t} \ll b_{n,t'}\} \in \mathcal{U}$

Plan

How do we translate this problem to the language of Cuntz Semigroups?

$$\Pi_n \mathrm{QT}(A_n)/\mathcal{U} \subseteq \mathrm{QT}(\Pi_{\mathcal{U}} A_n) \qquad \text{dense?}$$

$$\Downarrow \qquad \mathrm{QT}()\simeq \mathrm{F}(\mathrm{Cu}())$$

$$\Pi_n \mathrm{F}(\mathrm{Cu}(A_n))/\mathcal{U} \subseteq F(\Pi_{\mathcal{U}} \mathrm{Cu}(A_n))$$

- Describe $\mathrm{Cu}(\prod_{\mathcal{U}} A_n)$ as an ultraproduct of $\mathrm{Cu}(A_n)$. ✓
- Describe K, as in QT but with limit functionals. ✓
- Study density in cones of the form $\mathrm{F}(S)$, $S \in \mathrm{Cu}$. (in terms of properties of S).

Separation Results in $\mathrm{F}(S)$

Given $x \in S$, denote $\widehat{x}\colon \mathrm{F}(S) \longrightarrow [0,\infty]; \quad \lambda \mapsto \lambda(x)$
$(a \in (A \otimes \mathcal{K})_+, \ \widehat{[a]}\colon \mathrm{QT}(A) \longrightarrow [0,\infty]; \quad \tau \mapsto d_\tau([a]))$

Theorem (Antoine, Perera, Robert, Thiel)

- Let S be a Cu-semigroup, satisfying (O5), and $K \subseteq F(S)$ a closed (~~closed~~) subcone containing 0. Then $K = \mathrm{F}(S)$ (K is dense) if and only if:

$$\forall x, y \in S, \qquad \widehat{x}|_K \leq \widehat{y}|_K \implies \widehat{x} \leq \widehat{y}.$$

$$\forall x' \ll x, \quad y' \ll y \in S, \quad \gamma < 1, \qquad \widehat{x}|_K \leq \gamma\widehat{y'}|_K \implies \widehat{x'} \leq \widehat{y}.$$

- Let S be a Cu-semigroup, satisfying (O5), (O6), and the *Edwards' condition*, and $K \subseteq F(S)$ a closed (~~closed~~) subcone containing 0.
Then $K = \mathrm{F}(S)$ (K is dense) if $\exists\, M > 0$ such that:

$$\forall x, y \in S, \qquad \widehat{x}|_K \leq \widehat{y}|_K \implies \widehat{x} \leq M\widehat{y}.$$

$$\forall x, \quad y' \ll y \in S, \qquad \widehat{x}|_K \leq \widehat{y'}|_K \implies \widehat{x} \leq M\widehat{y}.$$

$S = \mathrm{Cu}(A)$ satisfies (O5), (O6) and the Edward's condition.

Comparability conditions

S a Cu-semigroup and $x, y \in S$, $\gamma < 1$:

$$\widehat{x} \leq \gamma \widehat{y} \quad \Rightarrow \quad \forall x' \ll x \quad (n+1)x' \leq ny \quad (n = n(x'))$$

$$\underset{(S \text{ almost unperf.})}{\Rightarrow} \quad x \leq y$$

(S, Σ) a scaled Cu-semigroup. Consider the following condition:

(*) $\forall \gamma < 1$, and $d \in \mathbb{N}$, there exists N such that:

$$\widehat{x} \leq \gamma \widehat{y} \implies Nx \leq Ny \quad \text{for all } x, y \in \Sigma^{(d)}.$$

$$d_\tau([a]) \leq \gamma d_\tau([b]) \quad \forall \tau \in \mathrm{QT}(A) \implies N[a] \leq N[b], \qquad \text{for all } a, b \in M_d(A)_+.$$

Characterization of density

Theorem (APRT):

Let A be a C*-algebra. The following conditions are equivalent:

(i) The set $\prod \mathrm{QT}(A)/\mathcal{U}$ is dense in $\mathrm{QT}(\prod_{\mathcal{U}} A)$.

(ii) $\forall \gamma < 1,\ d \in \mathbb{N}$, $\exists N$ such that

$$d_\tau([a]) \leq \gamma d_\tau([b])\ \forall \tau \in \mathrm{QT}(A) \implies N[a] \leq N[b]$$

for all positive $a, b \in M_d(A)$.

(iii) $\exists M \in \mathbb{N}$ such that $\forall d \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that

$$d_\tau([a]) \leq d_\tau([b])\ \forall \tau \in \mathrm{QT}(A) \implies k[a] \leq kM[b]\ \forall k \geq N$$

for all positive $a, b \in M_d(A)$.

