Quasitraces and AW^*-Bundles

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Introduction

A trace on a C^*-algebra A is a positive functional which satisfies the tracial property, so $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$, or equivalently, $\tau(xy) = \tau(yx)$ for all $x, y \in A$. Traces play an important role in understanding the structure of C^*-algebras, and moreover, they are very important in the theory of von Neumann algebra. For example, it is well known and very important that the II_1 factors admit a unique tracial state. But not every C^*-algebra possesses a trace. There is a generalization of traces to quasitraces. A quasitrace is a map $\tau \colon A \to \mathbb{C}$, such that the following properties hold:

- τ is linear on commutative C^*-subalgebras.
- $\tau(a + ib) = \tau(a) + i\tau(b)$ for all self-adjoint $a, b \in A$.
- $\tau(x^*x) = \tau(xx^*) \geq 0$.

A quasitrace τ is called an n-quasitrace if it extends to a quasitrace on $M_n(A)$. It is proved by Blackadar and Handelman in [BH82] that every 2-quasitrace is an n-quasitrace for every $n \in \mathbb{N}$.
It is noted by Haagerup in [Haa14] that Kirchberg gave an example of a quasitrace on a unital C^*-algebra which is not a 2-quasitrace. Haagerup has a positive answer to this question in the case that A is unital and exact.
An important tool in understanding quasitraces is the AW^*-algebras, which are an algebraic generalization of von Neumann algebras. These algebras have many properties of von Neumann algebras, so for example, also a decomposition into types analogous to the von Neumann case.
Similar to the II_1 factor in the von Neumann case, it is known that a II_1 AW^*-factor possesses a unique normalized quasitrace. Kaplansky asked the question if every AW^* factor of type II_1 is a von Neumann algebra, and this is still an open problem. This question has a positive answer if and only if the unique quasitrace is a trace. Also, Blackadar and Handelman showed in [BH82] that a positive answer to this question would imply that every 2-quasitrace on an arbitrary C^*-algebra is a trace.
We use quasitraces to develop AW^*-bundles in analog to W^*-bundles. Ozawa introduced W^*-bundles in [Oza13], and these are bundles over a compact Hausdorff space such that every fiber is a von Neumann algebra. The W^*-bundles were used in [Bos+15] to prove a special case of the Toms-Winter conjecture. The Toms-Winter conjecture is about the structure of unital, separable, simple, nuclear, non-elementary, and stably finite C^*-algebras. We want to note here that AW^*-bundles are not suitable to give similar results since on nuclear C^*-algebras, all 2-quasitraces are traces, and then, AW^*-bundles are, in some sense, isomorphic to W^*-bundles.

We will give some examples of AW^*-bundles and prove that the fibers are really AW^*-algebras. In the end, we will present a short outlook about the analogs of W^*-bundles that may be of interest.

Notation

We reserve the letters A and B for C^*-algebras, N and M for AW^*-algebras, von Neumann algebras and for the W^*-bundles and AW^*-bundles. With X we denote usually locally compact or compact Hausdorff spaces.
For a C^*-algebra A we denote with $\mathcal{P}(A)$ the set of projections in A. The set of projections is a subset of the set of positive elements in A, denoted by A_+. The projections inherit the usual ordering from A_+.
Two projections $p, q \in \mathcal{P}(A)$ are Murray-von Neumann equivalent, denoted with $p \sim q$ if there exists a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$. A projection p is Murray-von Neumann subequivalent to q if there exists a projection $p' \in \mathcal{P}(A)$ with

$$p \sim p' \leq q.$$

In this case we will write $p \precsim q$.
For a partial ordered set $(F, \leq)$ and $f_1, f_2 \in F$ we will sometimes write $f_1 \vee f_2$ instead of $\sup\{f_1, f_2\}$ and $f_1 \wedge f_2$ instead of $\inf\{f_1, f_2\}$ for better readability. Also for a family $(f_i)_{i \in I}$ of elements in F we write $\bigvee_{i \in I} f_i$ for $\sup_{i \in I} f_i$ respectively $\bigwedge_{i \in I} f_i$ for $\inf_{i \in I} f_i$.

1 AW*-Algebras

The purpose of this chapter is to introduce AW^*-algebras and prove some important properties of the class of these algebras. We start with the algebraic definition of an AW^*-algebra, and then we go further into the properties. Kaplasnky introduced AW^*-algebras as an algebraic generalization of von Neumann algebras in [Kap51]. In Theorem 1.15 we show that the most important definitions of AW^*-algebras are equivalent. We want to state some commonalities and differences between the von Neumann algebras and the AW^*-algebras, so we give an example of an abelian AW^*-algebra which is not isomorphic to a von Neumann algebra.
In the last part of this chapter, we prove the existence and uniqueness of the center-valued quasitrace on finite AW^*-algebras in analog to the center-valued trace on finite von Neumann algebras. In later chapters, the center-valued quasitrace plays an important role in proving some properties of quasitraces in arbitrary unital C^*-algebras.
The standard reference for AW^*-algebras is [Ber72], and most of the content in this chapter is based on this book.

Definition 1.1: Let A be a C^*-algebra, and $S \subseteq A$ an arbitrary subset. The right-annihilator of S is defined as

$$R(S) := \{x \in A : sx = 0 \ \ \forall s \in S\}.$$

And similarly, the left-annihilator of S is

$$L(S) := \{x \in A : xs = 0 \ \ \forall s \in S\}.$$

A C^*-algebra A is called an AW^*-algebra if for every subset $S \subseteq A$, there exists a projection $p \in A$ such that $R(S) = pA$.

Remark 1.2: Every AW^*-algebra is unital. The right-annihilator of the set $\{0\}$ is generated by a unit.
For every subset S, we get $R(S) = (L(S^*))^*$ and $L(S) = (R(S^*))^*$. So a C^*-algebra is an AW^*-algebra if and only if for every subset S, there exists a projection p such that $L(S) = Ap$.

Lemma 1.3: *The set of projections of an AW^*-algebra M forms a complete lattice.*

Proof. Let $(p_i)_{i\in I}$ be a family of projections. Then there exists a projection $p \in \mathcal{P}(M)$ such that

$$R(\{p_i | i \in I\}) = (1 - p)M.$$

We claim that $\sup_{i\in I} p_i = p$: First, with that, we see $(1-p) \in R(\{p_i|i \in I\})$, so for every $i \in I$, we got $p_i(1-p) = 0$. Thus, $p_i = p_i p$ and $p_i \leq p$ for every $i \in I$. Now we need to show that p is a least upper bound for $(p_i)_{i\in I}$. So, let $q \in \mathcal{P}(M)$ be another upper bound for $(p_i)_{i\in I}$. Then it is clear again that $(1-q) \in R(\{p_i|i \in I\}) = (1-p)M$. So there exists an $m \in M$ such that $(1-q) = (1-p)m$. But then:

$$(1-p)(1-q) = (1-p)(1-p)m = (1-p)m = (1-q).$$

So $(1-q) \leq (1-p)$, hence $p \leq q$, and this finishes the proof. □

Remark 1.4: Let M be an AW^*-algebra and $x \in M$ an arbitrary element. The right support projection of x is the projection $RP(x) \in M$ such that

(i) $xRP(x) = x$.

(ii) for every $y \in M$: $xy = 0$ if and only if $RP(x)y = 0$.

For every $x \in M$, the right support projection exists and is unique: Let p be the projection such that $R(\{x\}) = pA$. Then $RP(x) = 1-p$, and it is clear that $RP(x)$ has the desired properties. If q is another projection with properties (i) and (ii), then $x(1-q) = 0$, hence, $RP(x)(1-q) = 0$. So, $RP(x) \leq q$ and of course analog $q \leq RP(x)$, so $q = RP(x)$.
Analog the left support projection of x is the projection $LP(x) \in M$ such that

(i) $LP(x)x = 0$.

(ii) for every $y \in M$: $yx = 0$ if and only if $yLP(x) = 0$.

With the same argument as before, it is clear that the left support projection exists and is also unique.

Remark 1.5: Let M be an AW^*-algebra and $a \in M_{sa}$ a self-adjoint element. For $\lambda \in \mathbb{R}$, let $E_{(\lambda,\infty)}(a) := RP((x-\lambda 1)_+) = LP((x-\lambda 1)_+)$. So, it follows that $E_{(\lambda,\infty)}(a) \in M$ for every self-adjoint $a \in M$ and $\lambda \in \mathbb{R}$. We can define $E_{[\lambda,\infty)}(a) := \inf_{\lambda'>\lambda}\{E_{(\lambda,\infty)}(a)\}$. Since in an AW^*-algebra, supremas of projections exist and belong to the AW^*-algebra, it follows $E_{[\lambda,\infty)}(a) \in M$ for every $\lambda \in \mathbb{R}$. Then for $\lambda < \mu$, we define $E_{(\lambda,\mu)}(a) := E_{(\lambda,\infty)}(a) - E_{[\mu,\infty)}(a) \in M$. In a similar way, we can define the spectral projection $E_A(a)$ for every Borel set $A \subseteq \sigma(a)$. So, all spectral projections of a self-adjoint element belong to M.
Blackadar described this construction in I.6.1 in [Bla06] and he shows in III.5.2.13 that this definition coincides with the usual definition as an image of characteristic functions χ_B under functional calculus.
It easily follows that every AW^*-algebra has a real rank zero.

Lemma 1.6: *Let A be a C^*-algebra such that for every $x \in A$, there exists a projection $p \in \mathcal{P}(A)$ such that $R(\{x\}) = pA$ and that every family of orthogonal projections has a supremum. Then A is an AW^*-algebra.*

Proof. Let $S \subseteq A$ be an arbitrary subset of A. First, we may assume that $R(S) \neq \{0\}$ because, otherwise, it is obvious that $R(S) = \{0\} = 0A$ and 0 is a projection.
We may suppose that $R(S)$ contains some non-zero projection: If $0 \neq x \in R(S)$, then with Remark 1.4 we see that $0 \neq LP(x) \in A$ and $LP(x) \in R(S)$. So, we assume that $R(S) \neq \{0\}$.
With Zorn's Lemma, we choose a maximal family of orthogonal non-zero projections $\{p_i | i \in I\}$ in $R(S)$ and let $p := \sup_{i \in I} p_i$ which exists by assumption. It is then clear that $p \in R(S)$, hence, $pA \subseteq R(S)$. Now we want to show the inverse direction, so let $x \in R(S)$.
We set $y := x - px$ and show that $y = 0$. First, observe that $y \in R(S)$ so as stated above $LP(y) \in R(S)$. But then

$$p_i y = p_i x - p_i p x = p_i x - p_i x = 0 \text{ for all } i \in I,$$

and so, $p_i LP(y) = 0$ for all $i \in I$. So, $LP(y)$ is orthogonal to p_i for all $i \in I$. Since the family p_i is maximal, it follows that $LP(y) = 0$, so $y = 0$, and then $x = px \in pA$ and $pA = R(S)$. □

Definition and Lemma 1.7: Let M be an AW^*-algebra and N a C^*-subalgebra of M. We call N an AW^*-subalgebra if the following two conditions hold:

- For every $x \in N$, the right support projection of x in M belongs to N.
- The set of projections of N forms a complete lattice.

Then N is an AW^*-algebra.

Proof. First set $e := \sup\{RP(x) | x \in N\} \in N$, and it is clear that e is a unity element for N. Let $x \in N$, then it is clear by definition that $RP(x) \leq e$, and for another element $y \in N$ we compute:

$$xy = 0 \Leftrightarrow RP(x)y = 0 \Leftrightarrow (e - RP(x))y = y.$$

So, $R(\{x\}) = (e - RP(x))N$, and by the previous Lemma, N is an AW^*-algebra. □

Proposition 1.8: *Let M be an AW^*-algebra.*

(i) If $N \subseteq M$ is a *-algebra such that $N = N''$, then N is an AW^*-subalgebra.

(ii) $\mathcal{Z}(M)$ is an AW^*-subalgebra.

Proof. (i) : For an element $x \in N$, we need to show that $RP(x) \in N = N''$. For that, let $y \in N'$. Then:

$$x(y - yRP(x)) = xy - xyRP(x) = xy - yxRP(x) = xy - yx = 0.$$

But then, also $RP(x)(y - yRP(x)) = 0$ and so $RP(x)y = RP(x)yRP(x)$. Since N' is also a *-subalgebra, $y^* \in N'$, and we can compute the same for y^* and get

$$RP(x)y = RP(x)yRP(x) = (RP(x)y^*RP(x))^* = (RP(x)y^*)^* = yRP(x)$$

and can conclude $RP(x) \in N = N''$.
Next, we will show that the set of projections in N forms a complete lattice. Let $(p_i)_{i\in I}$ and $p := \sup_{i\in I} p_i \in M$. Again, we need to show that $p \in N = N''$. But this is the same idea as before: For an element $y \in N'$, the following holds:

$$p_i(y - yp) = p_i y - p_i yp = p_i y - yp_i p = p_i y - yp_i = 0 \text{ for every } i \in I.$$

Again, we get $p(y - yp) = 0$. Since $y^* \in N'$, we also obtain that $p(y^* - y^*p) = 0$ and finish the proof.
(ii) : This immediately follows from (i) since $\mathcal{Z}(M) = M' = M'''$. □

Now we want to characterize abelian AW^*-algebras. Therefore, we need the definition of a Stonean space. The theorems for abelian AW^*-algebras are in [Ber72] §7.

Definition 1.9: A compact Hausdorff space X is called Stonean (or extremely disconnected) if the closure of every open set is open again.

Theorem 1.10: *Let X be Stonean, then $M = C(X)$ is an AW^*-algebra.*

Proof. Let $(U_i)_{i\in I}$ be a family of clopen sets, and $U := \bigcup_{i\in I} U_i$. Then U is open and $\overline{U}$ is clopen by assumption. We claim that $\overline{U}$ is the supremum of $(U_i)_{i\in I}$ in the class of clopen sets ordered by inclusion. If $V \subseteq X$ is another clopen set with $U_i \subseteq V$ for all $i \in I$, then $U \subseteq V$ and also $\overline{U} \subseteq \overline{V} = V$.
The projections in $C(X)$ correspond exactly with the clopen sets in X, hence the set of projections in $C(X)$ forms a complete lattice.
Now we want to show for $f \in C(X)$ that there exists a projection $p \in C(X)$ such that $R(\{f\}) = (1 - p)C(X)$. We set

$$U := \{x \in X | f(x) \neq 0\}.$$

Since U is open, $\overline{U}$ is clopen, and we define $p := \chi_{\overline{U}}$. Then for $g \in C(X)$, we get that $fg = 0$ if and only if $g_{|U} \equiv 0$ if and only if $g_{|\overline{U}} \equiv 0$ if and only if $pg = 0$. Thus,

$$R(\{f\}) = R(\{p\}) = (1 - p)C(X),$$

as desired. □

Theorem 1.11: *Let $M = C(X)$ be an abelian AW^*-algebra, then X is Stonean.*

Proof. First, we show that the clopen sets separate the points of X: For $x, y \in X$, there exists open neighborhoods U of x and V of y, such that $U \cap V = \emptyset$. There exists continuous functions $f, g \in C(X)$ such that

$$f(x) \neq 0 \text{ and } f_{|X\setminus U} \equiv 0$$
$$g(y) \neq 0 \text{ and } g_{|X\setminus V} \equiv 0.$$

Now it is clear that $RP(f)g = 0$. Then consider the projection $RP(f)$, and notice that $RP(f)$ is of the form χ_P for a clopen set $P \subseteq X$. Then $p(x) = 1$ and $p(y) = 0$,

so $x \in P$ and $y \notin P$.
Next, we want to show that the clopen sets form a basis for the topology on X: Let U be an open set and $x \in U$. We seek a clopen set P with $x \in P \subseteq U$. For every $s \in X \backslash U$, we can choose a clopen set P_s with $s \in P_s$ and $x \notin P_s$. Since $X \backslash U$ is compact, there exist $s_1, ..., s_n \in X \backslash U$ with

$$X \backslash U \subseteq \bigcup_{k=1}^{n} P_{s_k}.$$

We set $P := \bigcap_{k=1}^{n}(X \backslash P_{s_k})$, and it is clear that this is a clopen set with $x \in P \subseteq U$.
Now let U be an open set again, and we want to show that $\overline{U}$ is open again. There exists a family $(P_i)_{i \in I}$ of clopen sets with $U = \bigcup_{i \in I} P_i$. Since $C(X)$ is an AW^*-algebra, the set of projections forms a complete lattice, and so does the set of clopen subsets of X. Denote with P the supremum of the P_i's. It immediately follows that $\overline{U} \subseteq P$. Now, it suffices to show that $\overline{U} = P$, so assume that $P \backslash \overline{U} \neq \emptyset$. Since $P \backslash \overline{U}$ is open, there exists a non-empty clopen set $V \subseteq P \backslash \overline{U}$. Then $V \cap P_i = \emptyset$ for all $i \in I$, which means $P_i \subseteq T \backslash V$. But $T \backslash V$ is clopen, hence $P \subseteq T \backslash V$. However, it follows that $P \cap V = \emptyset$, which contradicts $\emptyset \neq V \subseteq P$. □

Next, we want to give some equivalent definitions for AW^*-algebras. But before we can prove them, we need to do some work. First, recall the definition of monotone complete C^*-algebras.

Definition 1.12: A C^*-algebra A is called monotone complete if every upward directed and norm-bounded set of self-adjoint elements has a least upper bound.

Recall that a set is called meager if it is the countable union of nowhere dense sets. A set is nowhere dense if the interior of its closure is empty. The intersection of a meager set with another arbitrary set is meager again. Furthermore, an open set is called regular open if it coincides with the interior of its closure.
The following theorem is one direction of Theorem 2.3.7 in [SW15a]:

Theorem 1.13: *Let X be a Stonean space. Then $C(X)$ is monotone complete.*

Proof. Let $V \subseteq C(X)_{sa}$ be an upward directed and norm-bounded set. Without loss of generality, we may assume that $V \subseteq C(X)_+$. Let us define

$$f(x) := \sup\{a(x) | a \in V\}.$$

Then f is a bounded, non-negative, and lower semi-continuous function.
For $t \in \mathbb{R}^+$, let $F_t := \{x \in X | f(x) \leq t\}$, and since f is lower semi-continuous, each F_t is closed.
Let C_t be the interior of F_t: Then the interior of the closure of $F_t \backslash C_t$ is empty, so $F_t \backslash C_t$ is a nowhere dense set. Since X is extremely disconnected, also note that $\overline{C_t}$ is a clopen set. But then $\overline{C_t} \subseteq \overline{F_t} = F_t$, and so C_t is also clopen.

