Workshop Cuntz Semigroups 2021

University of Münster, July 12-16, 2021.

Titles and Abstracts

Ramon Antoine: *Traces and the Cuntz semigroup of ultraproduct C*-algebras.*

Abstract: We describe the computation of the Cuntz semigroup of ultra products C*-algebras through ultra products in the category Cu, and use these techniques to analyse the density of limit traces on ultapowers of C*-algebras. From the characterization obtained we prove that every simple C*-algebra that is (m, n)-pure in the sense of Winter is already pure.

This is joint work with F. Perera, L. Robert and H. Thiel.

M. Ali Asadi-Vasfi: *The Cuntz semigroup and crossed products by weakly tracially approximately representable actions.*

Abstract: The Cuntz semigroup is generally complicated, large, and difficult to compute or describe in any concrete terms. For example, the Cuntz semigroup of $C(X)$ is a wild beast even if X is a contractible metric space.

Explicit computations of the Cuntz semigroups of crossed products by group actions on C*-algebras are often quite interesting and depend on the technical features of the actions. One of the important classes of group actions is "approximately representable actions". The notion of approximate representability for finite group actions on C*-algebras was introduced by Izumi in 2004 and the tracial analog of approximate representability under the name "tracial approximate representability" was introduced by Phillips in 2011. In this talk, we define a "weak tracial" analog of the approximate representability for finite group actions and describe the Cuntz semigroups of the crossed products by this class of finite group actions.

Laurent Cantier: *The* Cu_1 *semigroup and the classification of non-simple C*-algebras.*

Abstract: The Cuntz semigroup has been playing an important role in the classification of C*-algebras and more particularly when considering the non-simple setting. For instance, it has been proved that the functor Cu is a complete invariant for unital C*algebras that can be written as inductive limits of 1-dimensional non-commutative CW complexes and that have a trivial K_1-group. The absence of K_1 information is a main drawback of this semigroup, built from classes of positive elements under a suitable equivalence relation.

First, I will briefly introduce a unitary version of the Cuntz semigroup, termed the unitary Cuntz semigroup and written Cu_1, that tends to remedy the lack of K_1 information in the classical Cuntz semigroup. After exposing some properties satisfied by this unitary version of the Cuntz semigroup, I will illustrate that a likewise invariant is needed in order to go beyond the trivial K_1 hypothesis in classification theorems by means of Cuntz semigroups. To do so, I will first define a notion of uniformly PoM-based Cuntz semigroup that will lead to an approximate intertwining theorem for Cu-semigroups. Then, I will construct two C*-algebras, whose K-Theory and Cu-semigroup are isomorphic, and that are nonetheless distinguished by their unitary Cuntz semigroup.

George A. Elliott: *Why are well-behaved C*-algebras so well behaved?*

Abstract: Well-behaved simple C*-algebras are classified (up to stable isomorphism) by a very simple invariant, which makes sense for an arbitrary C*-algebra—the even and odd Banach algebra K-groups, considered as abelian groups, together with the topological cone of densely defined lower semicontinuous traces, considered as functionals on the Pedersen ideal (with pointwise convergence), by cyclic homology paired with the even K-group of this ideal, which is the same as that of the whole algebra.

For a separable simple C*-algebra, the K-groups are countable and the cone is metrizable, with a compact base, which is a Choquet simplex (not unique—different bases are what might be called stably isomorphic). This system, consisting of two countable abelian groups, together with a metrizable simplex cone paired with one of them, is arbitrary.

The axiomatically specified class which has been completely classified in terms of this invariant (with minor additional structure in the non-stable case), namely, simple separable amenable (= nuclear) C*-algebras which are Jiang-Su stable (equivalently, have finite nuclear dimension) and (redundant?) satisfy the Universal Coefficient Theorem (UCT), also exhausts the possibilities for this invariant. (In particular, this invariant could not classify any larger class of separable simple C*-algebras.)

The invariant for an arbitrary simple C*-algebra, formulated in this bare-bones way, is not changed by tensoring with the Jiang-Su C*-algebra, sometimes called the non-type-I complex numbers—any more than it is by tensoring with the compact operators. This, together with the fact that the Jiang-Su algebra is itself Jiang-Su stable, gives another proof, independent of the invariant's being exhausted by the classifiable class mentioned, that this class is maximal—any larger class (in this case of simple UCT separable amenable C*-algebras, so only violating Jiang-Su stability) will (obviously) require the introduction of a finer invariant.

As it happens, there are lots of simple separable amenable C*-algebras which do not belong to this class, for example, both classes of simple C*-algebras constructed by Villadsen, as follows from consequences of Jiang-Su stability deduced by Gong-Jiang-Su and by Rordam. It is certainly a question to confront these non-well-behaved C*-algebras!

But, sticking with this classifiable class, and this almost ridiculously simple invariant, questions remain to be answered. For instance, why is the classification so simple?

Here is one point to bear in mind, perhaps. Certainly, one does not expect a simple classification of amenable groups. But, since the C*-algebra of an amenable group is not simple, there is no danger of having to consider just a single simple C*-algebra. One might want to consider the whole family of simple quotients of this C*-algebra. (Are these classifiable?) (The commutative case shows that these will not be enough.) At least, for a non-amenable group, the left regular representation C*-algebra can be simple, and since that is not amenable, and so not subject to a simple-minded classification, there could be some chance that that algebra (or, as in the Connes rigidity conjecture, even just the associated von Neumann algebra) determines the group.

Certainly, the C*-algebras in question have a deep structure, and it is not known for instance if the classification functor has a one-sided inverse. Already the CAR algebra is famously rich, as its name indicates. One consequence of the classification theorem (applied to the 1981 construction of Bratteli and Herman and me) is that

this algebra has a one-parameter group of automorphisms with an arbitrary KMS spectrum.

