The Cuntz semigroup and crossed products by weakly tracially approximately representable actions

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Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A.

1. The crossed product, denoted $C^*(G, A, \alpha)$, is the universal unital C*-algebra generated by a copy of A and a unitary representation $g \mapsto u_g$ of G which implements the action, in the sense that $u_g a u_g^* = \alpha_g(a)$ for all $g \in G$ and $a \in A$.
2. If G is abelian, the dual action $\widehat{\alpha}\colon \widehat{G} \to \mathrm{Aut}(C^*(G, A, \alpha))$ is defined by
$$\widehat{\alpha}_\sigma(a u_g) = \overline{\sigma(g)} a u_g.$$

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???: Weak Tracial Approximate Representability

Definition (Izumi 2004)

Let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. The action α has the Rokhlin property if, for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Example

Let A be a unital C*-algebra. Let $\alpha \colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A \oplus A)$ be the flip action. Then α has the Rokhlin property and Rokhlin projections are $(1_A, 0)$ and $(0, 1_A)$.

Definition (Izumi, 2004)

Let A be a unital C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. We say that α is approximately representable if, for every finite set $F \subset A$ and every $\varepsilon > 0$, there are $z_g \in \mathrm{U}(A)$ for $g \in G$ such that:

1. $\|\alpha_g(a) - z_g a z_g^*\| < \varepsilon$ for all $g \in G$ and $a \in F$.
2. $\|z_g z_h - z_{gh}\| < \varepsilon$ for all $g, h \in G$.
3. $\|\alpha_g(z_h) - z_{ghg^{-1}}\| < \varepsilon$ for all $g, h \in G$.

Example

Let G be a finite (nontrivial) group, let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be the trivial action. Then:

1. α is approximately representable.
2. α doesn't have the Rokhlin property.

Example

Let α be the action of $\mathbb{Z}/2\mathbb{Z}$ on the 2^∞ UHF algebra generated by $\bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Then:

1. α has the Rokhlin property.
2. α is approximately representable.

Example

Let A and B be unital C*-algebras. Let $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A \oplus A \oplus B)$ given by $\alpha(a, b, c) = (b, a, c)$.

1. α does not have the Rokhlin property.
2. α is not approximately representable.

Theorem (Izumi, 2004)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of finite abelian group G on a unital C*-algebra A. Then:

1. α has the Rokhlin property if and only if $\widehat{\alpha}$ is approximately representable.
2. α is approximately representable if and only if $\widehat{\alpha}$ dual has the Rokhlin property.

In 2015, Kodaka and Teruya introduced and studied the Rokhlin property and approximate representability in the setting of finite quantum group actions.

Using a different definition, the work of Izumi was generalized to coexact compact quantum groups by Barlak, Szabo, and Voigt in 2017.

The right version of the tracial Rokhlin property for actions of non-finite compact groups has not been defined yet.

Actions with the Rokhlin property are rare and many C*-algebras admit no finite group actions with the Rokhlin property.

Example

There is no action of any nontrivial finite group on $\mathcal{O}_\infty$ which has the Rokhlin property.

In 2011, Phillips defined the tracial analog of Rokhlin property under the name "tracial Rokhlin property".

Definition (Phillips 2011)

Let G be a finite group, let A be an infinite-dimensional simple unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. We say that α has the tracial Rokhlin property if for every $\varepsilon > 0$, every finite set $F \subseteq A$, and every positive element $x \in A$ with $\|x\| = 1$, there exist mutually orthogonal projections $e_g \in G$ in A for $g \in G$ such that, with $e = \sum_{g \in G} e_g$, the following hold:

1. $\|ae_g - e_g a\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
2. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
3. $1 - e \precsim_A x$.
4. $\|exe\| > 1 - \varepsilon$.

Definition (Phillips 2011)

Let A be an infinite-dimensional simple unital C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α is tracially approximately representable if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are a projection $e \in A$ and unitaries $w_g \in \mathrm{U}(eAe)$ for $g \in G$ such that:

1. $\|ea - ae\| < \varepsilon$ for all $a \in F$.
2. $\|\alpha_g(eae) - w_g eaew_g^*\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
3. $\|w_g w_h - w_{gh}\| < \varepsilon$ for all $g, h \in G$.
4. $\|\alpha_g(w_h) - w_{ghg^{-1}}\| < \varepsilon$ for all $g, h \in G$.
5. $1 - e \precsim_A x$.
6. $\|exe\| > 1 - \varepsilon$.

Theorem (Phillips 2011)

Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite abelian group G on A such that $C^*(G, A, \alpha)$ is also simple. Then:

1. α has the tracial Rokhlin property if and only if $\widehat{\alpha}$ is tracially approximately representable.
2. α is tracially approximately representable if and only if $\widehat{\alpha}$ has the tracial Rokhlin property.