For $0 < s < t$, it is clear that $F_s \subseteq F_t$, and so also $C_s \subseteq C_t$. We now define $g\colon X \to \mathbb{R}^+$ as follows:

$$g(x) := \inf\{t \in \mathbb{Q}^+ | x \in C_t\}.$$

Now, we need to show that g is a continuous function and the least upper bound for V. For the continuity of g, notice that for $s, r > 0$:

$$G_s := \{x \in X | g(x) < s\} = \bigcup\{C_t | t \in \mathbb{Q}^+ \text{ and } t < s\}$$
$$H_r := \{x \in X | g(x) \leq r\} = \bigcap\{C_t | t \in \mathbb{Q}^+ \text{ and } t > r\}.$$

For every $t \in \mathbb{R}^+$, the set C_t is clopen, so we note that G_s is open and H_r is closed. So, we conclude that for all $r < s$, the set

$$\{x \in X | r < g(x) < s\} = G_s \cap (X \backslash H_r)$$

is open as a finite intersection of open sets, so the function g is continuous.
Next, we want to show that g is an upper bound for V. For that, we show that $f \leq g$:
So, we fix $x \in X$. Then $x \in C_t$ for every $t > g(x)$ and $t \in \mathbb{Q}^+$. But then $f(x) \leq t$ for every $t > g(x)$, and then $f(x) \leq g(x)$. Since $x \in X$ was arbitrary, we can conclude that $f \leq g$ and g is an upper bound for V.
Recall that $F_t \backslash C_t$ is a nowhere dense set and define

$$M := \bigcup_{t \in \mathbb{Q}^+} F_t \backslash C_t.$$

Since $\mathbb{Q}^+$ is countable, M is a meager set as a countable union of nowhere dense sets. We want to show that $g \equiv f$ on $X \backslash M$:
Let $x \in X \backslash M$. Then $x \in F_t$ for all $t \in \mathbb{Q}^+$ with $t \geq f(x)$. But since $x \notin M$, it follows that $x \in C_t$ for all positive rational t with $t \geq f(x)$. So, we can conclude $g(x) \leq f(x)$ for $x \in X \backslash M$.
So, in summary, we have that g is a continuous function with $g \geq f$ and g equals f except on a meager set.
The last thing we need to show is that g is the least upper bound. Let $h \in C(X)$ with $a \leq h$ for all $a \in V$, then also $f \leq h$. Note that $\min\{h, g\}$ is again a continuous function and $g \geq \min\{h, g\} \geq f$. Now consider the set:

$$M_0 := \{x \in X | (g(x) - \min\{g(x), h(x)\}) > 0\}.$$

This set is open and meager. By the Baire Category Theorem, every open and meager set in a compact Hausdorff space is empty, so M_0 is empty. This delivers $g \leq h$, and g is the least upper bound for V. □

Also, we need the following Lemma 1.3 from [SW15b], which gives us the properties of C^*-algebras whose masas are monotone complete. A C^*-subalgebra B is called a masa if it is a maximal abelian self-adjoint subalgebra. So, there is no abelian C^*-subalgebra C with $B \subset C$.

Lemma 1.14: *Let A be a unital C^*-algebra such that every masa is monotone complete.*
Let P be a family of commuting projections and L be the set of all projections which are lower bounds for P. Then

(i) L is upward directed.

(ii) P has the greatest lower bound.

Proof. (i): Let $p, q \in L$ and $c \in P$. Then $cp = pc = p$ and $qc = cq = q$, and it follows that c commutes with $p+q$. Then the set $P \cup \{p+q\}$ is a set of commuting elements. This set is contained in a masa $B \subseteq A$.
Consider the sequence

$$\left(\frac{p+q}{2}\right)^{1/n}_{n\in\mathbb{N}}.$$

This is a monotone increasing, norm-bounded sequence in B, so it has a least upper bound f, which is a projection.
Let again $c \in P$, so $c \geq p$ and $c \geq q$ and also $c \geq \frac{p+q}{2}$. Since c is a projection, it follows that $c^{1/n} = c$, and we conclude:

$$c = c^{1/n} \geq \left(\frac{p+q}{2}\right)^{1/n}$$

for every $n \in \mathbb{N}$. Since f is a least upper bound for $\left(\frac{p+q}{2}\right)^{1/n}_{n\in\mathbb{N}}$, it follows that $c \geq f$ and so f is smaller than every projection in P, hence $f \in L$. The case $n = 1$ gives us that

$$f \geq \frac{p+q}{2} \geq \frac{p}{2}.$$

So, we compute

$$0 = (1-f)f(1-f) \geq \frac{1}{2}(1-f)p(1-f) \geq 0.$$

Then

$$0 = \|(1-f)p(1-f)\| = \|(1-f)pp^*(1-f)^*\| = \|(1-f)p\|^2 = 0.$$

So, it follows that $0 = (1-f)p$, and this implies that $f \geq p$. In the same way, we show that $f \geq q$, and this shows that L is upward directed.
(ii) : Suppose that C is a chain in L. Since we can compare every pair of projections in C, this is a commuting set and so is also $P \cup C$. This set is again contained in a masa B_2 in A. Let e be the least upper bound of C in B_2. We need to show that e is a projection: It is clear that $0 \leq e \leq 1$. But then $e^{1/2}$ is an upper bound for C, and so $e^{1/2} \geq e$, which implies $e \geq e^2$.
For the other inequality, note that e commutes with every $p \in C$, and this gives us

$$e^2 - p = e^2 - p^2 = (e+p)(e-p) \geq 0.$$

So e^2 is an upper bound for C, and we see that $e^2 \geq e$. Then $e^2 = e$, and e is a projection.
Now every $p \in P$ is bigger than e, hence $e \in L$. In other words, every chain in L has an upper bound, and we can apply Zorn's Lemma and find a maximal element for L. But this maximal element is the greatest element and so the greatest lower bound for P. □

The following theorem is a combination of §7 Proposition 2 in [Ber72], Theorem 2.3.7 in [SW15a], Proposition 1.4 in [SW15b], and Theorem 2.3 in [Kap51]. The assumption that A is unital can be dropped in the next theorem, but for the sake of simplification, we use this extra assumption.

Theorem 1.15: *Let A be a unital C^*-algebra. Then the following are equivalent:*

(i) A is an AW^*-algebra.

(ii) Every masa has Stonean spectrum.

(iii) Every masa is monotone complete.

(iv) a) Every masa is the closed linear span of its projections.

b) Every family of orthogonal projections has a supremum.

Proof. $(i) \Rightarrow (ii)$: Let $N \subseteq A$ be a masa. Then $N = N' = N''$, hence with Proposition 1.8, N is an AW^*-algebra. From Theorem 1.11, we know that the spectrum is Stonean.
$(ii) \Rightarrow (iii)$: This is Theorem 1.13.
$(iii) \Rightarrow (iv)$: a): Let $B \subseteq A$ be a masa, and write $B \cong C(X)$ for a compact Hausdorff space X. Then by assumption B is monotone complete. It can be shown that X is zero-dimensional, so the topology has a base of clopen sets (for details, see Lemma 2.3.1 and Proposition 2.3.2 in [SW15a]). Then for every point $x, y \in X$, we find clopen sets U such that $x \in U$ and $y \notin U$, and so the characteristic function of U fulfills $\chi_U(x) \neq \chi_U(y)$. We infer from the Stone-Weierstrass Theorem that the *-algebra generated by the projections is dense in $C(X)$.
b): Most of the work is done in Lemma 1.14, and the rest is Proposition 1.4 in [SW15b]: Let $(p_i)_{i \in I}$ be a family of orthogonal projections and set $P := \{1 - p_i | i \in I\}$. Then P is a family of commuting projections and has the greatest lower bound p, and then $(1 - p)$ is a supremum for $(p_i)_{i \in I}$.
$(iv) \Rightarrow (i)$: This direction can be found in [Kap51], and we outline the proof: First note that if $(p_i)_{i \in I}$ is a family of orthogonal projections in A with $\sup_{i \in I} p_i = p$, then for $x \in A$: If $xp_i = 0$ for all $i \in I$, then $xp = 0$.
With Lemma 1.6 we know that it suffices to show that for $x \in A$, there exists a projection p such that $R(\{x\}) = pA$. The idea of the proof is the same as before: With Zorn's Lemma, choose a maximal family of orthogonal projections $(p_i)_{i \in I}$ in $R(\{x\})$. Then we know that $p = \sup_{i \in I} p_i \in R(\{x\})$. Now let $y \in R(\{x\})$, and we want to show that $z := y - py = 0$. If $z \neq 0$, so is zz^*, and zz^* is contained in a

masa $C \subseteq A$. Then C is the closed linear span of its projections, which implies that the spectrum of C is totally disconnected. So, we can find a projection $f \in C$ which is a two-sided multiple of zz^*. But then f is also an element in $R(\{x\})$, which is orthogonal to p_i for every $i \in I$. But this contradicts the maximality of $(p_i)_{i\in I}$. □

We have seen that M is an AW^*-algebra if and only if every masa in M is monotone complete. It is stated by Saitô and Wright in [SW15b] that no one has ever seen an AW^*-algebra which is not monotone complete. It is an open problem if every AW^*-algebra is monotone complete.

Corollary 1.16: *Every von Neumann algebra is an AW^*-algebra.*

Proof. This follows immediately from Theorem 1.15 (*iv*). □

Next, we want to give an example of an AW^*-algebra which is not isomorphic to a von Neumann algebra. We want to restrict it here to the commutative case, but there are also examples in the non-commutative case. Before we can define this algebra, we need to do some basic work on regular open and on meager sets. The examples given here are Exercises 5.7.20 and 5.7.21 in [KR91]:

Lemma 1.17: *Let X be a complete metric space.*

(i) Interiors of closed sets are regular open.

(ii) For every open set, U exists a regular open set O such that symmetric difference of U and O

$$U\Delta O = U\backslash O \cup O\backslash U$$

is meager. We say that O differs from U on a meager set.

(iii) For every Borel set U, there exists a unique regular open set O such that U differs from U on a meager set.

(iv) Let $\mathcal{O}_0$ be the family of regular open subsets of X ordered by inclusion. Then $\mathcal{O}_0$ forms a complete lattice.

(v) Let $\mathcal{B}(X)$ be the set of Borel sets of X. We define an equivalence relation $\sim$ on $\mathcal{B}(X)$:

$$U \sim V \Leftrightarrow U\Delta V \text{ is meager.}$$

Let $\mathcal{F} := \mathcal{B}(X)/\sim$. Then for $\mathcal{U}, \mathcal{V} \in \mathcal{F}$, we define

$$\mathcal{U} \lesssim \mathcal{V} \Leftrightarrow \exists U \in \mathcal{U}, V \in \mathcal{V} : U \subseteq V.$$

Then $\lesssim$ is a partial ordering on $\mathcal{F}$. Furthermore, for every equivalence class $\mathcal{S} \in \mathcal{F}$, there exists a unique regular open set $O \in \mathcal{S}$. Furthermore, there exists an order isomorphism $\mathcal{F} \to \mathcal{O}_0$.

Proof. Since most of this is well known, we will only sketch the proof and give the ideas:
(i): Let $Y \subseteq X$ be closed, then $Y^\circ \subseteq (\overline{Y^\circ})^\circ$. But clearly, $(\overline{Y^\circ})^\circ \subseteq \overline{Y^\circ} \subseteq Y$ and since $(\overline{Y^\circ})^\circ$ is open, we get $(\overline{Y^\circ})^\circ \subseteq Y^\circ$, and so $Y^\circ = (\overline{Y^\circ})^\circ$ and Y° is regular open.
(ii): Let $U \subseteq X$ be open, then let $O := \overline{U}^\circ$ and O is regular open, and we see that

$$U \Delta O \subseteq \overline{O} \backslash O.$$

The set $\overline{O} \backslash O$ is nowhere dense. Hence, $U \Delta O$ is meager.
(iii): Let $\mathcal{B}(X)_0$ be the Borel sets that differ from a regular open set on a meager set. Then every open set is contained in $\mathcal{B}(X)_0$. We want to show that $\mathcal{B}(X)_0 = \mathcal{B}(X)$. For that, we show that $\mathcal{B}(X)_0$ is a σ-algebra:
Let $U \in \mathcal{B}(X)_0$ such that U differs from a regular open set O on a meager set. Then

$$U \Delta O = [(X \backslash O) \backslash (X \backslash U)] \cup [(X \backslash U) \backslash (X \backslash O)].$$

So, $X \backslash U$ differs from $X \backslash O$ on a meager set. From (i), we know that the interior of $X \backslash O$ is regular open, and it differs from $X \backslash U$ on a meager set, so $X \backslash U \in \mathcal{B}(X)_0$.
Let $(U_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{B}(X)_0$. Suppose that for $n \in \mathbb{N}$, O_n is a regular open set such that $U_n \Delta O_n$ is meager. Let O be the interior of the closure of $\bigcup_{n \in \mathbb{N}} O_n$. Then O is regular open, and one can show that $(\bigcup_{n \in \mathbb{N}} U_n) \Delta O$ is meager, and $\bigcup_{n \in \mathbb{N}} U_n \in \mathcal{B}(X)_0$.
So, $\mathcal{B}(X)_0$ is a σ-algebra containing all open sets. The open sets generate the σ-algebra $\mathcal{B}(X)$, and we deduce that $\mathcal{B}(X)_0 = \mathcal{B}(X)$.
For the uniqueness suppose that for a Borel set U, there exist regular open sets O_1, O_2 such that $U \Delta O_i$ is meager for $i = 1, 2$. Then we can show that $O_1 \backslash O_2$ is meager, and since X is complete, the interior of this set is empty by the Baire Category Theorem. So $O_1 \subset \overline{O_2}$, hence $O_1 \subseteq O_2 = (\overline{O_2})^\circ$. It follows symmetrically $O_2 \subseteq O_1$, and we conclude $O_1 = O_2$.
(iv): Let $(O_i)_{i \in I}$ be a family of regular open sets. Then the supremum is given by

$$\sup_{i \in I} O_i = \left(\overline{\bigcup_{i \in I} O_i} \right)^\circ.$$

(v): We show only that $\lesssim$ is anti-symmetric. Suppose $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{F}$ with $\mathcal{S}_1 \lesssim \mathcal{S}_2 \lesssim \mathcal{S}_1$. Then there are sets $S_1, S_1' \in \mathcal{S}_1, S_2, S_2' \in \mathcal{S}_2$ with $S_1 \subseteq S_2$ and $S_2' \subseteq S_1'$.
Since $S_1 \sim S_1'$ and $S_2 \sim S_2'$, the sets $M_1 := S_1 \Delta S_1'$ and $M_2 := S_2 \Delta S_2'$ are meager, and furthermore, $S_1 \cup M_1 = S_1' \bigcup M$ and $S_2 \cup M_2 = S_2' \cup M_2$. Then

$$\begin{aligned} S_2 \cup M_1 \cup M_2 &= S_2' \cup M_1 \cup M_2 \subseteq S_1' \cup M_1 \cup M_2 \\ &= S_1 \cup M_1 \cup M_2 \subseteq S_2 \cup M_1 \cup M_2 \end{aligned}$$

and this implies that $S_1 \sim S_2$ and $\mathcal{S}_1 = \mathcal{S}_2$. So $\lesssim$ is a partial ordering on $\mathcal{F}$.
We know that each $U \in \mathcal{B}(X)$ differs from a unique regular open set on a meager set. So, every equivalence class in $\mathcal{F}$ contains a unique regular open set.

Let $\mathcal{S}_i \in \mathcal{F}$ for $i = 1, 2$ and let $O_i \in \mathcal{S}_i$ be the unique regular open set. We show that $\mathcal{S}_1 \lesssim \mathcal{S}_2$ if and only if $O_1 \subseteq O_2$.
The second direction is clear, and we show the first one. Suppose that $\mathcal{S}_1 \lesssim \mathcal{S}_2$. Then there are sets $S_i \in \mathcal{S}_i$ for $i = 1, 2$ such that $S_1 \subseteq S_2$. Then let for $i = 1, 2$ again $M_i := S_i \Delta O_i$ the meager set, and we get

$$O_1 \cup M_1 \subseteq O_2 \cup M_2.$$

It follows that $O_1 \subseteq O_2 \cup M_2$ and $O_1 \backslash \overline{O}_2 \subseteq O_1 \backslash O_2 \subseteq M_2$. But $O_1 \backslash \overline{O}_2$ is an open set in a meager set, so it is empty per the Baire Category Theorem. Thus, $O_1 \subset \overline{O}_2$ and also, $O_1 \subseteq (\overline{O_2})^\circ = O_2$.
Then we can deduce that the map $\mathcal{F} \to \mathcal{O}_0$, which sends an equivalence class in $\mathcal{S}$ to the unique regular open set $O \in \mathcal{S}$, is an order isomorphism. Then with (iv), we see that $\mathcal{F}$ is also a complete lattice. □

Definition 1.18: A state τ on a von Neumann algebra M is called normal if the restriction of τ to the closed unit ball of M is continuous with respect to the strong operator topology on M.

Remark 1.19: If τ is a normal state on a von Neumann algebra, and $(p_i)_{i \in I}$ is a family of pairwise orthogonal projections, then $\sum_{i \in I} p_i$ converges strongly to $\sup_{i \in I} p_i$, see Proposition III.1.1 in [Bla06]. So, it follows that

$$\tau(\bigvee_{i \in I} p_i) = \sum_{i \in I} \tau(p_i).$$

We have shown that the spectrum of an AW^*-algebra is Stonean. There is a similar result for von Neumann algebras, which we want to state here without giving the proof.
Recall that a Stonean space X is called Hyperstonean if it admits many sufficiently positive normal measures; that is, for every nonzero positive $f \in C(X)$, there exists a positive normal measure μ with $\mu(f) \neq 0$. A positive Radon measure is called normal if $\mu(f) = \sup_{i \in I} \mu(f_i)$ for any increasing sequence $(f_i)_{i \in I}$ in $C_{\mathbb{R}}(X)$ with $\sup_{i \in I} f_i = f$.
For a proof of the following theorem, see Theorem 1.18 in [Tak02].

Theorem 1.20: *Let X be a compact Hausdorff space. Then $C(X)$ is isomorphic to a von Neumann algebra if and only if X is a Hyperstonean space.*

Theorem 1.20 shows that a commutative von Neumann algebra sufficiently admits many normal states. The AW^*-algebra we construct in the next theorem admits no normal state, so it is not isomorphic to a von Neumann algebra.