Ilan Hirshberg: *Circle actions on classifiable C*-algebras.*

Abstract: I will outline work in progress concerning a construction of an example of an action of the circle group on a simple, separable, unital, nuclear, stably finite and $\mathcal{Z}$-stable C*-algebra such that the crossed product is simple but not $\mathcal{Z}$-stable. This shows that the class of C*-algebras which satisfy the Toms-Winter regularity properties is not closed under crossed products by actions of amenable groups.

Bhishan Jacelon: *Metrics on trace spaces.*

Abstract: In many settings, the tracial state space of a separable C*-algebra A is naturally equipped with a metric. (For example, if A is a point-line algebra, or the algebra $C(X)$ of continuous functions on a metric space X, then traces are described as Borel probability measures and it is natural to consider the Wasserstein distances between them.) The corresponding "Lipschitz elements" of A then provide families with respect to which the Cuntz and unitary distances in $\mathrm{Hom}(A, B)$, for classifiable, stably finite C*-algebras B, can be measured and compared to the uniform tracial distance. I will describe how the notion of "continuous transport" (developed in joint work with Karen Strung and Alessandro Vignati) can be used to show that, in many cases, these distances are in fact equal.

Francesc Perera: *General quotient Cuntz semigroups and dynamical Cuntz semigroups.*

Abstract: We introduce a notion of normal preorders, which allows us to take quotients of W-semigroups (beyond the case of quotients by ideals) and is closely related to W-morphisms in a similar way as normal subgroups arise as kernels of group homomorphisms. We also study how to generate normal preorders from arbitrary relations that satisfy a continuity condition. In particular, it also allows us to construct quotients from group actions on W-semigroups, which leads to the notion of dynamical Cuntz semigroups.

This talk is based on joint work with Joan Bosa, Jianchao Wu, and Joachim Zacharias.

N. Christopher Phillips: *Lower bounds on the radius of comparison of crossed products by countable amenable groups.*

Abstract: We introduce "mean cohomological independence dimension" for actions of countable amenable groups on compact metric spaces, as a variant of mean dimension, and use it to obtain lower bounds for the radius of comparison of the associated crossed product C*-algebras. We hope this version of mean dimension is more computable in some cases. In some examples, including ones with arbitrarily large mean dimension and for arbitrarily countable amenable groups, our lower bound is close to half the mean dimension of the action.

This is joint work with Ilan Hirshberg.

Leonel Robert: *Simplicity, bounded normal generation, and automatic continuity of groups of unitaries.*

Abstract: I'll discuss results obtained jointly with Abhinav Chand on a number of properties of the special unitary group of a C*-algebra. Under standard regularity conditions, this group has local bounded normal generation, a unique polish group topology, and satisfies the invariant automatic continuity. These results are analogous to results by Dowerk and Thom in the setting of von Neumann algebra factors.

Aaron Tikuisis: *Tracially complete C*-algebras.*

Abstract: Lurking in many recent developments in the structure of C*-algebras is the notion of the (uniform) tracial completion of a C*-algebra. When this idea was introduced by Ozawa, he also defined W*-bundles, an abstract structural framework for these objects, albeit only in the case where the traces form a Bauer simplex (i.e., the extreme boundary is compact). In joint work with many people (Castillejos, Carrión, Evington, Gabe, Schafhauser, and White), we have set out an abstract framework for the tracial completions of C*-algebras, going beyond the Bauer setting. These objects interestingly interpolate between C*-algebras and tracial von Neumann algebras. It is a new field with some results and many open problems; I will discuss these.

Eduard Vilalta: *Covering dimension of Cuntz semigroups.*

Abstract: In this talk I will introduce a notion of covering dimension for Cuntz semigroups of C*-algebras and their abstract counterpart, Cu-semigroups. This dimension satisfies the expected permanence properties, and in the case of Cuntz semigroups it is always bounded by the nuclear dimension of the associated C*-algebra. For subhomogeneous C*-algebras, both dimensions agree.

Among other results, I will show that the Cuntz semigroups of $\mathcal{W}$-stable C*-algebras have dimension zero, that $\mathcal{Z}$-stable C*-algebras have dimension at most one, and that the Cuntz semigroup of a separable, simple, $\mathcal{Z}$-stable C*-algebra is zero-dimensional if and only if the algebra has real rank zero or if it is stably projectionless.

The talk is on joint work with Hannes Thiel.

Wilhelm Winter: *Stable rank one and Cuntz semigroup regularity.*

Abstract: A unital C*-algebra has stable rank one if its group of invertible elements is norm dense. While this plays an important role for many structural results especially for simple C*-algebras, its interplay with other regularity properties is subtle and somewhat mysterious. Thiel has shown that if a simple unital C*-algebra has stable rank one and strict comparison of positive elements, then its Cuntz semigroup is almost divisible. In this talk I will describe a partial converse: strict comparison and almost divisibility together imply stable rank one.

This is joint work in progress with Shirly Geffen.

Joachim Zacharias: *Almost finiteness, $\mathcal{Z}$-stability and the dynamical Cuntz-semigroup for non-commutative coefficients.*

Abstract: We propose an extension of Kerr's notion of almost finiteness to actions of discrete groups on possibly non-commutative C*-algebras by interpreting this concept as a simultaneous approximation of the action and the coefficient algebra. We call such actions almost elementary and show that they lead to $\mathcal{Z}$-stability for the corresponding crossed products. In case of no group action our condition is a weak form of tracial AF or tracial nuclear dimension 0, and, as we show, equivalent to Z-stability in many, conjecturally all simple nuclear cases. We describe versions of our definition and its connections to dynamical strict comparison, thereby establishing parts of a dynamical Toms-Winter conjecture.