The issue which is related to existence of projections is not solved even by the tracial Rokhlin property. So, we need to study a weaker versions of the tracial Rokhlin property.

One can generalize the definition of tracial Rokhlin property to weak tracial Rokhlin property. The idea is to replace Rokhlin projections by positive contractions.

Definition (Motivated by Hirshberg-Orovitz 2013)

Let G be a finite group, let A be a simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. We say that α has the weak tracial Rokhlin property if for every $\varepsilon > 0$, every finite set $F \subseteq A$, and every positive element $x \in A$ with $\|x\| = 1$, there exist orthogonal positive contractions $f_g \in G$ in A for $g \in G$ such that, with $f = \sum_{g \in G} f_g$, the following hold:

1. $\|af_g - f_g a\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
2. $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$.
3. $1 - f \precsim_A x$.
4. $\|fxf\| > 1 - \varepsilon$.

Example

Let $\mathcal{Z}$ be the Jiang-Su algebra and let $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathcal{Z} \otimes \mathcal{Z})$ be the flip action. Then α has the weak tracial Rokhlin property but it does not have the tracial Rokhlin property.

Definition (A. 2021)

Let A be an infinite-dimensional simple unital C*-algebra. An action $\alpha\colon G \to \mathrm{Aut}(A)$ of a finite group G on A is weakly tracially approximately representable if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are $c \in (A^{\alpha})_{+}$ with $\|c\| \leq 1$ and contractive elements $s_g \in \overline{cAc}$ for $g \in G$ such that:

1. $\|s_1 - c\| < \varepsilon$ and $\|s_g^* - s_{g^{-1}}\| < \varepsilon$ for all $g \in G$.
2. $\|s_g s_h - c s_{gh}\| < \varepsilon$ for all $g, h \in G$.
3. $\|ca - ac\| < \varepsilon$ for all $a \in F \cup \{s_g \colon g \in G\}$.
4. $\|\alpha_g(cac) - s_g a s_g^*\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
5. $\|\alpha_g(s_h) - s_{ghg^{-1}}\| < \varepsilon$ for all $g, h \in G$.
6. $(1 - c - \varepsilon)_{+} \precsim_A x$.
7. $\|cxc\| > 1 - \varepsilon$.

Theorem (A. 2021)

Let A be an infinite-dimensional simple unital C*-algebra and let G be a finite abelian group. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A which is pointwise outer. Then:

1. α has the weak tracial Rokhlin property if and only if $\widehat{\alpha}$ is weakly tracially approximately representable.
2. α is weakly tracially approximately representable if and only if $\widehat{\alpha}$ has the weak tracial Rokhlin property.

Proof of a special case.

α has the Rokhlin property $\Rightarrow \widehat{\alpha}$ is approximately representable. Let $\varepsilon > 0$ and let F be a finite set in $C^*(G, A, \alpha)$. Without loss of generality $F = F_0 \cup \{u_g : g \in G\}$. Applying th definition of Rokhlin property with F_0 and ε' related to ε, we can find mutually orthogonal projections $p_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|p_g a - a p_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} p_g = 1$.

For every $\tau \in \widehat{G}$, define $w_\tau = \sum_{g \in G} \tau(g) p_g$.
Now one can show that:

4. $w_\tau w_\tau^* = w_\tau^* w_\tau = 1$.
5. $\|\widehat{\alpha}_\tau(x) - w_\tau x w_\tau^*\| < \varepsilon$ for all $\tau \in \widehat{G}$ and $x \in F$.
6. $\|w_\tau w_\sigma - w_{\tau\sigma}\| < \varepsilon$ for all $\tau, \sigma \in \widehat{G}$.
7. $\|\widehat{\alpha}_\tau(w_\sigma) - w_\sigma\| < \varepsilon$ for all $\tau, \sigma \in \widehat{G}$.

Proof of a special case (Continued).

α is approximately representable $\Rightarrow \widehat{\alpha}$ has the Rokhlin property.
Let $\varepsilon > 0$ and let F be a finite set in $C^*(G, A, \alpha)$. Without loss of generality $F = F_0 \cup \{u_g : g \in G\}$. Applying th definition of approximate representability with F_0 and ε' related to ε, we can find $z_g \in \mathrm{U}(A)$ for $g \in G$ such that:

1. $\|\alpha_g(a) - z_g a z_g^*\| < \varepsilon'$ for all $g \in G$ and $a \in F$.
2. $\|z_g z_h - z_{gh}\| < \varepsilon'$ for all $g, h \in G$.
3. $\|\alpha_g(z_h) - z_h\| < \varepsilon'$ for all $g, h \in G$.

For every $\tau \in \widehat{G}$, define

$$p_\tau = \frac{1}{\mathrm{card}(G)} \sum_{g \in G} \tau(g) u_g z_g^*.$$

Proof of a special case (Continued).