Theorem 1.21: *There exists an abelian AW^*-algebra which is not isomorphic to a von Neumann algebra.*

Proof. Let $\mathcal{B}_b([0,1])$ be the C^*-algebra of bounded and Borel measurable complex-valued functions on $[0,1]$. Let

$$I([0,1]) := \{f \in \mathcal{B}_b([0,1]) | \mathrm{supp}(f) \text{ is meager}\}.$$

Then $I([0,1])$ is a closed, two-sided ideal in $\mathcal{B}_b([0,1])$:
Let $f, g \in I([0,1])$, then $\mathrm{supp}(f+g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$, and since $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ is meager, $\mathrm{supp}(f+g) \cap (\mathrm{supp}(f) \cup \mathrm{supp}(g))$ is meager. Hence $f+g \in I([0,1])$.
Now suppose that $f \in I([0,1])$ and $g \in \mathcal{B}_b([0,1])$. It is clear that $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f)$ and $\mathrm{supp}(f)$ is meager, and with the same argument as before, $\mathrm{supp}(fg)$ is meager and $fg \in I([0,1])$.
To show that $I([0,1])$ is closed, suppose a sequence $(f_n)_{n\in\mathbb{N}}$ in $I([0,1])$ that converges to $f \in \mathcal{B}_b([0,1])$. It follows that $\mathrm{supp}(f) \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{supp}(f_n)$, and the countable union of meager sets is meager again. We conclude that $\mathrm{supp}(f)$ is meager, and $f \in I([0,1])$.
Then $I([0,1])$ is a closed, two-sided ideal and we can define the Dixmier algebra

$$D([0,1]) := \mathcal{B}_b([0,1])/I([0,1]).$$

Let π be the quotient mapping. We now show that $D([0,1])$ is an AW^*-algebra:
It is well known that $\mathcal{B}_b([0,1])$ is the closed linear span of its projections. But then also, $D([0,1])$ is the closed linear span of its projections. So, with Theorem 1.15, we only need to show that every family of orthogonal projections in $D([0,1])$ possesses a supremum. We actually show that the set of projections forms a complete lattice:
Let $\mathcal{F} = \mathcal{B}([0,1])/\sim$ like we have defined in the previous Lemma 1.17. We want to construct an order isomorphism from $\mathcal{F}$ to $\mathcal{P}(D([0,1]))$:
Let $\mathcal{S} \in \mathcal{F}$ and let $S \in \mathcal{S}$. Let χ_S be the characteristic function of S and define

$$\eta\colon \mathcal{F} \to \mathcal{P}(D([0,1])),\ \mathcal{S} \mapsto \pi(\chi_S).$$

Then it is straightforward to prove that η is a well-defined order preserving map.
Let now p be a projection in $D([0,1])$ and e be a self-adjoint element in $\mathcal{B}_b([0,1])$ with $\pi(e) = p$. Then $\pi(e^2 - e) = 0$, which means that $e^2 - e$ vanishes outside a meager set M. We define

$$q\colon [0,1] \to \mathbb{C},\ t \mapsto \begin{cases} e(t) & \text{for } t \in [0,1] \backslash M \\ 0 & \text{for } t \in M \end{cases}.$$

q is an idempotent, self-adjoint element in $\mathcal{B}_b([0,1])$, hence a projection, and so it is the characteristic function of a Borel set S. Then it follows that $\eta([S]) = p$.
Now let p_1, p_2 be projections in $D([0,1])$ with $p_1 \leq p_2$. So, there are $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{F}$ such that $\eta(\mathcal{S}_1) = p_1$ and $\eta(\mathcal{S}_2) = p_2$. From the definition of η we know that there are Borel sets $S_i \in \mathcal{S}_i$ such that $\pi(\chi_{S_i}) = p_i$ for $i = 1, 2$.
Again, $\pi(\chi_{S_1} - \chi_{S_1}\chi_{S_2}) = p_1 - p_1 p_2 = 0$, and it follows again that $\chi_{S_1} - \chi_{S_1}\chi_{S_2}$ is zero outside a meager set M'. Then it immediately follows that $S_1 \subseteq S_2 \cup M' \in \mathcal{S}_2$. So, $\mathcal{S}_1 \lesssim \mathcal{S}_2$.

Now suppose that $\eta(\mathcal{S}_1) = \eta(\mathcal{S}_2)$. Then it follows that $\mathcal{S}_1 \lesssim \mathcal{S}_2 \lesssim \mathcal{S}_1$, and so $\mathcal{S}_1 = \mathcal{S}_2$. This clearly implies that η is an order preserving isomorphism from $\mathcal{F}$ to $D([0,1])$. So, the set of projections of $D([0,1])$ is a complete lattice, and we use Theorem 1.15 to see that $D([0,1])$ is an AW^*-algebra.

To show that $D([0,1])$ is not isomorphic to a von Neumann algebra, we show that there is no normal state on $D([0,1])$. Suppose that τ is a normal state on $D([0,1])$. Thus, for every sequence of orthogonal projections $(p_n)_{n\in\mathbb{N}}$, the following equation holds:

$$\tau\left(\bigvee_{n\in\mathbb{N}} p_n\right) = \sum_{n=1}^{\infty} \tau(p_n).$$

Then this implies that for every sequence of projections $(q_n)_{n\in\mathbb{N}}$, the inequality

$$\tau\left(\bigvee_{n\in\mathbb{N}} q_n\right) \leq \sum_{n=1}^{\infty} \tau(q_n)$$

holds. We will prove this in a more general case in Theorem 3.9.

Our next goal is to construct a countable family of projections $(p_n)_{n\in\mathbb{N}}$ such that

(i) $\sup_{n\in\mathbb{N}} p_n = 1$.

(ii) $\tau(p_n) \leq 2^{-(n+1)}$ for every $n \in \mathbb{N}$

We enumerate the open intervals with rational endpoints in $[0,1]$ and denote with $q_1, q_2, ...$ the image of the characteristic function of these intervals in $D([0,1])$. Suppose that there are projections $(p_n)_{n\in\mathbb{N}}$ in $D([0,1])$ such that $0 < p_n \leq q_n$ for every $n \in \mathbb{N}$. Let $\mathcal{S}_n := \eta^{-1}(p_n)$. From Lemma 1.17, we know that there exists a regular open set $O_n \in \mathcal{S}_n$. We set $O := \bigcup_{n\in\mathbb{N}} O_n$. Suppose that there is an $x \in [0,1]\backslash\overline{O}$. Then there exists an open interval with rational endpoints (a,b) such that $(a,b) \cap O = \emptyset$. Let $f_n := \eta([(a,b)])$, and we know that $0 < e_n \leq f_n$, thus $\mathcal{S}_n \lesssim [(a,b)]$. But again, using Lemma 1.17, we deduce that $O_n \subseteq (a,b)$, but this contradicts $O \cap (a,b) = \emptyset$, so $\overline{O}$ equals $[0,1]$. We have $\sup_{n\in\mathbb{N}} O_n = [0,1]$ and then $\sup_{n\in\mathbb{N}} p_n = 1$.

Next, note that there is no minimal nonzero projection in $D([0,1])$: Otherwise, if p would be non-zero and minimal, $\mathcal{S} = \eta^{-1}(p)$ would be minimal in $\mathcal{F}$. But there is a non-empty regular open set $O \in \mathcal{S}$, which is also minimal in the complete lattice of regular open sets $\mathcal{O}_0$. But O contains some open interval (a,b), which is clearly regular open.

Now let $(q_n)_{n\in\mathbb{N}}$ be our sequence of projections of open intervals with rational endpoints. If $\tau(q_n) < 2^{-(n+1)}$, there is nothing to do, and we set $p_n = q_n$. Otherwise, there exists a projection p_n' with

$$0 < p_n' < q_n.$$

We follow that $q_n - p_n'$ is a projection which is below q_n, and then one of $\tau(p_n')$ or $\tau(q_n - p_n')$ is smaller than $\frac{1}{2}\tau(q_n)$. We can continue this division until we find a projection p_n with $\tau(p_n) \leq 2^{-(n+1)}$.

So, we have a sequence $(p_n)_{n\in\mathbb{N}}$ of projections with $\tau(p_n) \leq 2^{-(n+1)}$ and $0 < p_n \leq q_n$. We showed that the second property implies that $\sup_{n\in\mathbb{N}} p_n = 1$.
Then we get the following inequality:

$$1 = \tau(1) = \tau\left(\sup_{n\in\mathbb{N}} p_n\right) \leq \sum_{n=1}^{\infty} \tau(p_n) \leq \sum_{n=1}^{\infty} 2^{-(n+1)} = \frac{1}{2}.$$

So τ cannot be normal, and it follows that $D([0,1])$ cannot be isomorphic to a von Neumann algebra. □

This is Theorem 5.4. from [Kap51], but we will only give the idea of the proof:

Theorem 1.22: *In any AW^*-algebra M, the Kaplansky identity holds, so for all projections $p, q \in M$:*

$$p \vee q - p \sim q - p \wedge q.$$

Proof. In AW^*-algebras, it is always true that $RP(x) \sim LP(x)$ for every $x \in M$, see, for example, Theorem 2.5 in [Ara89].
Then we can show that $RP(p(1-q)) = p \vee q - p$ and $LP(p(1-q)) = q - p \wedge q$. □

This last theorem is important for some later results, but we do not want to give the proof here since the proof uses the decomposition of AW^*-algebras in types as for von Neumann algebras. The decomposition of AW^*-algebras into types is Chapter 3 in [Ber72]. The next theorem is Corollary 1 in §62 in [Ber72].

Theorem 1.23: *Let M be an AW^*-algebra. Then $M_n(M)$ is, again, an AW^*-algebra*

1.1 The Center-Valued Quasitrace

Recall that a unital C^*-algebra A is called finite if every isometry is a unitary, i.e. that for $v \in A$ with $v^*v = 1$ implies that $vv^* = 1$.
It is a well-known fact that every finite von Neumann algebra M admits a center-valued trace $ctr\colon M \to \mathcal{Z}(M)$, and every trace on M is of the form $\varphi \circ ctr$ for a positive functional φ on $\mathcal{Z}(M)$. For further information, see, for example, Part III Chapter 4 in [Dix81].
The aim in this chapter is to construct a center-valued quasitrace for finite AW^*-algebras with similar properties. Therefore, we first need the following two lemmas, where the first is Lemma II.1.2 in [BH82]:

Lemma 1.24: *Let M be an AW^*-algebra and $0 \leq a \leq b$ be two positive elements of M. Let $0 < \lambda \leq \mu$ be two positive numbers and define $p := E_{(\mu,\infty)}(a)$ and $q := E_{(\lambda,\infty)}(b)$ to be the spectral projections as defined in Remark 1.5.*
Then $p \lesssim q$, and furthermore, we have

$$E_{(\lambda,\infty)}(a) \lesssim E_{(\lambda,\infty)}(b) \text{ and } E_{(\lambda,\infty)}(a) \lesssim E_{[\lambda,\infty)}(b)$$

Proof. First, we observe with functional calculus that $\mu p \leq a \leq b$ and also $\lambda(1-q) \geq b(1-q)$.

$$\begin{aligned}\|pqp - p\| &= \|p(1-q)p\| = \|(1-q)p(1-q)\| \\ &\leq \frac{1}{\mu}\|(1-q)b(1-q)\| \\ &= \frac{1}{\mu}\|(1-q)b\| \leq \frac{\lambda}{\mu} < 1.\end{aligned}$$

Then pqp is invertible in the hereditary subalgebra pMp, and it follows that $p \precsim q$. The rest then follows from the continuity of $\precsim$ under monotone limits. $\square$

Lemma 1.25: *Let M be an AW^*-algebra and $x \in M$ be an arbitrary element. Then for every $\lambda > 0$, we get $E_{(\lambda,\infty)}(x^*x) \sim E_{(\lambda,\infty)}(xx^*)$.*

Proof. Let $x = v|x|$ be the polar decomposition of $x \in M$ and note that $v \in M$. It is well known that the map

$$\varphi\colon \overline{x^*Mx} \to \overline{xMx^*},\ z \mapsto vzv^*$$

is an *-isomorphism and sends x^*x to xx^*. Since a *-homomorphism sends functional calculus to functional calculus, we get that $(x^*x - \lambda 1)_+$ maps to $(xx^* - \lambda 1)_+$.
Set $y := v(x^*x - \lambda 1)_+^{1/2}$, and thus, $y^*y = (x^*x - \lambda 1)_+$ and $yy^* = (xx^* - \lambda 1)_+$.
But again, the left and right support projections are equivalent, and it is easy to see that

$$RP(y^*y) = RP(y) \sim LP(y) = RP(yy^*).$$

This gives us the desired equivalence. $\square$

The main ingredient to construct the center-valued quasitrace is the following theorem from Chapter 6 in [Ber72], which we want to use without giving the proof:

Theorem 1.26: *Let M be a finite AW^*-algebra. Then, there exists a unique center-valued dimension function $D\colon \mathcal{P}(M) \to \mathcal{Z}(M)$ defined on the set of projections of M, satisfying:*

(i) $p \sim q$ *if and only if* $D(p) = D(q)$.

(ii) $p \precsim q$ *if and only if* $D(p) \leq D(q)$.

(iii) $0 \leq D(p) \leq 1$.

(iv) $D(1) = 1$.

(v) *If* $p \perp q$, *then* $D(p+q) = D(p) + D(q)$.

(vi) *If p is a central projection, we obtain $D(pq) = pD(q)$ for all projections $q \in \mathcal{P}(M)$.*

(vii) For all families $(p_i)_{i\in I}$ of orthogonal projections, we get $D(\sup_{i\in I} p_i) = \sum_{i\in I} D(p_i)$.

We now have all the requirements to prove the main theorem of this chapter. The result is known to experts, but we could not find it in the literature. So, we will give the proof here.

Theorem 1.27: *Let M be a finite AW^*-algebra. Then, there exists a unique center-valued quasitrace $T\colon M \to \mathcal{Z}(M)$ with the following properties:*

(i) T is linear on commutative C^*-subalgebras.

(ii) $T(a+ib) = T(a) + iT(b)$ for all self-adjoint $a, b \in M$.

(iii) $T(x^*x) = T(xx^*) \geq 0$.

(iv) $T(x^*x) = 0$ if and only if $x = 0$.

(v) $T_{|\mathcal{P}(M)} = D$ where D is the center-valued dimension function from the previous theorem.

(vi) $T(hm) = hT(m)$ for all self-adjoint $h \in \mathcal{Z}(M)$ and for all $m \in M$.

(vii) $T_{|\mathcal{Z}(M)} = \mathrm{id}_{\mathcal{Z}(M)}$.

(viii) T is order-preserving on M_{sa}.

(ix) T is continuous in norm, in particular, $\|T(x) - T(y)\| \leq 2\|x-y\|$ for all $x, y \in M$.

Proof. The strategy for this proof is to define T first on self-adjoint elements with finite spectrum and then show the continuity of T on these elements. We then use this and the real rank zero of M to define T on all self-adjoint elements. Then, we can prove all the desired properties of T.
Let $a = \sum_{k=1}^n \alpha_k p_k$ be a self-adjoint element with finite spectrum, so we have that $\alpha_k \in \mathbb{R}$ for all $k \in \{1, ..., n\}$ and the p_k are mutually orthogonal projections in M, and define $T(a) := \sum_{k=1}^n \alpha_k D(p_k)$.
Our next aim is to show that $T(a+b) = T(a) + T(b)$ for two self-adjoint, commuting elements with finite spectrum. So, we observe, that $a+b$ is also self-adjoint with finite spectrum: Suppose that $a = \sum_{k=1}^n \alpha_k p_k$ and $b = \sum_{j=1}^m \beta_j q_j$. Then all $p_k's$ commute with all $q_j's$, thus $p_k q_j$ is again a projection for all $k \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. We can now compute:

$$\begin{aligned} a+b &= \sum_{k=1}^n \alpha_k p_k + \sum_{j=1}^m \beta_j q_j \\ &= \sum_{k=1}^n \alpha_k \left(p_k - \sum_{j=1}^m p_k q_j \right) + \sum_{i=1}^m \beta_i \left(q_i - \sum_{l=1}^n p_l q_i \right) + \sum_{o=1}^n \sum_{r=1}^m (\alpha_o + \beta_r) p_o q_r, \end{aligned}$$

where all the projections $\left(p_k - \sum_{j=1}^m p_k q_j\right), \left(q_i - \sum_{l=1}^n p_l q_i\right), p_o q_r$ are mutually orthogonal. So now, we can compute $T(a+b)$:

$$\begin{aligned}
&T(a+b)\\
&= T\left(\sum_{k=1}^n \alpha_k \left(p_k - \sum_{j=1}^m p_k q_j\right) + \sum_{i=1}^m \beta_i \left(q_i - \sum_{l=1}^n p_l q_i\right) + \sum_{o=1}^n \sum_{r=1}^m (\alpha_o + \beta_r) p_o q_r\right)\\
&= \sum_{k=1}^n \alpha_k T\left(p_k - \sum_{j=1}^m p_k q_j\right) + \sum_{i=1}^m \beta_i T\left(q_i - \sum_{l=1}^n p_l q_i\right) + \sum_{o=1}^n \sum_{r=1}^m (\alpha_o + \beta_r) T(p_o q_r)\\
&= \sum_{k=1}^n \alpha_k T(p_k) + \sum_{i=1}^m \beta_i T(q_i) = T(a) + T(b).
\end{aligned}$$

Since all the spectral projections of a commute, we can compute

$$T(a) = \int D(E_{(\lambda,\infty)}(a))d\lambda,$$

where we interpret the integral as a Riemann integral. Now we want to show that T is order-preserving on self-adjoint elements with finite spectrum. For that, let $0 \leq a \leq b$, then we can compute with Lemma 1.24

$$T(a) = \int D(E_{(\lambda,\infty)}(a))d\lambda \leq \int D(E_{(\lambda,\infty)}(b))d\lambda = T(b).$$

Now, we can show the continuity of T. Let $a, b \in M$ again self-adjoint elements with finite spectrum. Then the following implications hold:

$$\begin{aligned}
&-\|a-b\|1 \leq a - b \leq \|a-b\|1\\
\Rightarrow\ &b - \|a-b\|1 \leq a \leq \|a-b\|1 + b\\
\Rightarrow\ &T(b - \|a-b\|1) \leq T(a) \leq T(\|a-b\|1 + b).
\end{aligned}$$

Since $\|a-b\|1$ and b commute, and $T(1) = 1$, we obtain $\|T(a) - T(b)\| \leq \|a-b\|$.
Now we can define T for all self-adjoint elements: Let $a \in M$ be a self-adjoint element. M has real rank zero, hence, there exists a sequence of self-adjoint elements with finite spectrum $(a_n)_{n\in\mathbb{N}}$, which converge in norm to a.
The sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, thus with the inequality above $(T(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, and so it converges in norm. We can define

$$T(a) := \lim_{n\to\infty} T(a_n).$$

The next step is to show the independence of the chosen sequence: If $(b_n)_{n\in\mathbb{N}}$ is another sequence of self-adjoint elements with finite spectrum which converges to a in norm, then we have the following inequality:

$$\|T(a_n) - T(b_n)\| \leq \|a_n - b_n\| \leq \|a_n - a\| + \|a - b_n\|.$$

So, we have the independence from the chosen sequence.
Let $x \in M$ be an arbitrary element. We can decompose x into its real and imaginary parts $Re(x)$ and $Im(x)$, and then define $T(x) = T(Re(x)) + i(T(Im(x))$.
Now let $a, b \in M$ be two commuting self-adjoint elements. Then there are sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in the commutative AW^*-subalgebra $\{a, b\}''$, generated by a and b which converge to a and b, respectively. Then it is clear that $(a_n + b_n)_{n\in\mathbb{N}}$ is again a self-adjoint sequence with finite spectrum, it converges to $a + b$, and we see that $T(a_n + b_n) = T(a_n) + T(b_n)$ for every $n \in \mathbb{N}$ because a_n and b_n commute for every $n \in \mathbb{N}$. We see the additivity of T on the commuting self-adjoint elements.
If now $x, y \in M$ are two normal commuting elements such that the $x^*y = yx^*$, then also the real and imaginary parts of both elements commute, and we obtain:

$$\begin{aligned} T(x+y) &= T(Re(x) + iIm(x) + Re(y) + iIm(y)) \\ &= T(Re(x) + Re(y)) + iT(Im(x) + Im(y)) \\ &= T(Re(x)) + T(Re(y)) + iT(Im(x)) + iT(Im(y)) = T(x) + T(y). \end{aligned}$$

Now we want to show that $T(\lambda x) = \lambda T(x)$ for all $\lambda \in \mathbb{C}$ and all normal $x \in M$:
It is easy to see that $T(\lambda x) = \lambda T(x)$ if $\lambda \in \mathbb{R}$ and x is a self-adjoint element.
Suppose now that $x \in M$ is normal, thus the real and imaginary parts of x commute, and then for $\lambda \in \mathbb{C}$

$$\begin{aligned} T(\lambda x) &= T((Re(\lambda) + iIm(\lambda))(Re(x) + iIm(x)) \\ &= T((Re(\lambda)Re(x) - Im(\lambda)Im(x) + i(Re(\lambda)Im(x) + Im(\lambda)Re(x))) \\ &= T((Re(\lambda)Re(x) - Im(\lambda)Im(x)) + iT(Re(\lambda)Im(x) + Im(\lambda)Re(x))) \\ &= T((Re(\lambda)Re(x)) - T(Im(\lambda)Im(x)) + iT(Re(\lambda)Im(x)) + iT(Im(\lambda)Re(x))) \\ &= Re(\lambda)T(Re(x)) - Im(\lambda)T(Im(x)) + iRe(\lambda)T(Im(x)) + iIm(\lambda)T(Re(x)) \\ &= (Re(\lambda) + Im(\lambda))(T(Re(x)) + iT(Im(\lambda)) \\ &= \lambda T(x). \end{aligned}$$

So, we have shown the first property of T, namely that T is linear on commutative subalgebras.
Now we want to prove the tracial property of T. For that, it is first clear that $T(x^*x) \geq 0$. We again use the equality $T(x^*x) = \int D(E_{(\lambda,\infty)}(x^*x))d\lambda$. We now use Lemma 1.25 and get the desired property:

$$T(x^*x) = \int D(E_{(\lambda,\infty)}(x^*x))d\lambda = \int D(E_{(\lambda,\infty)}(xx^*))d\lambda = T(xx^*).$$

In the next step, we want to prove property (vi): If $h \in \mathcal{Z}(M)$ and $m \in M$ are both self-adjoint elements with finite spectrum, then it is clear that $T(hm) = hT(m)$ since D has the property. The general case then follows because from Proposition 1.8, we know that $\mathcal{Z}(M)$ is an AW^*-algebra, and we can approximate every self-adjoint $h \in \mathcal{Z}(M)$ with self-adjoint elements with finite spectrum in $\mathcal{Z}(M)$. □

2 Quasitraces and the AW^*-Completion

Traces are an important tool to understand von Neumann algebras. In the case of AW^*-algebras, we cannot resort to traces, but we can use quasitraces. In the first part of this chapter, we will introduce quasitraces and state some results that we need about quasitraces. In the second part, we will introduce a metric that arises from quasitraces and prove similar results for AW^*-algebras as for the von Neumann algebras.