Some particular cases (stable, unital)

Corollary:

If A is stable, then the following are equivalent:

1. (i) The set $\prod \mathrm{QT}(A)/\mathcal{U}$ is dense in $\mathrm{QT}(\prod_{\mathcal{U}} A)$.
2. (ii) $\exists M$ such that $d_\tau([a]) \leq d_\tau([b])\ \forall \tau \in \mathrm{QT}(A) \implies [a] \leq M[b]$ for all $a, b \in A_+$.

Theorem:

If A is unital, then the following are equivalent:

1. (i) The set $\prod \mathrm{QT}_1(A)/\mathcal{U}$ is dense in $\mathrm{QT}_1(\prod_{\mathcal{U}} A)$.
2. (ii) $\forall \gamma < 1$, $d \in \mathbb{N}$, $\exists N$ s.t. if $[a], [b] \in \mathrm{Cu}(A)$ with $[a], [b] \leq d[1]$, $[b]$ full:

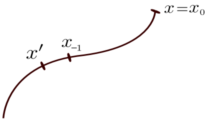
$$d_\tau([a]) \leq \gamma d_\tau([b])\ \forall \tau \implies N[a] \leq N[b]$$

Proof (sketch):

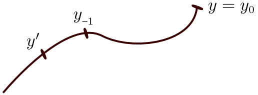
•$((*) \implies$ density) Need to show:

$$\begin{array}{l} x' \ll x \\ y' \ll y \end{array} \qquad \widehat{x}|_K \leq \gamma \widehat{y'}|_K \overset{?}{\implies} \widehat{x'} \leq \widehat{y}$$

$x = [(v_{t,j})_{t,j}]$



$y = [(w_{t,j})_{t,j}]$



$$\gamma' < \gamma, \qquad \{j \in \mathbb{N} : \widehat{v_{-1,j}} \leq \gamma' \widehat{w_{-1,j}}\} \overset{?}{\in} \mathcal{U}.$$

if not $\rightsquigarrow$ $\lambda_j \in \mathrm{QT}(A)$ (some $\lambda_j = 0$) $\rightsquigarrow$ $\lambda_{\mathcal{U}} \in K$ s.t. $\lambda_{\mathcal{U}}(x) \not\leq \gamma \lambda_{\mathcal{U}}(y)$!!
$\exists\, d,\ v_{-1,j}, w_{-1,j} \in \Sigma^{(d)}$. (hypothesis) $\rightsquigarrow$, $Nx' \leq Ny \Rightarrow \widehat{x'} \leq \widehat{y}$ $\quad\checkmark$

Proof (sketch):

- (density $\implies$ (*))

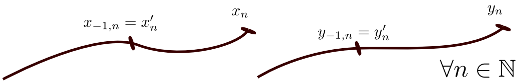
$(*) \sim (**):\quad \forall \gamma < 1, d \in \mathbb{N}, \quad \exists N$ s.t. $\widehat{x} \leq \gamma \widehat{y'} \Rightarrow (N+1)x \leq Ny$ in $\Sigma^{(d)}$.

Suppose not.

$\exists \gamma < 1,\ d \in \mathbb{N},$ and $x'_n, x_n, y'_n, y_n \in \Sigma^{(d)}$ $n = 1, 2, 3 \ldots$ s.t

$\widehat{x_n} \leq \gamma \widehat{y'_n}$ and $(n+1)x'_n \not\leq n y_n$

$\rightsquigarrow$ Can build paths $x = [(x_{n,t})_{n,t}]$ and $y = [(y_{n,t})_{n,t}]$ such that



we can build $x, y \in \Pi_{\mathcal{U}}(S, \Sigma)$, since $x'_n, y'_n \in \Sigma^{(d)}$ $\forall n$.

$\rightsquigarrow$ for all $\lambda \in K$, $\lambda(x) \leq \gamma \lambda(y_{-1})$. $\rightsquigarrow$ (density) $\widehat{x} \leq \gamma \widehat{y}$.
Given $x_{-1} \ll x$, $\exists\, N$ s.t. $(N+1)x_{-1} \leq Ny$ $\rightsquigarrow$ for some large enough n!! $\square$

Application. Pure C*-algebras

Definition (Winter 2012):

A C*-algebra A is (m,n)-*pure* if $\mathrm{Cu}(A)$ has

- m-comparison: $(k_i+1)x \leq k_i y_i$ for $i=0,\ldots,m \implies x \leq \sum_{i=0}^m y_i$.
- n-almost divisible: given $x' \ll x$, $k \in \mathbb{N}$, there is y such that $ky \leq x$, $x' \leq (k+1)(n+1)y$.

A is *pure* if it is $(0,0)$-pure.

▸ Thm.

Theorem (APRT):

If A is simple and (m,n)-pure, then A is pure.

Thank You!