Now one can show that:

4. $\|p_\tau - p_\tau^*\| < \varepsilon$ for all $\tau \in \widehat{G}$.
5. $\|p_\tau^2 - p_\tau\| < \varepsilon$ for all $\tau \in \widehat{G}$.
6. $\|p_\sigma p_\tau\| < \varepsilon$ for all $\sigma, \tau \in \widehat{G}$ with $\sigma \neq \tau$.
7. $\|\widehat{\alpha}_\sigma(p_\tau) - p_{\sigma\tau}\| < \varepsilon$ for all $\sigma, \tau \in \widehat{G}$.
8. $\left\|\sum_{\tau \in \widehat{G}} p_\tau - 1\right\| < \varepsilon$.

A slight weakening of approximate representability is called weak tracial strict approximate innerness.

Question

Does there exist a weakly tracially strictly approximately inner action of a nontrivial finite group on a unital C*-algebra which is not weakly tracially approximately representable?

Theorem (A. 21)

Let A be an infinite-dimensional simple unital C*-algebra and let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of a finite group G on A which is weakly tracially strictly approximately inner. Then for every finite set $F \subset C^*(G, A, \alpha)$, every $\varepsilon > 0$, and every $a \in A_+$ with $\|a\| = 1$, there are $c \in (A^\alpha)_+$ with $\|c\| \leq 1$ and a surjective linear map $\psi\colon \overline{cC^*(G, A, \alpha)c} \to \overline{cAc}$ such that:

1. $\|\psi\| \leq \operatorname{card}(G)$.
2. $\left\|\psi(cbc) - c^{3/2}bc^{3/2}\right\| < \varepsilon$ and $\left\|\psi(c^4bc^4) - c^{9/2}bc^{9/2}\right\| < \varepsilon$ for all $b \in A$.
3. $\left\|\psi(c^4xyz^*c^4) - \psi(cxc)\psi(cyc)\psi(czc)^*\right\| < \varepsilon$ for all $x, y, z \in F$.
4. $(1 - c - \varepsilon)_+ \precsim_A a$.
5. $\|cac\| > 1 - \varepsilon$.

Theorem (A. 21)

Let A be an infinite-dimensional simple unital C*-algebra, let $a, b \in A_+$, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which is weakly tracially strictly approximately inner. Assume 0 is a limit point of $\mathrm{sp}(b)$. Then $a \precsim_{C^*(G,A,\alpha)} b$ if and only if $a \precsim_A b$.

If α is approximately representable, the assumption "0 is a limit point of $\mathrm{sp}(b)$" is not necessary and it was shown in my Ph.D. dissertation. The problem occurs only with tracial versions of approximate representability.

Notation

Let A be a stably finite exact unital C*-algebra. For every $\tau \in \mathrm{T}(A)$ and every $a \in \bigcup_{k=1}^{\infty} M_k(A)_+$, set

$$d_\tau(a) = \lim_{n \to \infty} \tau(a^{\frac{1}{n}}).$$

Definition (Toms 2006)

Let A be a stably finite exact unital C*-algebra.

1. For $r \in [0, \infty)$, A has *r-comparison* if whenever $a, b \in \bigcup_{k=1}^{\infty} M_k(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim_A b$.
2. The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\big(\big\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\big\}\big).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

Notation

Let A be a C*-algebra. We define

$$A_{++} = \big\{ a \in A_{+} \colon \text{there is no projection } q \in M_{\infty}(A) \text{ such that } \langle a \rangle_A = \langle q \rangle_A \big\}.$$

and

$$\mathrm{Cu}_{+}(A) = \big\{ \langle a \rangle_A \colon a \in (\mathcal{K} \otimes A)_{++} \big\}.$$

The elements of A_{++} are called *purely positive*.

Remark

If A is a stably finite simple unital C*-algebra, then

$$(\mathcal{K} \otimes A)_{++} = \big\{ a \in (\mathcal{K} \otimes A)_{+} \colon 0 \text{ is a limit point of } \mathrm{sp}(a) \big\},$$

and $\mathrm{Cu}_{+}(A) \cup \{0\}$ is a unital subsemigroup of $\mathrm{Cu}(A)$.

Theorem (A. 21)

Let A be a unital stably finite simple C*-algebra which is not of type I and let G be a finite group. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action which is weakly tracially strictly approximately inner. Let $\iota\colon A \to C^*(G, A, \alpha)$ be the inclusion map. Then the map

$$\mathrm{Cu}(\iota)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(C^*(G, A, \alpha))$$

induces an isomorphism of ordered semigroups from $\mathrm{Cu}_+(A) \cup \{0\}$ to its image in $\mathrm{Cu}(C^*(G, A, \alpha))$.