In the third part, we will use this metric to construct an AW^*-completion of a unital C^*-algebra. Both the construction of the metric and the completion is made by Haagerup in [Haa14].

The last part of this chapter is about a tensor product of a finite AW^*-factor with the hyperfinite II_1-factor $\mathcal{R}$.

2.1 Properties of Quasitraces

In this section, we will give the definition of quasitraces and prove some results about them. This chapter is based on [BH82] and uses the center-valued quasitrace of the previous chapter.

Definition 2.1: Let A be a C^*-algebra. A 1-quasitrace on A is a map $\tau\colon A \to \mathbb{C}$ such that the following properties hold:

- τ is linear on commutative C^*-subalgebras.
- $\tau(a + ib) = \tau(a) + i\tau(b)$ for all self-adjoint $a, b \in A$.
- $\tau(x^*x) = \tau(xx^*) \geq 0$.

Furthermore, τ is called an n-quasitrace for $n \in \mathbb{N}, n \geq 2$ if there exists a 1-quasitrace $\tau_n\colon M_n(A) \to \mathbb{C}$ such that

$$\tau(x) = \tau_n(x \otimes e_{11}).$$

Furthermore, we call an n-quasitrace τ

- faithful if $\tau(x^*x) = 0$ implies that $x = 0$.
- normalized if $\tau(1) = 1$ in the case that A is unital.

Haagerup mentioned in [Haa14] that Kirchberg proved the existence of a unital C^*-algebra with a 1-quasitrace τ which is not a 2-quasitrace. So τ is not a trace. Later, we will see that every 2-quasitrace is an n-quasitrace for every $n \in \mathbb{N}$, and it is still an open question whether every 2-quasitrace is a trace. The main result in [Haa14] is that 2-quasitraces on exact unital C^*-algebras are traces.

Corollary 2.2: *Let M be an AW^*-algebra. Then M is finite if and only if there exists a faithful family of quasitraces, that is, a family $(\tau_i)_{i\in I}$ of quasitraces such that $\tau_i(x^*x) = 0$ for all $i \in I$ if and only if $x = 0$.*

Proof. If M is finite, the existence of a faithful family of quasitraces follows directly from the center-valued quasitrace.
If, otherwise, $(\tau_i)_{i\in I}$ is a faithful family of quasitraces, and $v \in M$ such that $v^*v = 1$, then for every $i \in I$,

$$\tau_i(1 - vv^*) = \tau_i(1) - \tau_i(vv^*) = \tau_i(1) - \tau_i(v^*v) = \tau_i(1) - \tau_i(1) = 0.$$

Since $1 - vv^* \geq 0$, it follows from the faithfulness that $1 - vv^* = 0$, and so $1 = vv^*$, and M is finite. □

First, we want to examine quasitraces on AW^*-algebras, then we want to use a result from [BH82] and the center-valued quasitrace from Theorem 1.27 to combine these for 2-quasitraces on arbitrary C^*-algebras.

Lemma 2.3: *Let τ be a 1-quasitrace on an AW^*-algebra M. Then τ is order-preserving on M_{sa}. Furthermore, τ is continuous, in particular, $|\tau(x) - \tau(y)| \leq 2\|x - y\|$ for all $x, y \in M$.*

Proof. The idea of the proof is the same as for the center-valued quasitrace in Theorem 1.27:
We will again use Lemma 1.24: Let $a, b \in M_{sa}$ be self-adjoint elements with $a \leq b$. Then for $\lambda > 0$ we get $E_{(\lambda,\infty)}(a) \precsim E_{(\lambda,\infty)}(b)$. So there exists $v \in M$ such that $E_{(\lambda,\infty)}(a) = v^*v$ and $vv^* \leq E_{(\lambda,\infty)}(b)$, so we can compute

$$\tau(E_{(\lambda,\infty)}(a)) = \tau(v^*v) = \tau(vv^*) \leq \tau(E_{(\lambda,\infty)}(b)),$$

where we obtain the last inequality from the fact that vv^* and $E_{(\lambda,\infty)}(b)$ commute, so $0 \leq \tau(E_{(\lambda,\infty)}(b) - vv^*) = \tau(E_{(\lambda,\infty)}(b)) - \tau(vv^*)$.
We again get the inequality $\tau(a) \leq \tau(b)$ by integrating over the spectral projections. The proof of the continuity of τ is then the same as for the center-valued quasitrace. □

Remark 2.4: Let M be an AW^*-algebra and τ a 1-quasitrace on M. As stated in [BH82], it follows from [Ber72], §17, Theorem 1, that there exists a largest finite central projection $p \in M$, such that $\tau(px) = \tau(x)$ for every $x \in M$. So, it suffices to look at 1-quasitraces on finite AW^*-algebras instead of arbitrary AW^*-algebras.

The following is Theorem II.1.7 in [BH82]. The proof needs some background information about the construction of the center-valued dimension function from Theorem 1.26 and simple projections, so we only sketch the proof.

Theorem 2.5: *Let τ be a 1-quasitrace on a finite AW^*-algebra M, and let T be the center-valued quasitrace constructed in Theorem 1.27. Then τ is uniquely expressible in the form $\tau = \varphi \circ T$ for a positive functional φ on $\mathcal{Z}(M)$.*

Proof. Recall that the central cover for a projection p is the smallest central projection $C(p)$ such that $C(p)p = p$ and that in AW^*-algebras, a central cover for a projection always exists. For more details, see §6 in [Ber72].
Let p be a projection in M. We call p simple if there exist projections $p_1, ..., p_n$ in M, mutually orthogonal such that

$$C(p) = p_1 + ... + p_n \text{ and } p_i \sim p$$

for every $i = 1, ..., n$. Let D be the center-valued dimension function of Theorem 1.26. From the construction of T and D, we know that for simple projections p with decomposition as above, $T(p) = D(p) := \frac{1}{n}C(p)$.
Furthermore, for an arbitrary non-zero projection $p \in M$, there exist simple projections $(p_i)_{i\in I}$ with $p = \sup_{i\in I} p_i$ and then $T(p) = D(p) := \sum_{i\in I} p_i$. This is done in Chapter 6 of [Ber72].
Now, suppose that τ is a quasitrace on M. Define $\tau_1 := \tau_{\mathcal{Z}(M)} \circ T$.
It is easy to see that τ and τ_1 agree on simple projections and also on finite linear combinations of orthogonal simple projections.
But then, one can show that for every $\varepsilon > 0$ and every projection $p \in M$, there exist finite families of orthogonal simple projections $(q_i)_{i\in I}, (r_j)_{j\in J}$ with

$$\sum_{i\in I} q_i \leq p \leq \sum_{j\in J} r_j \text{ and } \left|\tau_1\left(\sum_{i\in I} q_i - \sum_{j\in J} r_j\right)\right| < \varepsilon.$$

It follows that $|\tau(p) - \tau_1(p)| < \varepsilon$, which implies that τ and τ_1 agree on projections. Since quasitraces are continuous, and AW^*-algebras have real rank zero, we conclude that $\tau = \tau_1$. □

Now we get to an important corollary. There are some more results which follow from the last theorem, but we will only state the following one. For more details, see [BH82].

Corollary 2.6: *Let τ be a 1-quasitrace on a finite AW^*-algebra M, then τ is an n-quasitrace for every $n \in \mathbb{N}$.*

Proof. By Theorem 1.23 $M_n(M)$ is again a finite AW^*-algebra, so there is a center-valued quasitrace $T_n : M_n(M) \to \mathcal{Z}(M_n(M))$. But it is standard that $\mathcal{Z}(M_n(M)) \cong \mathcal{Z}(M)$, and so it follows that τ is an n-quasitrace for every $n \in \mathbb{N}$.

□

Definition 2.7: A rank function D is a map $D : A \to [0, 1]$ such that

(i) D is normalized, that is $\sup_{a\in A} D(a) = 1$.

(ii) For all $a, b \in A$ with $a \perp b$, we have $D(a + b) = D(a) + D(b)$.

(iii) For all $a \in A$: $D(a) = D(a^*a) = D(aa^*) = D(a^*)$.

(iv) For all positive elements $0 \leq a \leq b$: $D(a) \leq D(b)$.

(v) For all $a, b \in A$ with $a \lesssim_c b$, say that a is Cuntz sub-equivalent to b, that means that there exist sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ of elements in A such that $(x_n b y_n)_{n\in\mathbb{N}}$ converges in norm to a, we have got that $D(a) \leq D(b)$.

A dimension function is a map $D\colon \bigcup_{n\in\mathbb{N}} M_n(A) \to [0,\infty)$ which satisfies properties $(i) - (v)$ above.
A rank (dimension) function is called subadditive if $D(a + b) \leq D(a) + D(b)$ for all $a, b \in A$.
A rank (dimension) function is called weakly subadditive if $D(a + b) \leq D(a) + D(b)$ for all positive, commuting $a, b \in A$.

The next Theorem combines Theorems II.2.2 and II.3.1 of [BH82].

Theorem 2.8: *There is a natural one-to-one correspondence between quasitraces and lower semi-continuous weakly additive rank functions on a C^*-algebra. Furthermore, 2-quasitraces correspond exactly to the lower semi-continuous subadditive dimension functions.*

Proof. We only sketch the proof: For $\varepsilon > 0$ define $f_\varepsilon\colon [0,\infty) \to [0,1]$ via

$$f_\varepsilon(t) = \begin{cases} 0 & 0 \leq t \leq \frac{\varepsilon}{2} \\ \frac{\varepsilon}{2}t - 1 & \frac{\varepsilon}{2} \leq t \leq \varepsilon \\ 1 & \varepsilon \leq t \end{cases}.$$

If now τ is a quasitrace, the rank function D_τ is given by

$$D(x) = \sup_{\varepsilon>0} \tau(f(|x|)).$$

Suppose now that D is a rank function. Let $B \subseteq A$ be a commutative subalgebra. Then $D_{|B}$ is a rank function. Proposition 1.2.1 in [BH82] gives us a one-to-one correspondence between rank functions and positive functionals on commutative C^*-algebras. So, $D_{|B}$ induces a positive functional τ_D^B on B. This defines τ_D on all normal elements. For an arbitrary element $a \in A$ define τ_D as follows

$$\tau_D(a) = \tau_D(Re(a)) + i\tau_D(Im(a)).$$

Then τ_D will be a well-defined quasitrace on A. □

We need an induced quasitrace on quotient C^*-algebras, so we need the following theorem. This is Proposition 3.3 in [Haa14] and uses Theorem 1.1.17 in [BH82].

Theorem 2.9: *Let τ be a 2-quasitrace on a C^*-algebra A. We define the kernel of τ:*

$$I_\tau := \{a \in A | \tau(a^*a) = 0$$

Then I_τ is a two-sided closed ideal of A, and there exists a quasitrace $\overline{\tau}$ on A/I_τ such that

$$\tau(a) = \overline{\tau}(\pi(a))$$

for all $a \in A$ where $\pi\colon A \to A/I_\tau$ is the quotient map.

Proof. Let D_τ be the corresponding dimension function to τ, and then define $\ker(D_\tau) := \{x \in A | D_\tau(x) = 0\}$. Theorem 1.1.17 in [BH82] shows that $\ker(D_\tau)$ is a two-sided closed ideal in A and that D_τ induces a lower semi-continuous dimension function $\overline{D}_\tau$ on $A \backslash \ker(D_\tau)$.
We note that $I_\tau = \ker D_\tau$ and the quasitrace corresponding to $\overline{D}_\tau$ is exactly the desired quasitrace $\overline{\tau}$. □

The following is Corollary II.2.4 in [BH82]. The proof of this Corollary is based on an ultraproduct construction for D_τ, the dimension function of A corresponding to the 2-quasitrace τ, which turns out to be a finite AW^*-algebra. The construction of the AW^*-algebra is very long, so we omit it here and later give an AW^*-completion for 2-quasitraces on unital C^*-algebras.

Theorem 2.10: *Let τ be a 2-quasitrace on a C^*-algebra A, then there exists a finite AW^*-algebra M, a 2-quasitrace $\overline{\tau}$ on M, and a unital *-homomorphism $\theta\colon A \to M$ such that $\tau = \theta \circ \overline{\tau}$*

Corollary 2.11: *Let τ be a 2-quasitrace on a C^*-algebra A.*

(i) *τ is an n-quasitrace for every $n \in \mathbb{N}$.*

(ii) *τ is order-preserving on A_{sa}.*

(iii) *τ is continuous.*

(iv) *τ is bounded.*

Proof. (i): The quasitrace τ is of the form $\tau = \theta \circ \overline{\tau}$ for a *-homomorphism $\theta\colon A \to M$ and an n-quasitrace $\overline{\tau}$ on M. Since θ is completely positive, it follows that τ is also an n-quasitrace.
(ii): Again, $\overline{\tau}$ and θ are both order-preserving, and so is τ.
(iii): This again follows from (ii) or again from the fact that τ is the composition from continuous maps.
(iv): From Remark I.1.19(b) in [BH82], we know that dimension functions are bounded, and then it follows from the fact that $\tau(a) \leq D_\tau(a)$ for all positive $a \in A$ with $\|a\| \leq 1$. □

Remark 2.12: From now on, when we write quasitrace, we will always mean 2-quasitraces. If A is a unital C^*-algebra, we write $QT(A)$ for the set of normalized 2-quasitraces on A.

We want to state this last corollary without the proof. It is a combination of Theorem II.4.4 and Proposition II.4.5 in [BH82].

Corollary 2.13: *If A is a unital C^*-algebra, then $QT(A)$ is a compact convex set. Furthermore, $QT(A)$ is a simplex, and the set $T(A)$ of normalized traces is a closed face in $QT(A)$.*

2.2 Metric from Quasitraces

In this section, we want to construct a metric from a quasitrace like it is done in [Haa14]. Furthermore, we want to describe the connection from this metric to AW^*-algebras like from the 2-norm and von Neumann algebras.

Definition 2.14: Let τ be a quasitrace on a C^*-algebra A. Then for $x \in A$, we define

$$\|x\|_{2,\tau} := \tau(x^*x)^{1/2}.$$

Remark 2.15: Obviously $\|.\|_{2,\tau}$ is not a norm in general because it does not satisfy the triangle inequality. But if τ is a trace, this is of course true. We will fix this somehow so that we get a metric instead of a norm.

The next Lemma is an important tool which we will use frequently.

Lemma 2.16: *Let A be a unital C^*-algebra, and τ a quasitrace on A. Then*

(i) $\tau(a+b)^{\frac{1}{2}} \leq \tau(a)^{\frac{1}{2}} + \tau(b)^{\frac{1}{2}}$ *for all positive $a, b \in A$.*

(ii) $\|x+y\|_{2,\tau}^{2/3} \leq \|x\|_{2,\tau}^{2/3} + \|y\|_{2,\tau}^{2/3}$ *for all $x, y \in A$.*

(iii) $\|xy\|_{2,\tau} \leq \|x\|_{2,\tau}\|y\|$ *and* $\|xy\|_{2,\tau} \leq \|x\|\|y\|_{2,\tau}$ *for all $x, y \in A$.*

Proof. (i) : This is essentially a matrix trick: Define

$$x := \begin{pmatrix} a^{1/2} & 0 \\ b^{1/2} & 0 \end{pmatrix}.$$

Then

$$x^*x = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix} \text{ and } xx^* = \begin{pmatrix} a & a^{1/2}b^{1/2} \\ b^{1/2}a^{1/2} & b \end{pmatrix}.$$

For $\lambda > 0$, we define

$$x_\lambda := \begin{pmatrix} \lambda^{1/2}a^{1/2} & 0 \\ -\lambda^{1/2}b^{1/2} & 0 \end{pmatrix}.$$

Then,

$$x_\lambda x_\lambda^* = \begin{pmatrix} \lambda a & -a^{1/2}b^{1/2} \\ -b^{1/2}a^{1/2} & \lambda^{-1}b \end{pmatrix}.$$

We obtain the following inequality:

$$xx^* \leq xx^* + x_\lambda x_\lambda^* = \begin{pmatrix} (1+\lambda)a & 0 \\ 0 & (1+\lambda^{-1})b \end{pmatrix}.$$

Since τ_2 is order-preserving, we get:

$$\tau(a+b) = \tau_2(x^*x) = \tau_2(xx^*) \leq \tau_2\left(\begin{pmatrix} (1+\lambda)a & 0 \\ 0 & (1+\lambda^{-1})b \end{pmatrix}\right).$$

Let $y := b^{1/2} \otimes e_{12}$, then $\tau_2(b \otimes e_{22}) = \tau_2(y^*y) = \tau_2(yy^*) = \tau_2(b \otimes e_{11}) = \tau(b)$.
Since $(1+\lambda)a \otimes e_{11}$ and $(1+\lambda^{-1})b \otimes e_{22}$ commute in $M_2(A)$, we get:

$$\tau(a+b) \leq \tau_2\left(\begin{pmatrix} (1+\lambda)a & 0 \\ 0 & (1+\lambda^{-1})b \end{pmatrix}\right) = (1+\lambda)\tau(a) + (1+\lambda^{-1})\tau(b).$$

Suppose first that $\tau(a) > 0$ and $\tau(b) > 0$: Then a routine calculus shows that the function $f\colon (0,\infty) \to (0,\infty)$, $\lambda \mapsto (1+\lambda)\tau(a) + (1+\lambda^{-1})\tau(b)$ takes its minimum at $\lambda = \left(\frac{\tau(b)}{\tau(a)}\right)^{1/2}$. Then:

$$\begin{aligned}
\tau(a+b) &\leq f\left(\left(\frac{\tau(b)}{\tau(a)}\right)^{1/2}\right) \\
&= \left(1 + \left(\frac{\tau(b)}{\tau(a)}\right)^{1/2}\right)\tau(a) + \left(1 + \left(\frac{\tau(a)}{\tau(b)}\right)^{1/2}\right)\tau(b) \\
&= \tau(a) + \tau(b)^{1/2}\tau(a)^{1/2} + \tau(b) + \tau(a)^{1/2}\tau(b)^{1/2} \\
&= (\tau(a)^{1/2} + \tau(b)^{1/2})^2.
\end{aligned}$$

This clearly implies, $\tau(a+b)^{1/2} \leq \tau(a)^{1/2} + \tau(b)^{1/2}$.
Suppose now that $\tau(a) = 0$, then $\tau(a+b) \leq (1+\lambda^{-1})\tau(b)$ for every $\lambda > 0$. Now letting $\lambda \to \infty$, we get $\tau(a+b) \leq \tau(b)$, which clearly implies $\tau(a+b)^{1/2} \leq \tau(a)^{1/2} + \tau(b)^{1/2}$.
For the case $\tau(b) = 0$, we let $\lambda \to 0$ and get the result.
(ii) : This is similar to the previous part. Let $x, y \in A$ and $\lambda > 0$:

$$\begin{aligned}
(x+y)^*(x+y) &\leq (x+y)^*(x+y) + (\lambda^{1/2}x - \lambda^{-1/2}y)^*(\lambda^{1/2}x - \lambda^{-1/2}y) \\
&= (1+\lambda)x^*x + (1+\lambda^{-1})y^*y.
\end{aligned}$$

We again consider that quasitraces are order-preserving and part (i) of this Lemma:

$$\begin{aligned}
\|x+y\|_{2,\tau} &= \tau((x+y)^*(x+y))^{1/2} \leq \tau((1+\lambda)x^*x + (1+\lambda^{-1})y^*y)^{1/2} \\
&\leq (1+\lambda)^{1/2}\tau(x^*x)^{1/2} + (1+\lambda^{-1})^{1/2}\tau(y^*y)^{1/2} \\
&= (1+\lambda)^{1/2}\|x\|_{2,\tau} + (1+\lambda^{-1})^{1/2}\|y\|_{2,\tau}.
\end{aligned}$$

Now suppose again that $\|x\|_{2,\tau} > 0$ and $\|y\|_{2,\tau} > 0$ and look at the function

$$g\colon (0,\infty) \to (0,\infty),\ \lambda \mapsto (1+\lambda)^{1/2}\|x\|_{2,\tau} + (1+\lambda^{-1})^{1/2}\|y\|_{2,\tau}.$$

The function g takes its minimum at $\lambda = \left(\frac{\|y\|_{2,\tau}}{\|x\|_{2,\tau}}\right)^{2/3}$ and the minimum is

$$\begin{aligned} g\left(\left(\frac{\|y\|_{2,\tau}}{\|x\|_{2,\tau}}\right)^{2/3}\right) &= \left(1+\left(\frac{\|y\|_{2,\tau}}{\|x\|_{2,\tau}}\right)^{2/3}\right)^{1/2}\|x\|_{2,\tau} + \left(1+\left(\frac{\|x\|_{2,\tau}}{\|y\|_{2,\tau}}\right)^{2/3}\right)^{1/2}\|y\|_{2,\tau} \\ &= (\|x\|_{2,\tau}^2 + \|x\|_{2,\tau}^{4/3}\|y\|_{2,\tau}^{2/3})^{1/2} + (\|y\|_{2,\tau}^2 + \|y\|_{2,\tau}^{4/3}\|x\|_{2,\tau}^{2/3})^{1/2} \\ &= \|x\|_{2,\tau}^{2/3}(\|x\|_{2,\tau}^{2/3} + \|y\|_{2,\tau}^{2/3})^{1/2} + \|y\|_{2,\tau}^{2/3}(\|x\|_{2,\tau}^{2/3} + \|y\|_{2,\tau}^{2/3})^{1/2} \\ &= (\|x\|_{2,\tau}^{2/3} + \|y\|_{2,\tau}^{2/3})^{1/2}(\|x\|_{2,\tau}^{2/3} + \|y\|_{2,\tau}^{2/3}) = (\|x\|_{2,\tau}^{2/3} + \|y\|_{2,\tau}^{2/3})^{3/2}. \end{aligned}$$

This shows the claimed inequality in the first case. The cases $\|x\|_{2,\tau} = 0$ and $\|y\|_{2,\tau} = 0$ follow again by letting $\lambda \to \infty$ and $\lambda \to 0$, respectively.

(iii) : Since $y^*x^*xy \le \|x\|^2 y^*y$ and τ is linear on $C^*(y^*y)$, the first inequality follows from the fact that τ is order-preserving. The second inequality then follows by using $\|z\|_{2,\tau} = \|z^*\|_{2,\tau}$ for every $z \in A$. □

Definition 2.17: Let τ be a quasitrace on a C^*-algebra A. For $x, y \in A$, we define:

$$d_\tau(x,y) := \|x-y\|_{2,\tau}^{2/3}.$$

From the previous lemma, we see that d_τ is quasimetric. If τ is faithful, this defines a metric on A.

The definition of the metric d_τ and all the properties of this metric are from Haagerup and were introduced in [Haa14].

Proposition 2.18: *Let τ be a faithful quasitrace on a C^*-algebra A.*

(i) The sum is continuous in d_τ.

(ii) Involution is continuous in d_τ.

(iii) The product is continuous on norm-bounded subsets in d_τ.

(iv) The map $a \mapsto \tau(a)$ is continuous in d_τ on A_+.

Proof. (i) : Is clear since $d_\tau(x+x', y+y') \le d_\tau(x,y) + d_\tau(x',y')$ for all $x, x', y, y' \in A$.
(ii) : Is also clear by the fact that

$$\|x\|_{2,\tau} = \tau(x^*x)^{1/2} = \tau(xx^*)^{2/3} = \|x^*\|_{2,\tau}.$$

(iii) : Let $x, y, y', y' \in A$, then with Lemma 2.16, we see:

$$\begin{aligned}\|xy - x'y'\|_{2,\tau}^{2/3} &\leq \|xy - x'y\|_{2,\tau}^{2/3} + \|x'y - x'y'\|_{2,\tau}^{2/3} \\ &\leq \|x - x'\|_{2,\tau}^{2/3}\|y\|^{2/3} + \|x'\|^{2/3}\|y - y'\|_{2,\tau}^{2/3}.\end{aligned}$$

This inequality shows the continuity of the product on norm-bounded subsets.
(iv) : Let $a, b \in A_+$ be two positive elements, then

$$a \leq b + |a - b| \text{ and } b \leq a + |a - b|.$$

Since τ is order-preserving, we got $\tau(a) \leq \tau(b+|a-b|)$ and also $\tau(a)^{1/2} \leq \tau(b+|a-b|)^{1/2}$. With Lemma 2.16 (i), we get $\tau(b + |a - b|)^{1/2} \leq \tau(b)^{1/2} + \tau(|a - b|)^{1/2}$. So, it is clear that $\tau(a)^{1/2} - \tau(b)^{1/2} \leq \tau(|a - b|)^{1/2}$. Doing the same for b delivers $\tau(b)^{1/2} - \tau(a)^{1/2} \leq \tau(|a - b|)^{1/2}$, so it follows that

$$|\tau(a)^{1/2} - \tau(b)^{1/2}| \leq \tau(|a - b|)^{1/2}.$$

The quasitrace τ is linear on $C^*(|a - b|, 1)$, hence, we can use the Cauchy Schwarz inequality on $C^*(|a - b|, 1)$ and conclude

$$\begin{aligned}|\tau(a)^{1/2} - \tau(b)^{1/2}| &\leq \tau(|a - b|1)^{1/2} \\ &\leq \tau(|a - b|^2)^{1/4}\tau(1)^{1/4} = \|a - b\|_{2,\tau}^{1/2}\tau(1)^{1/4}.\end{aligned}$$

This proves the claim. □

Lemma 2.19: *Let A be a C^*-algebra and τ a faithful quasitrace on A. Then the closed unit ball of A is closed in d_τ.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the closed unit ball of A, converging to x in d_τ. Define $a_n := x_n^* x_n$ and $a := x^* x$. Since the product is continuous in d_τ on norm-bounded sets, it is obvious that a_n converges to a in d_τ, and we can also deduce that for every $p \in \mathbb{N}$, a_n^p converges to a^p in d_τ. With the previous Lemma 2.18, we see that

$$\tau(a_n^p) \to \tau(a^p) \text{ for every } p \in \mathbb{N}.$$

Let μ_n be the measure on $\sigma(a_n)$ given by the linear functional $\tau_{|C^*(a_n,1)}$, and let μ be the measure on $\sigma(a)$ given by the linear functional $\tau_{|C^*(a,1)}$.
We can consider all the measures as measures in the interval $J := [0, \max\{1, \|a\|\}]$ because all a_n are in the closed unit ball of A. Since $\tau(a_n^p) \to \tau(a^p)$ for all $p \in \mathbb{N}$, we see that μ_n converges to μ in the w^*-topology on $C(J)^*$. Furthermore, μ_n has support in $[0, 1]$ for all $n \in \mathbb{N}$, hence, μ has also support in $[0, 1]$. From the fact that τ is faithful, we obtain that $\operatorname{supp}(\mu) = \sigma(a)$, and the C^*-equation then delivers $\|x\|^2 = \|a\| \leq 1$. □

The following is Lemma A.3.3 in [SS08]:

Lemma 2.20: *Let A be a C^*-algebra and τ a faithful normalized trace on A.*

- *If the closed unit ball of A is complete in the $\|.\|_{2,\tau}$-norm, then A is a von Neumann algebra and τ is normal.*
- *If τ is normal and A is a von Neumann algebra, then the closed unit ball of A is complete in the $\|.\|_{2,\tau}$-norm.*

Definition 2.21: Let τ be a quasitrace on an AW^*-algebra M. Then τ is called normal if for every orthogonal family of projections $(p_i)_{i\in I}$, the following holds:

$$\tau\left(\sup_{i\in I} p_i\right) = \sum_{i\in I} \tau(p_i).$$

Remark 2.22: With Remark 1.19, it is easy to see that a normal trace on a von Neumann algebra is also a normal quasitrace.

The following four theorems are all from Haagerup and can be found in [Haa14].

Theorem 2.23: *Let A be a C^*-algebra with faithful quasitrace τ. If the closed unit ball of A is complete in d_τ, then A is an AW^*-algebra and τ is normal.*

Proof. From Theorem 1.15, we know that it suffices to show that every masa has Stonean spectrum. So, let B be a masa in A. Since the closed unit ball of B is closed in d_τ, the closed unit ball of B is also complete in d_τ. Since τ is linear on B, $\|.\|_{2,\tau}$ is a norm, and the closed unit ball is also complete in this norm. So, by Lemma 2.20, we have that B is a von Neumann algebra, and $\tau_{|B}$ is normal. With Theorem 1.20, we see that B has Hyperstonean spectrum, in particular, it is Stonean. So, A is an AW^*-algebra. The normality of τ on every masa as a trace ensures that τ is also normal as a quasitrace. □

This is Proposition 3.10 in [Haa14]. We will not give the proof now since it is essentially the same as the proof in Theorem 3.9. There we show that the closed unit ball of a finite AW^*-algebra is complete in a universal 2-metric which arises from the center-valued quasitrace in Theorem 1.27.

Theorem 2.24: *Let τ be a faithful, normal quasitrace on an AW^*-algebra M. Then the closed unit ball of M is complete in the d_τ-metric.*

The following is a version of Kaplansky density theorem:

Theorem 2.25: *Let A be a unital C^*-algebra with a faithful quasitrace τ and B be a unital C^*-subalgebra. Then the following two conditions are equivalent:*

(i) B is dense in A in the d_τ-metric.

(ii) The closed unit ball of B is dense in the closed unit ball of A in the d_τ-metric.

Proof. The direction $(ii) \Rightarrow (i)$ is trivial, so we show the other direction: Let

$$f\colon \mathbb{R} \to \mathbb{R}\, t \mapsto \frac{2t}{1+t^2}.$$

Then $\|f\|_\infty \leq 1$, and the restriction of f to $[-1,1]$ is a bijective map to $[-1,1]$, and so it is a homeomorphism. Let g be the inverse map of $f_{|[-1,1]}$. Since f is an odd function on $\mathbb{R}$, so also is g.
Let $x \in A_1$ and

$$a = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in M_2(A).$$

Then clearly a is self-adjoint and contained in the closed unit ball of $M_2(A)$. We know that g is an odd function, so there exists a $y \in A$ such that

$$g(a) = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}.$$

f is the inverse of g, so $f(g(a)) = a$, which implies

$$x = 2y(1+y^*y)^{-1} = 2(1+yy^*)^{-1}y.$$

Now choose a sequence $(y_n)_{n\in\mathbb{N}}$ such that $d_\tau(y_n, y) \to 0$. Then let

$$x_n := 2y_n(1+y_n^*y_n)^{-1}.$$

From $\sup_{t\geq 0} \frac{4t}{(1+t)^2} = 1$, we can conclude that

$$x_n^*x_n = 4(1+y_n^*y_n)^{-1}y_n^*y_n(1+y_ny_n^*)^{-} \leq 1.$$

So $\|x_n\|^2 = \|x_n^*x_n\| \leq 1$ and x_n is in the unit ball of A.
We get

$$\begin{aligned} x_n - x =& 2y_n(1+y_n^*y_n)^{-1} - 2(1+yy^*)^{-1}y \\ =& 2(1+yy^*)^{-1}((1+yy^*)y_n - y(1+y_n^*y_n))(1+y_n^*y_n)^{-1} \\ =& 2(1+yy^*)^{-1}(y_n - y)(1+y_n^*y_n)^{-1} \\ &+ 2(1+yy*)^{-1}y(y^* - y_n^*)y_n(1+y_n^*y_n)^{-1}. \end{aligned}$$

Now, recall that $(1+yy^*)^{-1}, (1+y_n^*y_n)^{-1}, 2(1+yy^*)^{-1}y$, and $2y_n(1+y_n^*y_n)^{-1}$ all have C^*-norm at most 1. Then we can apply Lemma 2.16 and 2.18, and get:

$$d_\tau(x_n, x) \leq 2^{2/3}d_\tau(y_n, y) + 2^{-2/3}d_\tau(y_n^*, y^*) \to 0.$$

So, x is in the d_τ closure of the closed unit ball of B and this finishes the proof. □

Theorem 2.26: *Let M be a finite AW^*-algebra with a faithful normal quasitrace τ. Let A be a unital C^*-subalgebra of M. Then the d_τ-closure of A is the smallest AW^*-subalgebra of M containing A.*

Proof. Denote with $B := \overline{A}^{d_\tau}$ the d_τ-closure of A in M. With Proposition 2.18, we see that B is closed under sums and involution. For $a, b \in B$, we can use Theorem 2.25 to find bounded sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, which converge to a and b in d_τ. But then again, using Proposition 2.18, we see that $a_n b_n$ is converging to ab, hence, B is closed under products and also a *-subalgebra.
To see that B is a C^*-subalgebra, note that norm-convergence implies convergence in d_τ.
Again, with Theorem 2.25 and Proposition 2.19, we see that B itself is an AW^*-algebra.
To be an AW^*-subalgebra of M, it is necessary that the supremum of orthogonal families of projections $(p_i)_{i\in I}$ is the same whether it is calculated in M or in B: But therefore, we note that $(\sum_{i\in J} p_i)_{J\in\mathcal{J}}$, where $\mathcal{J}$ is the family of finite subsets of I, converges in d_τ to $\sup_{i\in I} p_i$.
Now, let C be another AW^*-subalgebra of M, containing A. Then by Theorem 2.24, the closed unit ball of C is complete in d_τ. Again, using Theorem 2.25, we see that C is closed in d_τ and this implies $B \subseteq C$. □

2.3 The AW^*-Completion

In this chapter, we want to construct an AW^*-algebra from a unital C^*-algebra with a faithful quasitrace. For this, we need to construct a huge AW^*-algebra as an ultraproduct of C^*-algebras and then take the d_τ-closure of A in this AW^*-algebra. This construction is due to [Haa14] and this chapter is based on this work.

Theorem 2.27: *Let (X_n, d_n) be a sequence of metric spaces with*

$$\sup_{n\in\mathbb{N}} diam(X_n) < \infty.$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and set $X := \prod_{n\in\mathbb{N}} X_n$. We define an equivalence relation on X:

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \Leftrightarrow \lim_{\mathcal{U}} d_n(x_n, y_n) = 0.$$

Then the space $X/\sim$ is complete in the metric $d([(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}]) = \lim_{\mathcal{U}} d_n(x_n, y_n)$.

Proof. The proof that $\sim$ is an equivalence relation is standard, also that d is a metric and is omitted here. We only show that $X/\sim$ is complete in d. So, let $(z^i)_{i\in\mathbb{N}}$ be a Cauchy sequence in d. To show that z^i converges, it suffices to show that a subsequence converges, so we may assume, without loss of generality, that

$$d(z^i, z^{i+1}) < 2^{-i} \text{ for every } i \in \mathbb{N}.$$

Choose a sequence $x^i = (x^i_n)_{n\in\mathbb{N}}$ in X such that $[(x^i_n)_{n\in\mathbb{N}}] = z^i$. Then

$$d(x^i, x^{i+1}) = \lim_{\mathcal{U},n} d(x^i_n, x^{i+1}_n) < 2^{-i}.$$

So, from the definition of the convergence, there exist sets $\tilde{F}_i \in \mathcal{U}$ such that for every $n \in \tilde{F}_i$, then

$$d_n(x_n^i, x_n^{i+1}) < 2^{-i}.$$

So, we can choose sets $F_1 \supseteq F_2 \supseteq ... \supseteq F_i \supseteq ...$ in $\mathcal{U}$ such that for all $n \in F_i$:

$$d_n(x_n^i, x_n^{i+1}) < 2^{-i}.$$

Since $\mathcal{U}$ is a free ultrafilter, the filter of cofinite sets is contained in $\mathcal{U}$, so we can replace F_i with $F_i \cap \{i, i+1, ...\}$ and get that

$$\bigcap_{i \in \mathbb{N}} F_i = \emptyset.$$

Define $F_0 := \mathbb{N}$, and we see that $\mathbb{N}$ is the disjoint union of the $F_{i-1} \backslash F_i$, so

$$\mathbb{N} = \dot{\bigcup}_{i=1}^{\infty} F_{i-1} \backslash F_i.$$

Now we can define our limit $x = (x_n)_{n \in \mathbb{N}}$: Let $n \in \mathbb{N}$, then $n \in F_{i-1} \backslash F_i$ for some $i \in \mathbb{N}$. Then define $x_n := x_n^i$.
Let now $n \in F_i$. Then $n \in F_{j-1} \backslash F_j$ for some $j > i$, and for this j we compute:

$$\begin{aligned} d_n(x_n^i, x_n) &= d_n(x_n^i, x_n^j) \\ &\leq \sum_{k=i}^{j-1} d_n(x_n^k, x_n^{k+1}) \leq 2^{1-i} \end{aligned}$$

So, we can follow that

$$d([x^i], [x]) \leq \sup_{n \in F_i} d_n(x_n^i, x_n) \leq 2^{1-i}$$

and then it follows that z_i converges to $[x]$ in $X/\sim$. $\square$

Theorem 2.28: *Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of unital C^*-algebras with faithful normalized quasitraces τ_n. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Define*

$$\ell^\infty(A_n) := \{(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mid \sup_{n \in \mathbb{N}} \|x_n\| < \infty\}$$

and

$$\mathcal{J}_{\mathcal{U}} := \{(x_n)_{n \in \mathbb{N}} \in \ell^\infty(A_n) \mid \lim_{\mathcal{U}} \tau_n(x_n^* x_n) = 0\}.$$

Then $\mathcal{J}_{\mathcal{U}}$ is a norm-closed, two-sided ideal in $\ell^\infty(A)$. Furthermore, $\ell^\infty(A)/\mathcal{J}_{\mathcal{U}}$ is a finite AW^*-algebra with faithful normal quasitrace $\tau_{\mathcal{U}}([(x_n)_{n \in \mathbb{N}}]) := \lim_{\mathcal{U}} \tau_n(x_n)$.