Theorem (A. 2021)

Let A be an infinite-dimensional simple unital stably finite C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a weakly tracially strictly approximately inner action of a finite group G on A. Then:

1. $\mathrm{rc}(A) \leq \mathrm{rc}\big(C^*(G, A, \alpha)\big)$.
2. If $C^*(G, A, \alpha)$ is simple, then $\mathrm{rc}(A) \leq \mathrm{rc}\big(C^*(G, A, \alpha)\big) \leq \mathrm{rc}(A^{\alpha})$.

Proof of a special case.

Suppose that α is approximately representable. Assume $\mathrm{rc}\big(C^*(G, A, \alpha)\big) < \infty$. Let $r \in [0, \infty)$. Suppose that $C^*(G, A, \alpha)$ has r-comparison. Let $a, b \in M_\infty(A)_+$ satisfy

$$d_\rho(a) + r < d_\rho(b)$$

for all $\rho \in \mathrm{QT}(A)$. Since every quasitrace on $C^*(G, A, \alpha)$ restricts to a quasitrace on A, it follows that

$$d_\rho(a) + r < d_\rho(b)$$

for all $\rho \in \mathrm{QT}(C^*(G, A, \alpha))$. Now, since $C^*(G, A, \alpha)$ has r-comparison, we get $a \precsim_{C^*(G,A,\alpha)} b$. Then $a \precsim_A b$. Therefore $\mathrm{rc}(A) \leq r$. Taking the infimum over $r \in [0, \infty)$ such that $C^*(G, A, \alpha)$ has r-comparison, we get

$$\mathrm{rc}(A) \leq \mathrm{rc}\big(C^*(G, A, \alpha)\big).$$

■

Question

Let G be a finite group, let A be an infinite-dimensional stably finite simple unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a weakly tracially approximately inner action. Does it follow that

$$\mathrm{rc}(A) = \mathrm{rc}\big(C^*(G, A, \alpha)\big)?$$

Proposition (A. 2021)

Let A be an infinite-dimensional simple unital stably finite C*-algebra with $0 < \mathrm{rc}(A) < \infty$. Then there is no action of any nontrivial finite group on A which both has the weak tracial Rokhlin property and is weakly tracially approximately representable.

The above proposition is true in the more general setting of finite groups by the work of Hirshberg, but our method is different and based on the radius of comparison of the crossed product.

Definition (Bosa, Perera, Wu, and Zacharias, in preparation)

Let A be a C*-algebra, let G be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. Let $a, b \in M_\infty(A)_+$. We say that a is *dynamically Cuntz subequivalent* to b, written $a \precsim_{A,G} b$ or $a \precsim_\alpha b$, if $a \precsim_A b$ or for any $\varepsilon > 0$, there are $m, n_1, n_2, \ldots, n_m \in \mathbb{Z}_{>0}$ and $g_{j,k} \in G$ and $x_{j,k} \in M_\infty(A)_+$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n_j$, such that

$$(a-\varepsilon)_+ \ll_A \bigoplus_{k=1}^{n_1} x_{1,k}, \quad \bigoplus_{k=1}^{n_1} \alpha_{g_{1,k}}(x_{1,k}) \ll_A \bigoplus_{k=1}^{n_2} x_{2,k},$$

$$\bigoplus_{k=1}^{n_2} \alpha_{g_{2,k}}(x_{2,k}) \ll_A \bigoplus_{k=1}^{n_3} x_{3,k}, \ldots,$$

$$\bigoplus_{k=1}^{n_{m-1}} \alpha_{g_{m-1,k}}(x_{m-1,k}) \ll_A \bigoplus_{k=1}^{n_m} x_{m,k}, \quad \bigoplus_{k=1}^{n_m} \alpha_{g_{m,k}}(x_{m,k}) \ll_A b.$$

We further say that $a, b \in M_\infty(A)_+$ are *dynamically Cuntz equivalent*, written $a \sim_{A,G} b$ or $a \sim_\alpha b$, if $a \precsim_\alpha b$ and $b \precsim_\alpha a$.

Definition (A.-Phillips, in preparation)

Let A be a stably finite unital C*-algebra. Let G be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A.

1. Let $r \in [0, \infty)$. We say that A has *dynamical r-comparison* if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)^G$, then $a \precsim_{A,G} b$.
2. The *dynamical radius of comparison* of α (or of A when α is understood), denoted $\mathrm{rc}_\alpha(A)$ or $\mathrm{rc}_G(A)$, is

$$\mathrm{rc}_\alpha(A) = \inf\big(\big\{r \in [0, \infty)\colon A \text{ has dynamical } r\text{-comparison}\big\}\big)$$

   if it exists, and ∞ otherwise.

Theorem (A.-Phillips, in preparation)

Let A be an infinite dimensional simple unital C*-algebra, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A which is weekly tracially strictly approximately inner. Then $\mathrm{rc}_\alpha(A) = \mathrm{rc}(A)$.

Thank you for your attention!