Proof. First, we show that $\tau_{\mathcal{U}}\colon \ell^\infty(A_n) \to \mathbb{C}$, $(x_n)_{n\in\mathbb{N}} \mapsto \lim_{\mathcal{U}} \tau_n(x_n)$ is really a quasitrace. We first note that $|\tau(x_n)| \leq 2\|x_n\|$, hence, $\sup_{n\in\mathbb{N}} |\tau_n(x_n)| \leq \sup_{n\in\mathbb{N}} 2\|x_n\| < \infty$. So $\lim_{\mathcal{U}} \tau_n(x_n)$ exists. Then, it is easy to see that $\tau_{\mathcal{U}}$ is a 1-quasitrace because τ_n is a quasitrace for every $n \in \mathbb{N}$.
The quasitrace $\tau_{\mathcal{U},2}$ on $M_2(\ell^\infty(A_n) \cong \ell^\infty(M_2(A))$ is given by the formula

$$\tau_{\mathcal{U},2}((a_n)_{n\in\mathbb{N}})) = \lim_{\mathcal{U}} \tau_{n,2}(a_n),$$

where $\tau_{n,2}$ is the unique quasitrace on $M_2(A_n)$ and $(a_n)_{n\in\mathbb{N}} \in \ell^\infty(M_2(A_n))$. Then, we see that $\tau_{\mathcal{U},2}$ is again a quasitrace and for every $(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A_n)$ the following holds:

$$\tau_{\mathcal{U},2}((a_n)_{n\in\mathbb{N}} \otimes e_{11}) = \lim_{\mathcal{U}} \tau_{n,2}(a_n \otimes e_{11}) = \lim_{\mathcal{U}} \tau_n(a_n) = \tau_{\mathcal{U}}((a_n)_{n\in\mathbb{N}}).$$

We see that $\tau_{\mathcal{U},2}$ is the desired extension of $\tau_{\mathcal{U}}$, and $\tau_{\mathcal{U}}$ is a (2-)quasitrace.
We now apply Theorem 2.9 and see that $\ker(\tau_{\mathcal{U}}) = \mathcal{J}_{\mathcal{U}}$ is a norm-closed, two-sided ideal in $\ell^\infty(A_n)$. Also, we got an induced faithful quasitrace $\tau_{\mathcal{U}}$ on $\ell^\infty(A)/\mathcal{J}_{\mathcal{U}}$, and the following equation holds

$$\tau_{\mathcal{U}}([(x_n)_{n\in\mathbb{N}}] = \lim_{\mathcal{U}} \tau_n(x_n) \text{ for every } (x_n)_{n\in\mathbb{N}} \in \ell^\infty(A_n).$$

The last thing we need to show is that $\ell^\infty(A)/\mathcal{J}_{\mathcal{U}}$ is an AW^*-algebra. Therefore, we want to apply Theorem 2.23 and show that the closed unit ball of $\ell^\infty(A)/\mathcal{J}_{\mathcal{U}}$ is complete in $d_{\tau_{\mathcal{U}}}$:
So, we note that

$$\begin{aligned} d_{\tau_{\mathcal{U}}}([(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}]) &= \|[(x_n - y_n)_{n\in\mathbb{N}}]\|_{2,\tau_{\mathcal{U}}}^{2/3} \\ &= \left(\tau_{\mathcal{U}}([((x_n - y_n)(x_n - y_n)^*)_{n\in\mathbb{N}}])^{1/2}\right)^{2/3} \\ &= \lim_{\mathcal{U}} \left(\tau_n((x_n - y_n)(x_n - y_n)^*)^{1/2}\right)^{2/3} \\ &= \lim_{\mathcal{U}} \|x_n - y_n\|_{2,\tau_n}^{2/3} = \lim_{\mathcal{U}} d_{\tau_n}(x_n, y_n). \end{aligned}$$

Let $A_{n,1}$ be the closed unit ball of A_n. Then the closed unit ball of $\ell^\infty(A_n)$ is exactly $\ell^\infty(A_{n,1})$. Since surjective *-homomorphisms map the closed unit ball onto the closed unit ball, we see by Theorem 2.27 that the closed unit ball of $\ell^\infty(A_n)/\mathcal{J}_{\mathcal{U}}$ is complete in $\lim_{\mathcal{U}} d_{\tau_n} = d_{\tau_{\mathcal{U}}}$. So, by Theorem 2.23, $\ell^\infty(A_n)/\mathcal{J}_{\mathcal{U}}$ is a finite AW^*-algebra with a faithful normal quasitrace $\tau_{\mathcal{U}}$. □

The following corollary is originally from Haagerup, and it is a slight extension of Theorem 2.10 of Blackadar and Handelman:

Corollary 2.29: *Let A be a unital C^*-algebra with faithful quasitrace τ. Then there exists a finite AW^*-algebra M with faithful normal quasitrace $\overline{\tau}$ and an injective *-homomorphism $\theta\colon A \to M$ such that*

$$\tau = \overline{\tau} \circ \theta.$$

Proof. Apply the previous Theorem 2.28 to the sequence $A_n = A$ and $\tau_n = \tau$ for every $n \in \mathbb{N}$. Then set $M := \ell^\infty(A)/\mathcal{J}_\mathcal{U}$. The injective *-homomorphism $\theta\colon A \to M$ is given by $\theta(a) = [(a, a, a, ...)]$. □

Proposition 2.30: *Let A be a unital C^*-algebra with faithful quasitrace τ. Let (M_i, τ_i, θ_i) for $i = 1, 2$ be two triples satisfying the conditions of the previous Corollary 2.29. Let B_i denote the smallest AW^*-subalgebra generated by $\theta_i(A)$. Then there exists a unique *-isomorphism*

$$\pi\colon B_1 \to B_2$$

such that $\theta_1 = \pi \circ \theta_2$ and $\tau_1 = \pi \circ \tau_2$.

Proof. We will show that B_1 and B_2 are isomorphic to the same quotient C^*-algebra: From Theorem 2.26, we know that the smallest AW^*-subalgebra in M_i containing A is the d_{τ_i}-closure of A. So, with our version of the Kaplansky density theorem 2.25, we know that every element in B_i is the d_τ-limit of a bounded sequence of elements in $\theta_i(A)$. So, let

$$\tilde{A} := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(A) | x_n \text{ is a } d_\tau - \text{Cauchysequence}\}$$
$$\tilde{I} := \{(x_n)_{n\in\mathbb{N}} \in \tilde{A} | \lim_{n\to\infty} d_\tau(x_n, 0) = 0\}.$$

Then, both B_1 and B_2 are naturally isomorphic to $B := \tilde{A}/\tilde{I}$, and the faithful normal quasitrace $\overline{\tau}$ on B is given by $\overline{\tau}([(x_n)_{n\in\mathbb{N}}]) = \lim_{n\to\infty} \tau(x_n)$.

□

Definition 2.31: Let A be a unital C^*-algebra with a faithful quasitrace τ. Let $M_\tau := \tilde{A}/\tilde{I}$ be the AW^*-algebra described in the previous proposition. Then, we call $(M_\tau, \overline{\tau})$ the AW^*-completion of A with respect to τ. But often, we abuse the notation and just write τ for $\overline{\tau}$. Sometimes, we also write $\overline{A}^{d_\tau}$ for the AW^*-completion of A.

The following is Proposition 2.7.7 in [Evi18], which we will use without the proof.

Proposition 2.32: *Let M be a von Neumann algebra and $(\varphi_i)_{i\in I}$ a separating family of normal states. Then the strong operator topology on bounded subsets is induced by the family of seminorms $(\|.\|_{2,\varphi_i})_{i\in I}$.*

Corollary 2.33: *Let A be a unital C^*-algebra with faithful normalized trace τ. Then the AW^*-completion of A with respect to τ is isomorphic to the strong closure of the image under the GNS-representation π_τ.*

Proof. From Proposition 2.8.13 in [Evi18], we know that the trace τ extends to a faithful normal trace τ on $\pi_\tau(A)'' = \overline{\pi_\tau(A)}^{sot}$. From Proposition 2.32, we know that the strong operator topology on bounded subsets of $\pi_\tau(A)''$ is induced by the 2-norm $\|.\|_{2,\tau}$. Then with Kaplansky's density theorem for traces, it follows that $\overline{\pi_\tau(A)}^{sot} \cong \overline{\pi_\tau(A)}^{\|.\|_{2,\tau}}$. But the triple $(\overline{\pi_\tau(A)}^{\|.\|_{2,\tau}}, \pi_\tau, \tau)$ clearly satisfies the conditions of Corollary 2.30. So, the AW^*-completion of A is isomorphic to the closure of $\pi_\tau(A)$ in d_τ, which is isomorphic to $\overline{\pi_\tau(A)}^{\|.\|_{2,\tau}} \cong \overline{\pi_\tau(A)}^{sot}$. □

The first direction toward of next proposition is Proposition 3.12 of [Haa14]. For the sake of completeness, we give the proof of both directions.

Proposition 2.34: *Let A be a unital C^*-algebra with a faithful, normalized quasitrace τ. Then the AW^*-completion M_τ is a factor if and only if τ is an extreme point in $QT(A)$.*

Proof. Suppose first that τ is an extreme point in $QT(A)$.
If M_τ is not a factor, then there exists a central projection $p \in B$ with $p \neq 0$ and $p \neq 1$. Let $\theta\colon A \to M_\tau$ be the embedding, and let $\overline{\tau}$ be the quasitrace on M_τ. For $x \in A$, we define

$$\tau_1(x) = \overline{\tau}(p\theta(x))$$
$$\tau_2(x) = \overline{\tau}((1-p)\theta(x)).$$

It is obvious that $\tau = \tau_1 + \tau_2$ and then, after normalizing τ_1 and τ_2, we see that τ is a non-trivial convex combination of elements in $QT(A)$, which is a contradiction to the assumption that τ is an extreme point.
Now suppose that M_τ is a factor. Note that $\overline{\tau}$ is the unique normalized quasitrace on M_τ. We show that τ is an extreme point by contradiction: So suppose that $\tau = \lambda\tau_1 + (1-\lambda)\tau_2$ with $\tau_1, \tau_2 \in QT(A)$ and $\lambda \in (0,1)$. Then we get the following inequalities:

$$\lambda\tau_1(x^*x) \leq \tau(x^*x) \text{ and } (1-\lambda)\tau_2(x^*x) \leq \tau(x^*x) \text{ for every } x \in A.$$

So, we can extend τ_1 and τ_2 to normalized quasitraces $\overline{\tau}_1$ and $\overline{\tau}_2$ on M_τ, and $\overline{\tau}$ is a convex combination of $\overline{\tau}_1$ and $\overline{\tau_2}$, in particular, $\overline{\tau} = \lambda\overline{\tau_1} + (1-\lambda)\overline{\tau_2}$. Since $\overline{\tau}$ is the unique normalized quasitrace on M_τ, it follows that $\overline{\tau} = \overline{\tau}_1 = \overline{\tau}_2$. Thus, $\tau = \tau_1 = \tau_2$ and τ is an extreme point in $QT(A)$. □

2.4 A Tensor Product and McDuff Property for Finite AW^*-Factors

Let M, N be two finite von Neumann factors with faithful normal traces τ_N and τ_M. Let $L^2(M, \tau_M)$ and $L^2(N, \tau_N)$ be the two GNS representations of M and N with respect to the traces. The tensor product $M_1 \odot M_2$ acts naturally on $\mathcal{L}(L^2(M,\tau_M) \otimes L^2(N,\tau_N))$, and then the von Neumann tensor product $M\overline{\otimes}N$ of M and N is given by the weak closure of the image of $M \odot N$ in $\mathcal{L}(L^2(M,\tau_M) \otimes L^2(N,\tau_N))$. For details, see Chapter III.1.5 in [Bla06].
Let M_{2^∞} be the CAR algebra and let tr be the unique tracial state on M_{2^∞}. Then the unique hyperfinite II_1-factor $\mathcal{R}$ is given by $\mathcal{R} := \overline{\pi_{\mathrm{tr}}(M_{2^\infty})}^{sot}$. It is well known that $\mathcal{R}$ has a unique faithful tracial state which we also denote with tr. For this construction, see III.1.4 in [Bla06].
An important result is that $\mathcal{R}\overline{\otimes}\mathcal{R} \cong \mathcal{R}$.
Now let M be a finite AW^*-factor and let τ be the unique normalized quasitrace

on M. The goal of this chapter is with use of the AW^*-completion to construct a tensor product $M\tilde{\otimes}\mathcal{R}$, similarly to the von Neumann tensor product, such that the following holds:

(i) There are embeddings $M \hookrightarrow M\tilde{\otimes}\mathcal{R}$ and $\mathcal{R} \hookrightarrow M\tilde{\otimes}\mathcal{R}$.

(ii) $M\tilde{\otimes}\mathcal{R} \cong M\overline{\otimes}\mathcal{R}$ if M is a von Neumann algebra with trace τ.

(iii) $(M\tilde{\otimes}\mathcal{R})\tilde{\otimes}\mathcal{R} \cong M\tilde{\otimes}\mathcal{R}(\tilde{\otimes}\mathcal{R})$.

We need to normalize the quasitraces on $M_n(A)$ and therefore, we use Lemma 5.6 of [Haa14]:

Lemma 2.35: *Let A be a unital C^*-algebra and τ a normalized quasitrace on A. Let τ_n be the unique quasitrace on $M_n(A)$ such that $\tau_n(x \otimes e_{11}) = \tau(x)$ for all $x \in A$. Denote the unique normalized trace on M_n with tr_n. Set $\tau_n' := \frac{1}{n}\tau_n$. Then τ_n' is a quasitrace on $M_n(A)$ such that:*

(i) $\tau_n'(x \otimes 1) = \tau(x)$ for all $x \in A$.

(ii) $\tau_n'(1 \otimes y) = tr_n(y)$ for all $y \in M_n(\mathbb{C})$.

If τ is faithful, then τ_n' is faithful.

Proof. (i): First, we note that $\tau_n(x \otimes e_{ii}) = \tau(x)$ for every $1 \leq i \leq n$: We prove this in three steps: First, for positive x, then for self-adjoint elements, and then the general case. Let $x \in A$ be a positive element. Define $y := x^{1/2} \otimes e_{ij}$. Then

$$yy^* = x \otimes e_{jj} \text{ and } yy^* = x \otimes e_{ii}.$$

It follows

$$\tau_n(x \otimes e_{jj}) = \tau_n(yy^*) = \tau_n(y^*y) = \tau_n(x \otimes e_{ii}),$$

and we can conclude that $\tau_n(x \otimes e_{11}) = \tau_n(x \otimes e_{ii})$ for every $1 \leq i \leq n$.
The case for self-adjoint elements follows from decomposing into a positive and a negative part. So, let $a \in A$ be a self-adjoint element. Then a_+ and a_- commute and then

$$\begin{aligned}\tau_n(a \otimes e_{ii}) =& \tau_n((a_+ - a_-) \otimes e_{ii}) = \tau_n(a_+ \otimes e_{ii}) - \tau_n(a_- \otimes e_{ii}) \\ =& \tau_n(a_+ \otimes e_{jj}) - \tau_n(a_- \otimes e_{jj}) = \tau_n((a_+ - a_-) \otimes e_{jj}) = \tau_n(a \otimes e_{jj}).\end{aligned}$$

The general case follows by decomposition into real and imaginary parts.
Now, let $x \in A$, then

$$\tau_n'(x \otimes 1) = \frac{1}{n}\tau_n(x \otimes 1) = \frac{1}{n}\sum_{i=1}^{n}\tau_n(x \otimes e_{ii}) = \tau(x),$$

as desired.
(ii): This is clear since tr_n is the unique normalized trace on $M_n(\mathbb{C})$.
(iii): Suppose that τ is faithful. Let

$$x = \sum_{i,j=1}^{n} x_{ij} e_{ij}$$

such that $\tau_n'(xx^*) = 0$. Thus, $\|x\|_{2,\tau_n'} = 0$. We use Proposition 2.16 (iii) to see:

$$\| \|x_{ij} \otimes e_{11}\|_{2,\tau_n'} = \|(1 \otimes e_{1i})x(1 \otimes e_{j1})\|_{2,\tau_n'} \leq \|(1 \otimes e_{1i})\| \|x\|_{2,\tau_n'} \|(1 \otimes e_{j1})\| = 0.$$

With (i), we obtain

$$\tau(x_{ij}x_{ij}^*) = n\tau_n'(x_{ij}x_{ij}^* \otimes e_{11}) = 0$$

for all $1 \leq i,j \leq n$ and the faithfulness of τ, this implies $x_{ij} = 0$, hence $x = 0$. □

Let $\varphi_n : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C}),\ x \mapsto x \otimes (e_{11} + e_{22})$ be the connecting maps for the CAR algebra M_{2^∞}.
Now let M be a finite AW^*-factor with quasitrace τ_M. With the previous Lemma, we obtain a well-defined quasitrace

$$\tau : M \odot \left(\bigcup_{n \in \mathbb{N}} M_{2^n} \right),\ x \otimes e \mapsto \tau_{M,2^n}'(x \otimes e) \text{ for } e \in M_{2^n}(\mathbb{C}).$$

With Lemma 2.35, it is easy to check that $\tau(x \otimes e) = \tau(x \otimes \varphi_n(e))$ for $e \in M_{2^n}(\mathbb{C})$. The CAR algebra is nuclear as a direct limit of nuclear C^*-algebras, hence, there is only one C^*-norm $\|.\|$ on $M \odot M_{2^\infty}$. We equip $M \odot \bigcup_{n\in\mathbb{N}} M_{2^n}$ with this norm. Since τ is order-preserving, it is also continuous in this norm with the usual argument used in the proof of Theorem 1.27.
With the usual embedding

$$\bigcup_{n \in \mathbb{N}} M_{2^n} \hookrightarrow M \odot \bigcup_{n \in \mathbb{N}} M_{2^n},$$

we see that the completion of $M \odot \bigcup_{n\in\mathbb{N}} M_{2^n}$ is isomorphic to $M \otimes M_{2^\infty}$, and τ extends to a quasitrace on $M \otimes M_{2^\infty}$.

We define $M\tilde{\otimes}\mathcal{R}$ as the AW^*-completion $\overline{M \otimes M_{2^\infty}}^{d_\tau}$ of $M \otimes M_{2^\infty}$ with respect to τ.

First, we want to show the existence of the embeddings. The embedding of M is clear since there are embeddings

$$M \hookrightarrow M \otimes M_{2^\infty} \hookrightarrow M\tilde{\otimes}\mathcal{R}.$$

Next, we want to show the embedding of $\mathcal{R}$, so let $x \in \mathcal{R}$ be an arbitrary element. From Proposition 2.32, we know that the strong operator topology on bounded subsets on $\mathcal{R}$ is induced by $\|.\|_{2,tr}$. With the Kaplansky density theorem, we can choose a bounded sequence $(x_n)_{n\in\mathbb{N}}$, which converges to x in $\|.\|_{2,tr}$. Since there is an embedding

$$M_{2^\infty} \hookrightarrow M \otimes M_{2^\infty} \hookrightarrow M\tilde{\otimes}\mathcal{R},$$

it is easily seen that $(1 \otimes x_n)_{n\in\mathbb{N}}$ is a bounded Cauchy sequence in d_τ, and we can define an embedding

$$\mathcal{R} \hookrightarrow M\tilde{\otimes}\mathcal{R},$$

which sends x to the equivalence class of the bounded d_τ-Cauchy sequence $(1\otimes x_n)_{n\in\mathbb{N}}$.

Suppose now that M is a finite von Neumann factor with trace τ_M. Then the map τ on $M \otimes M_{2^\infty}$ is also a trace, and it follows that the AW^*-completion is isomorphic to the strong closure of $\pi_\tau(M \otimes M_{2^\infty})$. Then by the Bicommutant Theorem, we see that $\overline{\pi_\tau(M \otimes M_{2^\infty})}^{sot} = \overline{\pi_\tau(M \otimes M_{2^\infty})}^{wot} = \pi_\tau(M \otimes M_{2^\infty})''$, which is isomorphic to $M\overline{\otimes}\mathcal{R}$.

The associativity of the tensor product follows easily from the associativity of the algebraic tensor product and the usual embeddings.
Now we have proved all the properties we desired at the beginning of this section, and we can define what it means to be AW^*-McDuff:

Definition 2.36: Let M be a finite AW^*-factor. We say that M is AW^*-McDuff if

$$M \cong M\tilde{\otimes}\mathcal{R}.$$

Example 2.37: There are some obvious examples of AW^*-McDuff algebras. For every finite AW^*-factor M, the AW^*-factor $M\tilde{\otimes}\mathcal{R}$ is AW^*-McDuff:

$$M\tilde{\otimes}\mathcal{R}\tilde{\otimes}\mathcal{R} \cong M\tilde{\otimes}(\mathcal{R}\tilde{\otimes}\mathcal{R}) \cong M\tilde{\otimes}(\mathcal{R}\overline{\otimes}\mathcal{R}) \cong M\tilde{\otimes}\mathcal{R}.$$

Other obvious examples are the von Neumann-McDuff factors, which follow easily from the property (ii) of the AW^*-tensor product.

3 AW^*-Bundles

In this chapter, we want to generalize W^*-bundles over a compact Hausdorff space X. These types of bundles were introduced by Ozawa in [Oza13]. They were used in [Bos+15] to prove one direction in a special case of the Toms-Winter conjecture. A good reference for the theory of W^*-bundles is the Ph.D. thesis of Samuel Evington [Evi18].

3.1 Definition and First Examples

Definition 3.1: An AW^*-bundle over a compact Hausdorff space X is a unital C^*-algebra M, together with a unital embedding $C(X) \hookrightarrow \mathcal{Z}(M)$ and a map $E\colon M \to C(X)$ with the following properties:

(i) E is linear on commutative C^*-subalgebras.

(ii) $E(a + ib) = E(a) + iE(b)$ for all self-adjoint $a, b \in M$.

(iii) $E(x^*x) = E(xx^*) \geq 0$ for all $x \in M$.

(iv) E is faithful, which means $E(x^*x) = 0$ if and only if $x = 0$.

(v) $E(fx) = fE(x)$ for all $x \in M$ and all self-adjoint $f \in C(X)$.

(vi) E is norm continuous.

(vii) E extends uniquely to $E_n\colon M_n(M) \to C(X)$ for all $n \in \mathbb{N}$ such that $(i) - (vi)$ holds for E_n and that $E_n(x \otimes e_{11}) = E(x)$ for all $x \in M$.

For all $x \in M$, we obtain a quasitrace τ_x on M via

$$\tau_x(m) = E(m)(x).$$

Now, we can define a metric on M: Set $d_{2,u}(a, b) := \sup_{x\in X} d_{\tau_x}(a, b)$. Property (iv) then ensures that $d_{2,u}$ is positive definite. Then the last property of an AW^*-bundle is:

(viii) The closed unit ball M_1 is complete with respect to $d_{2,u}$.

We call X the base space and M the section algebra of the bundle. Furthermore, for $x \in X$, let $I_{\tau_x} := \{m \in M | \tau_x(m^*m) = 0\}$, the kernel of τ_x. From Theorem 2.9, we know that I_{τ_x} is a closed two-sided ideal in M, so we define the fiber over $x \in X$ as $M_x := M/I_{\tau_x}$. For $m \in M$, with $m(x)$, we denote the image of m in M/I_{τ_x} under the quotient map. Also, from Theorem 2.9, we know that there is a faithful quasitrace $\overline{\tau}_x$ on M_x such that $\tau_x(m) = \overline{\tau}_x(m(x))$ for all $m \in M$. But we make no difference between τ_x and $\overline{\tau}_x$, and we will just write τ_x for this quasitrace.

We now want to give a series of examples of AW^*-bundles.

Example 3.2: The easiest example for an AW^*-bundle is the bundle over the one-point space $\{x\}$. Then the section algebra M is a finite AW^*-algebra, together with a faithful quasitrace τ. The embedding is $C(\{x\}) \cong \mathbb{C} \hookrightarrow \mathcal{Z}(M),\ x \mapsto x1_M$, and the map $E\colon M \to C(\{x\}),\ m \mapsto \tau(m)$.

Example 3.3: Another easy examples are commutative AW^*-bundles. It is easy to see that every commutative C^*-algebra $C(X)$ is an AW^*-bundle over X with fibers isomorphic to $\mathbb{C}$.

Next, we want to give the original definition of W^*-bundles. This definition is originally from Ozawa and was introduced in [Oza13].

Example 3.4 (W^*-bundles)**:** A W^*-bundle over a compact Hausdorff space X is a unital C^*-algebra M, together with a unital embedding $C(X) \hookrightarrow M$ and a conditional expectation $E\colon M \to C(X)$ such that

(T) $E(x^*x) = E(xx^*) \geq 0$.

(F) E is faithful in the usual sense.

(C) The unit ball M_1 is complete with respect to the universal 2-norm defined by $\|a\|_{2,u} := \|E(a^*a)\|_{C(X)}$.

Then it is clear that this is also an AW^*-bundle.
There are three different typical constructions for W^*-bundles:

(i) Let X be a compact Hausdorff space, and M a von Neumann algebra with a faithful quasitrace τ. Then the trivial bundle over X with fibers M is defined as

$$C_\sigma(X, M) := \{f\colon X \to M | f \text{ is norm-bounded and continuous in } d_\tau\}.$$

The embedding is given by $C(X) \hookrightarrow \mathcal{Z}(C_\sigma(X, M)),\ f(.) \mapsto f(.)1$ and the conditional expectation E by the formula $E(f)(x) = \tau(f(x))$.

(ii) Let M be a finite von Neumann algebra together with the center-valued trace. Then M is a W^*-bundle over the spectrum of its center.

(iii) Let A be a unital, separable C^*-algebra, and assume that $T(A)$ is non-empty and a Bauer-simplex. Then the completion of A with respect to the norm $\|a\|_{2,u} := \sup_{\tau \in T(A)} \tau(a^*a)^{1/2}$ is a W^*-bundle over $\partial_e T(A)$.
This W^*-bundle is called the strict completion of A.

The first and third examples are due to Ozawa and were introduced in [Oza13]. The second example is from [Evi18]. Now we want to generalize the first two examples.

Proposition 3.5 (Trivial AW^*-bundle)**:** *Let X be a compact Hausdorff space and M an AW^*-algebra with a normal, faithful, normalized quasitrace τ. We define*

$$C_\sigma(X,M) := \{f\colon X \to M | f \text{ is norm-bounded and continuous in } d_\tau\}.$$

Then $C_\sigma(X,M)$, together with the embedding

$$\begin{aligned} C(X) &\hookrightarrow \mathcal{Z}(C_\sigma(X,M)) \\ f &\mapsto (x \mapsto f(x)1_M) \end{aligned}$$

and the map $E\colon C_\sigma(X,M) \to C(X)$, defined by $E(f)(x) = \tau(f(x))$, is an AW^*-bundle over X. We call it the trivial bundle over X with fiber M.

Proof. The first thing we prove is that $C_\sigma(X,M)$ is really a C^*-algebra: With Proposition 2.18, we see that $C_\sigma(X,M)$ is a *-algebra. So, the next thing we need to show is that $C_\sigma(X,M)$ is closed in $\ell^\infty(X,M)$. So, suppose a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_\sigma(X,M)$ converges in norm to $f \in \ell^\infty(X,M)$. We show now that f is continuous in d_τ, so let $(x_i)_{i\in I}$ a net in X, converging to an $x \in X$, and let $\varepsilon > 0$.
First note that there is an $n \in \mathbb{N}$ such that $\|f_n - f\|^{2/3} \leq \frac{\varepsilon}{3}$.
Since f_n is continuous in d_τ, there exists a $j \in I$ such that for every $i \geq j$:

$$d_\tau(f_n(x_i), f_n(x)) \leq \frac{\varepsilon}{3}.$$

Let $n \in \mathbb{N}$ as above, and $i \geq j$, and we compute with Lemma 2.16:

$$\begin{aligned} d_\tau(f(x_i), f(x)) &\leq d_\tau(f(x_i), f_n(x_i)) + d_\tau(f_n(x_i), f_n(x)) + d_\tau(f_n(x), f(x)) \\ &\leq \|f - f_n\|^{2/3} + d_\tau(f_n(x_i), f_n(x)) + \|f - f_n\|^{2/3} \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So, f is continuous in d_τ, and $C_\sigma(X,M)$ is really a unital C^*-algebra.
Now we want to show that the map E is well defined because it is not obvious that E takes it values in $C(X)$.
From Proposition 2.18, we know that the map $x \mapsto \tau(f(x))$ is continuous in the case that f is positive, so let $f \in C_\sigma(X,M)$ be an arbitrary element. We can decompose f into its real and imaginary parts Ref, Imf. Then we can decompose both in its positive and negative parts. Then the maps

$$\begin{aligned} x \mapsto \tau((Ref)_+(x)) \quad x \mapsto \tau((Ref)_-(x)) \\ x \mapsto \tau((Imf)_+(x)) \quad x \mapsto \tau((Imf)_-(x)) \end{aligned}$$

are all d_τ-continuous. But since the positive parts of a self-adjoint element commute, we get that the following map is d_τ-continuous:

$$x \mapsto \tau(f(x)) = \tau((Ref)_+(x)) - \tau((Ref)_-(x)) + i(\tau((Imf)_+(x)) - \tau((Imf)_-)(x))).$$

So, the map E is well-defined, and the other properties of E follow because τ is a quasitrace.
The last thing we have to prove is that the closed unit ball of $C_\sigma(X,M)$ is complete in $d_{2,u}$, so let now $(f_n)_{n\in\mathbb{N}}$ be a $d_{2,u}$-Cauchy sequence. Then for every $x\in X$, it is clear that $(f_n(x))_{n\in\mathbb{N}}$ is a d_τ Cauchy sequence in M_1. Since M is an AW^*-algebra and τ is normal, we know from Theorem 2.24 that the closed unit ball of M is complete in d_τ. So $(f_n(x))_{n\in\mathbb{N}}$ converges to a $f(x)\in M_1$ in d_τ, and it is standard that $(f_n)_{n\in\mathbb{N}}$ converges in $d_{2,u}$ to f. Since $\|f(x)\|\leq 1$, it is trivial that $\sup_{x\in X}\|f(x)\|_M\leq 1$.
We show the continuity of f in d_τ. This is again a standard $\frac{\varepsilon}{3}$ argument. Let $(x_i)_{i\in I}$ be a net converging to $x\in X$ and $\varepsilon>0$. There exists an $n\in\mathbb{N}$ such that $d_{2,u}(f_n,f)\leq\frac{\varepsilon}{3}$. For this $n\in\mathbb{N}$, there exists a $j\in I$ such that for every $i\geq j$, we have that $d_\tau(f_n(x_i),f_n(x))\leq\frac{\varepsilon}{3}$. Then we have:

$$\begin{aligned}d_\tau(f(x_i),f(x)) &\leq d_\tau(f(x_i),f_n(x_i))+d_\tau(f_n(x_i),f_n(x))+d_\tau(f_n(x),f(x))\\ &\leq d_{2,u}(f,f_n)+d_\tau(f_n(x_i),f_n(x))+d_{2,u}(f_n,f)\\ &\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\end{aligned}$$

So, f is bounded and d_τ continuous, hence, $f\in C_\sigma(X,M)$, and the closed unit ball of $C_\sigma(X,M)$ is complete in $d_{2,u}$.
This was the last thing to prove, and now we have that $C_\sigma(X,M)$ is an AW^*-bundle. □

A slight modification of the proof of Proposition 3.5 gives us a similar result as Proposition 2.18 *(iv)*:

Corollary 3.6: *Let A be a unital C^*-algebra with faithful, normalized quasitrace τ. The map $x\mapsto\tau(x)$ is continuous with respect to d_τ on norm-bounded subsets of A.*

Proof. Denote with (A_1,d_τ) the closed unit ball of A equipped with the topology from d_τ. Then $C_\sigma((A_1,d_\tau),A)$ is again a C^*-algebra, and the embedding $\iota\colon(A_1,d_\tau)\to A$ is in $C_\sigma((A_1,d_\tau),A)$. The proof of the continuity of the map $x\mapsto\tau(\iota(x))=\tau(x)$ is essentially the same as in the previous proof. □

Before giving the next example, we need two more lemmas: The first is Lemma 20 in [Kap52], and we skip the proof.

Lemma 3.7: *Let M be an AW^*-algebra and $(p_i)_{i\in I},(q_i)_{i\in I}$ two families of orthogonal projections with supremum p and q, respectively. Furthermore, let $p_i\sim q_i$ for every $i\in I$ via the partial isometry v_i. Then there exists a partial isometry $v\in M$ such that $v^*v=p$, $vv^*=q$, and $q_iv=vp_i=v_i$ for every $i\in I$.*

Lemma 3.8: *Let M be a unital C^*-algebra, X a compact Hausdorff space, and $E\colon M\to C(X)$ a map satisfying conditions $(i)-(vii)$ from definition 3.1. Then the* *map*

$$\Phi\colon M \to \prod_{x\in X} M_x$$
$$m \mapsto (m(x))_{x\in X}$$

is an isometric **-homomorphism. In particular, we have* $\|m\|_M = \sup_{x\in X}\|m(x)\|_{M_x}$ *for all* $m \in M$.
Furthermore, the closed unit ball of M *is closed in* $d_{2,u}$.

Proof. It is clear that for every $x \in X$, the map $m \mapsto m(x)$ is a *-homomorphism, so the map Φ is also a *-homomorphism. Now it suffices to show that Φ is injective. Therefore, let $m \in M$ such that $\Phi(m) = 0$. But this implies that $m \in I_{\tau_x}$ for every $x \in X$, hence, $E(m^*m)(x) = 0$ for every $x \in X$, so $E(m^*m) = 0$, but E is faithful, so $m = 0$.
Now let $(a_n)_{n\in\mathbb{N}}$ be a sequence in the closed unit ball of M, converging to $a \in M$ in $d_{2,u}$. Then for every $x \in X$, $(a_n(x))_{n\in\mathbb{N}}$ is a sequence in the closed unit ball of M_x, converging in d_{τ_x} to $a(x)$. From Theorem 2.19, we know that the unit ball of M_x is closed in d_{τ_x}, so we have that $\|a(x)\|_{M_x} \leq 1$ for every $x \in X$. Then, from the previous part of this lemma, we get that $\|a\|_M = \sup_{x\in X}\|a(x)\|_{M_x} \leq 1$. So, a is contained in the closed unit ball of M, and M_1 is closed under $d_{2,u}$. □

The next theorem shows that finite AW^*-algebras are AW^*-bundles over the spectrum of its center. The difficult part of the proof is the completeness of the closed unit ball, and the proof is a slight modification of the proof of Proposition 3.10 in [Haa14].

Theorem 3.9: *Let* M *be a finite* AW^**-algebra with center-valued quasitrace* $T\colon M \to \mathcal{Z}(M)$, *and let* $\varphi\colon \mathcal{Z}(M) \to C(X)$ *be a* **-isomorphism where* X *is a Stonean space. Then* M *is an* AW^**-bundle over* X *with embedding* φ^{-1}, *and the map* $E = \varphi \circ T$.

Proof. As stated before, we only show that the unit ball of M is complete in $d_{2,u}$. The proof is divided into four main parts. First, note again that for projections $p, q \in \mathcal{P}(M)$, the Kaplansky identity from Theorem 1.22 $p - p \wedge q \sim p \vee q - q$ holds. Since p and $p \wedge q$, respectively q and $q \vee p$, commute, we obtain that

$$T(p) - T(p \wedge q) = T(p \vee q) - T(q).$$

Hence, $T(p \vee q) \leq T(p) + T(q)$. If for a sequence $(p_n)_{n\in\mathbb{N}}$, $\sum_{n\in\mathbb{N}} T(p_n)$ exists, we want to show that the following holds:

$$T\left(\bigvee_{n\in\mathbb{N}} p_n\right) \leq \sum_{n\in\mathbb{N}} T(p_n).$$

We inductively construct a sequence of mutually orthogonal projections $(q_n)_{n\in\mathbb{N}}$ such that

- $\sup_{n\in\mathbb{N}} p_n = \sup_{n\in\mathbb{N}} q_n$.

- $T(q_n) \leq T(p_n)$.

Set $q_1 := p_1$, and suppose that we constructed $q_1, ..., q_{n-1}$. Then define

$$q_n := \left(\bigvee_{k=1}^{n-1} q_k \right) \vee p_n - \sum_{k=1}^{n-1} q_n.$$

So, it is clear that the q_n are mutually orthogonal and also that $\sup_{k=1,\dots,n} q_n = \sup_{k=1,\dots,n} p_n$, and we see that $\sup_{n\in\mathbb{N}} q_n = \sup_{n\in\mathbb{N}} p_n$. With the inequality mentioned at the beginning the following holds:

$$T(q_n) = T \left(\bigvee_{k=1}^{n-1} q_k \vee p_n - \sum_{k=1}^{n-1} q_n \right) \leq \sum_{k=1}^{n-1} T(q_k) + T(p_n) - \sum_{k=1}^{n-1} T(q_k) = T(p_n).$$

Putting all these things together, we obtain

$$T(\sup_{n\in\mathbb{N}} p_n) = T(\sup_{n\in\mathbb{N}} q_n) = \sum_{n=1}^{\infty} T(q_n) \leq \sum_{n=1}^{\infty} T(p_n).$$

If now $\sum_{n\in\mathbb{N}} T(p_n)$ exists, then it is clear that

$$\varphi\left(\sum_{n=1}^{\infty} T(p_n) \right) = \sum_{n=1}^{\infty} \varphi(T(p_n)).$$

So, for the rest of the proof, we drop φ and will only write T instead of $\varphi \circ T$.
The next step is to show that $\mathcal{U}(M)$ is complete in $d_{2,u}$. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $d_{2,u}$. Without loss of generality, we may assume that

$$d_{2,u}(u_n, u_{n+1})^{3/2} = \|T((u_n - u_{n+1})^*(u_n - u_{n+1}))^{1/2}\|_{C(X)} < 4^{-n}.$$

Now define $e_n := E_{[0,2^{-n}]}(|u_n - u_{n+1}|)$, and observe that

$$\|(u_n - u_{n+1})e_n\| \leq 2^{-n} \text{ and } e_n^{\perp} \leq 2^n |u_n - u_{n+1}|.$$

Since T is order-preserving, we can compute:

$$\begin{aligned} T(e_n^{\perp}) &\leq 2^n T(|u_n - u_{n+1}|) \\ &\leq 2^n T(1)^{1/2} T(|u_n - u_{n+1}||u_n - u_{n+1}|)^{1/2} \\ &= 2^n T((u_n - u_{n+1})^*(u_n - u_{n+1}))^{1/2}. \end{aligned}$$

Now computing this in norm, we get:

$$\begin{aligned} \|T(e_n^{\perp})\|_{C(X)} &\leq 2^n \|T((u_n - u_{n+1})^*(u_n - u_{n+1}))^{1/2}\|_{C(X)} \\ &= 2^n d_{2,u}(u_n, u_{n+1})^{3/2} \leq 2^{-n}. \end{aligned}$$

Thus, for every $n \in \mathbb{N}$, the series $\sum_{k=n}^{\infty} T(e_k^{\perp})$ exists and converges in norm.
Now set $f_n := \bigwedge_{k \geq n} e_k$. We claim that $T(f_n^{\perp})$ converges to 0:

$$\begin{aligned}
\|T(f_n^{\perp})\|_{C(X)} &= \left\| T\left(1 - \bigwedge_{k \geq n} e_k\right)\right\|_{C(X)} = \left\| T\left(1 - \left(1 - \bigvee_{k \geq n}(1 - e_k)\right)\right)\right\|_{C(X)} \\
&= \left\| T\left(\bigvee_{k \geq n} e_k^{\perp}\right)\right\|_{C(X)} \leq \left\| \sum_{k \geq n} T(e_k^{\perp})\right\|_{C(X)} \\
&\leq \sum_{k \geq n} \|T(e_k^{\perp})\|_{C(X)} \leq 2^{1-n}.
\end{aligned}$$

So, we see that $\|T(f_n^{\perp})\| \to 0$ for $n \to \infty$.
For two natural numbers $k \geq n$, it is clear that $e_k \geq f_n$, so the following inequality holds

$$\|(u_k - u_{k+1})f_n\| \leq \|(u_k - u_{k+1})e_k\| \leq 2^{-k}.$$

Thus, the series $\sum_{k=n}^{\infty}(u_k - u_{k+1})f_n$ converges for every $n \in \mathbb{N}$ in C^*-norm. This implies that

$$v_n := \lim_{k \to \infty} u_k f_n$$

exists in C^*-norm for every $n \in \mathbb{N}$. Furthermore, every v_n is a partial isometry because

$$v_n^* v_n = \lim_{k \to \infty} f_n u_k^* u_k f_n = f_n.$$

Since $(f_n)_{n \in \mathbb{N}}$ is an increasing sequence of projections, it follows that

$$v_n f_m = \lim_{k \to \infty} u_k f_n f_m = \lim_{k \to \infty} u_k f_m = v_m \text{ for } m \leq n.$$

Now set $v_0 = 0 = f_0$ and define $w_n := v_n - v_{n-1}$. Then the w_n are partial isometries such that

$$(w_n^* w_n)_{n \in \mathbb{N}} = (f_n - f_{n-1})_{n \in \mathbb{N}} \text{ and } (w_n w_n^*) = v_n v_n^* - v_{n-1} v_{n-1}^*$$

are sequences of mutually orthogonal projections. With Lemma 3.7, we get an element $w \in M$ such that

$$w^* w = \bigvee_{n \in \mathbb{N}} w_n^* w_n^* \text{ and } w w^* = \bigwedge_{n \in \mathbb{N}} w_n w_n^*$$

and $w_n = w w_n^* w_n = w(f_n - f_{n-1})$.
So, from $w_n^* w_n = f_n - f_{n-1}$ and $T(f_n^{\perp}) \to 0$, we conclude that $w^* w = 1$. M is finite, hence, $w w^* = 1$ and so, $w \in \mathcal{U}(M)$. So, it remains to show that $(u_n)_{n \in \mathbb{N}}$ converges in $d_{2,u}$ to w.
We observe that

$$v_n = \sum_{k=1}^{n}(v_k - v_{k-1}) = \sum_{k=1}^{n} w_k = \sum_{k=1}^{n} w(f_k - f_{k-1}) = w f_n$$

and with this equality, we get

$$\|(u_k - w)f_n\| = \|u_k f_n - w f_n\| = \|u_k f_n - v_n\| \to 0 \text{ for } k \to \infty$$

and so with Lemma 2.16, we obtain

$$\lim_{k\to\infty} \sup_{x\in X} \|(u_k - w)f_n\|_{2,\tau_x}^{2/3} \leq \lim_{k\to\infty} \|(u_k - w)f_n\|^{2/3} = 0.$$

Now let $\varepsilon > 0$. Then there is $n \in \mathbb{N}$ such that $\|T(f_n^\perp)\|_{C(X)} < \varepsilon$. Again, we use Lemma 2.16 and compute

$$\sup_{x\in X} \|(u_k - w)f_n^\perp\|_{2,\tau_x} \leq 2 \sup_{x\in X} \|f_n^\perp\|_{2,\tau_x} = 2 \sup_{x\in X} |T(f_n^\perp)(x)^{1/2}| = \|T(f_n^\perp)\|_{C(X)}^{1/2} \leq 2\varepsilon^{1/2}.$$

Finally, we again use Lemma 2.16 and can conclude:

$$\begin{aligned} \limsup_{k\to\infty} \sup_{x\in X} \|u_k - w\|_{2,\tau_x}^{2/3} &\leq \limsup_{k\to\infty} \left(\sup_{x\in X} \|(u_k - w)f_n\|_{2,\tau_x}^{2/3} + \sup_{x\in X} \|(u_k - w)f_n^\perp\|_{2,\tau_x}^{2/3} \right) \\ &\leq 0 + 2^{2/3}\varepsilon^{2/6}. \end{aligned}$$

So, it follows that $(u_n)_{n\in\mathbb{N}}$ converges to w in $d_{2,u}$, and so $\mathcal{U}(M)$ is complete in $d_{2,u}$.
Now we want to show that the self-adjoint part of the closed unit ball $(M_{sa})_1$ is complete in the $d_{2,u}$-metric, so let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(M_{sa})_1$. Let u_n be the Cayley transform of a_n:

$$u_n := (a_n + i1)(a_n - i1)^{-1}.$$

Then u_n is a unitary for every $n \in \mathbb{N}$, and

$$u_n - u_m = 2i(a_n - i1)^{-1}(a_m - a_n)(a_m - i1)^{-1}.$$

With the use of Lemma 2.16, we obtain

$$d_{2,u}(u_n, u_m) = \sup_{x\in X} \|u_n - u_n\|_{2,\tau_x}^{2/3} \leq 2^{2/3} \sup_{x\in X} \|a_m - a_n\|_{2,\tau_x}^{2/3} = 2^{2/3} d_{2,u}(a_m, a_n).$$

Hence $(u_n)_{n\in\mathbb{N}}$ converges in $d_{2,u}$-metric to a unitary $u \in \mathcal{U}(M)$. Recall that $\sigma(a_n) \subseteq [-1, 1]$, and observe that for $f\colon [-1, 1] \to \mathbb{T}$, $x \mapsto \frac{x+i}{x-i}$

$$Re(f(x)) = \frac{x^2 - 1}{x^2 + 1} \leq 0$$

thus, $\sigma(u_n) \subseteq \{t \in \mathbb{T} | Re(t) \leq 0\}$.
Since $\sup_{x\in\sigma(u_n)} |1 + x| \leq \sqrt{2}$, it follows that $\|1 + u_n\| \leq \sqrt{2}$. In Lemma 3.8, we proved that the closed unit ball of M is closed in $d_{2,u}$-metric, so $\|1 + u\| \leq \sqrt{2}$. So, we can again conclude that $\sigma(u) \subseteq \{t \in \mathbb{T} | Re(t) \leq 0\}$.
Let a be the inverse Cayley transform of u, so

$$a := i(u + 1)(u - 1)^{-1} \text{ and } a_n = i(u_n + 1)(u_n - 1)^{-1}.$$

Then $\sigma(a) \subseteq [-1,1]$, and so, $a \in (M_{sa})_1$. Again, we compute

$$a_n - a = 2i(u_n - 1)^{-1}(u - u_n)(u-1)^{-1}.$$

We need to control the norm of $(u_n - 1)^{-1}$ and $(u-1)^{-1}$. But with the conditions on $\sigma(u_n)$ and $\sigma(u)$ and functional calculus, we get

$$\|(u_n - 1)^{-1}\| \leq \frac{1}{\sqrt{2}} \text{ and } \|(u-1)^{-1}\| \leq \frac{1}{\sqrt{2}}.$$

Now we can show that $a_n \to a$ in $d_{2,u}$-metric, again using Lemma 2.16:

$$\begin{aligned} d_{2,u}(a_n, a) &= \sup_{x\in X} \|a_n - a\|_{2,\tau_x}^{2/3} = \sup_{x\in X} \|2i(u_n - 1)^{-1}(u - u_n)(u-1)^{-1}\|_{2,\tau_x}^{2/3} \\ &\leq \left(2\frac{1}{\sqrt{2}}\right)^{2/3} \sup_{x\in X} \|u_n - u\|_{2,\tau_x}^{2/3} \left(\frac{1}{\sqrt{2}}\right)^{2/3} = d_{2,u}(u_n, u) \to 0. \end{aligned}$$

Now we can prove the completeness of M_1: Let x_n be a $d_{2,u}$-Cauchy sequence in M_1. Let

$$a_n := \frac{1}{2}(x_n + x_n^*) \text{ and } b_n := \frac{1}{2i}(x_n - x_n^*)$$

be the real respectively imaginary part of x. Then a_n, b_n are Cauchy sequences in $(M_{sa})_1$, hence converging to some $a, b \in (M_{sa})_1$. But then we use Lemma 2.16:

$$\begin{aligned} d_{2,u}(x_n, a+ib) &= \sup_{x\in X} \|a_n + ib_n - a - b\|_{2,\tau_x}^{2/3} \\ &\leq \sup_{x\in X} \|a_n - a\|_{2,\tau_x}^{2/3} + \|i(b_n - b)\|_{2,\tau_x}^{2/3} \\ &\leq \sup_{x\in X} \|a_n - a\|_{2,\tau_x}^{2/3} + \sup_{x\in X} \|b_n - b\|_{2,\tau_x}^{2/3} \to 0 \end{aligned}$$

So x_n converges to $x = a + ib$, and since the closed unit ball of M is closed in $d_{2,u}$, we conclude that $\|x\| \leq 1$, and this finishes the proof, showing that M is really an AW^*-bundle over X. □

3.2 The Fibers of an AW^*-Bundle

In this chapter, we want to justify the name AW^*-bundle and show that the fibers of an AW^*-bundle are AW^*-algebras. The proof is essentially the same as for the W^*-bundles given in [EP16] or [Evi18]. But we first need one more lemma:

Lemma 3.10: *Let M be an AW^*-bundle over X. Then for every $m \in M$, the map $x \mapsto \|m(x)\|_{2,\tau_x}$ is continuous. In particular, for every $m, n \in M$, the map $x \mapsto d_{\tau_x}(m(x), n(x))$ is continuous.*

Proof. Clear, because $\|m(x)\|_{2,\tau_x} = E(m^*m)(x)^{1/2}$ and E take their values in $C(X)$. □

Theorem 3.11: *Let M be an AW^*-bundle over X. Then for every $x \in X$, the fiber M_x is an AW^*-algebra.*

Proof. We will use Theorem 2.23 and show that the closed unit ball of M_x is complete in d_{τ_x}:
So, let $(b_n)_{n\in\mathbb{N}}$ be a d_{τ_x}-Cauchy sequence in the closed unit ball of M_x. Without loss of generality, we may assume that $d_{\tau_x}(b_{n+1}, b_n) < 2^{-n}$. Now we want to construct a sequence $(a_n)_{n\in\mathbb{N}}$ in M such that

- $a_n(x) = b_n$ for every $n \in \mathbb{N}$.
- $\|a_n\| \leq 1$ for every $n \in \mathbb{N}$.
- $d_{2,u}(a_{n+1}, a_n) \leq 2^{-n}$ for every $n \in \mathbb{N}$.

We can lift b_1 to an element $a_1 \in M$ such that $\|a_1\| \leq 1$. Now assume that we had constructed elements $a_1, a_2, ..., a_n$ with the desired properties. Now we lift b_{n+1} to an element $\tilde{a}_{n+1} \in M$ such that $\|\tilde{a}_{n+1}\| \leq 1$.
It is clear that $d_{\tau_x}(\tilde{a}_{n+1}(x), a_n(x)) < 2^{-n}$, and from Lemma 3.10, we know that the map $y \mapsto d_{\tau_y}(\tilde{a}_{n+1}(y), a_n(y))$ is continuous. So there exists an open neighborhood U of x such that

$$\sup_{y\in U} d_{\tau_y}(\tilde{a}_{n+1}(y), a_n(y)) \leq 2^{-n}.$$

Let $f\colon X \to [0,1]$ be a continuous function with $f(x) = 1$ and $f(X\backslash U) \subseteq \{0\}$. Now we define

$$a_{n+1} := f\tilde{a}_{n+1} + (1-f)a_n.$$

We see that a_{n+1} is a lift for b_{n+1} because

$$a_{n+1}(x) = f(x)\tilde{a}_{n+1}(x) + (1-f(x))a_n(x) = f(x)\tilde{a}_{n+1}(x) = \tilde{a}_{n+1}(x) = b_{n+1}.$$

Now we use Lemma 3.8 to show that a_{n+1} lies in the closed unit ball of M:

$$\begin{aligned}\|a_{n+1}\| &= \sup_{y\in X} \|f(y)\tilde{a}_{n+1} + (1-f(y))a_n(y)\|_{M_y} \\ &= \sup_{y\in X} |f(y)|\|\tilde{a}_{n+1}(y)\|_{M_y} + |1-f(y)|\|a_n(y)\|_{M_y} \\ &\leq 1.\end{aligned}$$

Next, we show that $(a_n)_{n\in\mathbb{N}}$ is a $d_{2,u}$-Cauchy sequence. We regard two cases. For the first one, let $y \in U$:

$$\begin{aligned}d_{\tau_y}(a_{n+1}(y), a_n(y)) &= \|f(y)\tilde{a}_{n+1}(y) + (1-f(y))a_n(y) - a_n(y)\|_{2,\tau_y}^{2/3} \\ &= \|f(y)\tilde{a}_{n+1}(y) - f(y)a_n(y)\|_{2,\tau_y}^{2/3} \\ &\leq \|\tilde{a}_{n+1}(y) - a_n(y)\|_{2,\tau_y}^{2/3} = d_{\tau_y}(\tilde{a}_{n+1}(y), a_n(y)) \leq 2^{-n}.\end{aligned}$$

The other case is $y \in X \backslash U$, and we observe that $f(y) = 0$ and compute

$$\begin{aligned} d_{\tau_y}(a_{n+1}(y), a_n(y)) &= \|f(y)\tilde{a}_{n+1}(y) + (1 - f(y))a_n(y) - a_n(y)\|_{2,\tau_y}^{2/3} \\ &= \|a_n(y) - a_n(y)\|_{2,\tau_y}^{2/3} = 0. \end{aligned}$$

So, we got $d_{2,u}(a_{n+1}, a_n) = \sup_{y\in X} d_{\tau_y}(a_{n+1}(y), a_n(y)) \leq 2^{-n}$, and so $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $d_{2,u}$. From the completeness, it follows that $(a_n)_{n\in\mathbb{N}}$ converges to an a in the unit ball of M. But then we conclude that $(b_n)_{n\in\mathbb{N}}$ converges to $a(x)$ in d_{τ_x} because

$$d_{\tau_x}(b_n, a(x)) = d_{\tau_x}(a_n(x), a(x)) \leq d_{2,u}(a_n, a).$$

So, the unit ball of M_x is complete in d_{τ_x}, and so with Theorem 2.23, we deduce that M is an AW^*-algebra. □

3.3 A Short Outlook

In this final short section, we want to present some ideas that may prove W^*-bundle theorems, perhaps for AW^*-bundles as well.
Let A be a unital and separable C^*-algebra such that $T(A)$ is non-empty and a Bauer simplex. In 3.4, we mentioned the W^*-bundle of the strict completion of A. Is it possible to get an AW^*-bundle in the same way if we assume that $QT(A)$ is a non-empty Bauer simplex? An important step in proving that the strict completion of A is a W^*-bundle is Lemma 10 in [Oza13]:

Lemma 3.12: *For every $\tau \in T(A)$, there is a normal *-isomorphism*

$$\theta_\tau \colon L^\infty(\partial_e T(A), \mu_\tau) \to \mathcal{Z}(\pi_\tau(A)''),$$

such that

$$\tau(\theta_\tau(f)a) = \int f(\rho)\rho(a)d\mu_\tau(\rho)$$

for $a \in A$.

The proof of this Lemma uses Sakai's Radon-Nikodym theorem. Is there an analog in Sakai's Radon-Nikodym theorem for quasilinear maps?
What happens in the Lemma if we assume that τ is a faithful extreme point in $QT(A)$ and replace $\pi_\tau(A)''$ with the AW^*-completion of A with respect to τ? The AW^*-completion of A with respect to τ is a factor by Proposition 2.34. Then we would obtain a map $\theta_\tau \colon L^\infty(\partial_e QT(A), \mu_\tau) \to \mathbb{C}$ such that

$$\tau(\theta_\tau(f)a) = \int f(\rho)\rho(a)d\mu_\tau(\rho)$$

for all $a \in A$. This would imply that $\tau(\lambda a) = \lambda\tau(a)$ for all $a \in A, \lambda \in \mathbb{C}$. In [Nag02], Nagy noted that, for the unique normalized quasitrace on an AW^*-algebra of type

II_1, this implies that τ is linear and, hence, a trace.

Other important results from Ozawa are Theorem 15 and Corollary 16 in [Oza13]. They both state under which assumptions a (strictly separable) W^*-bundle is isomorphic to the trivial bundle $C_\sigma(X, \mathcal{R})$. For AW^*-bundles, it is difficult to prove this in a similar way since there is no AW^*-analog to the hyperfinite II_1 factor $\mathcal{R}$. But it is may be possible to make some steps towards this and perhaps prove an analog of Corollary 12 in [Oza13]. This could be done in the following way:

Conjecture 3.13: Let M be an AW^*-bundle X. Assume that every fiber M_x is AW^*-McDuff and that X has a finite covering dimension. Then, for every $k \in \mathbb{N}$, there is an approximately central, approximately multiplicative embedding

$$\varphi_n \colon M_k(\mathbb{C}) \hookrightarrow M$$

such that

(i) For all $x, y \in M_k(\mathbb{C})$ the following holds: $\limsup_n d_{2,u}(\varphi_n(xy), \varphi_n(x)\varphi(y)) = 0$.

(ii) For all $x \in M_k(\mathbb{C})$ and $m \in M$: $\limsup_n d_{2,u}(\varphi_n(x)a, a\varphi_n(x)) = 0$.

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