Generalized quotients of $\mathbb{W}$-semigroups and dynamical Cuntz semigroups

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Basics

Composition of relations

Given relations R_1, R_2 on a set X, then $R_1 \circ R_2$ is the relation given by $(a, b) \in R_1 \circ R_2$ provided there is $c \in X$ with aR_1cR_2b.

Definition:

Let $(X, 0)$ be a pointed set. A relation $\prec$ on X is *idempotent* provided $\prec \circ \prec = \prec$ (i.e. transitive and dense).
We say $\prec$ is *positive* if $0 \prec a$ for any $a \in X$.

Definition: (Auxiliary relation)

If $(X, 0)$ has a preorder $\leq$, we say $\prec$ is *auxiliary* for $\leq$ provided $\prec \subseteq \leq$, $\leq \circ \prec = \prec$, and $\prec \circ \leq = \prec$.

The W axioms

Definition:(**Antoine, P, Thiel 2018**)

Let $(S, 0, \prec)$ be a semigroup, where $\prec$ is transitive and positive. Consider:

(W1) For any a, the set $a^{\prec}$ has a countable cofinal $\prec$-increasing subset.

(W2) If S has a preorder $\leq$ for which $\prec$ is auxiliary, for any $a \in S$, we have $a = \sup^{\leq} a^{\prec}$.

(W3) The relation $\prec$ is compatible with addition.

(W4) If $a \prec b + c$, there are $b' \prec b$ and $c' \prec c$ such that $a \prec b' + c'$.

Definition:(**Antoine, P, Thiel 2018-2020**)

A W-semigroup is a semigroup $(S, 0, \prec)$ that satisfies axioms (W1), (W3), (W4). A morphism $f\colon S \to T$ must be additive, preserve 0, the relation $\prec$, and *continuous*:

$$b \prec f(a) \implies \exists a' \prec a \text{ such that } b \prec f(a').$$

We denote the category of W-semigroups by W.

Two examples

Example:

Let A be a C*-algebra. Then $W(A) = M_\infty(A)_+/\sim$ is a W-semigroup, with

$$[a] \prec [b] \iff a \precsim (b-\varepsilon)_+$$

for some $\varepsilon > 0$. With $[a] \leq [b] \iff a \precsim b$, $W(A)$ is ordered and satisfies (W1)–(W4). Also, $\leq$ is induced by $\prec$, i.e. $[a] \leq [b] \iff [a]^{\prec} \subseteq [b]^{\prec}$.

Example:

Let A be a C*-algebra. For $a, b \in A_+$, define $a \prec_{\text{her}} b$ if $a \in \text{her}((b-\varepsilon)_+)$ for some $\varepsilon > 0$. Then $(A_+, 0, \prec_{\text{her}})$ is a W-set. One also has a preorder given by

$$a \precsim_{\text{her}} b \iff a \in \text{her}(b),$$

for which in fact $\prec_{\text{her}}$ is the way-below relation. Thus $(A_+, 0, \prec_{\text{her}}, \precsim_{\text{her}})$ satisfies (W1)–(W2).

The forgetful and reconstruction functors:

Let $(S, 0, \prec)$ be a W-semigroup. Set $a \leq_{\mathrm{ind}} b \iff a^{\prec} \subseteq b^{\prec}$. Set also $\prec_{+} = \leq \circ \prec$. Then $(S, 0, \prec_{+}, \leq_{\mathrm{ind}})$ is a(n enriched) W-semigroup, i.e. one satisfying (W1)–(W4).
Conversely, if $(S, 0, \prec, \leq)$ is an enriched W-semigroup, then forgetting $\leq$ we obtain a W-semigroup.
This yields two functors:

$$F\colon \mathrm{W}_{\mathrm{enr}} \to \mathrm{W} \text{ and } G\colon \mathrm{W} \to \mathrm{W}_{\mathrm{enr}}$$

given by $F(S, 0, \prec, \leq) = (S, 0, \prec)$ and $G(S, 0, \prec) = (S, 0, \prec_{+}, \leq_{\mathrm{ind}})$.
One further has $G \circ F \cong \mathrm{id}$ but $F \circ G \ncong \mathrm{id}$.
(Take, e.g. $S = \mathbb{N}$ with $x \prec y$ if and only if $x = 0$. Then $(F \circ G)(S) = 0$.)

Strongly prenormal relations

Notation:

Given $(X, 0, \prec)$, with $\prec$ a transitive relation, denote by $\leq_{\mathrm{ind}}$ the preorder *induced* by $\prec$.

Definition: (Continuous relation)

Given $(X, 0, \prec)$, with $\prec$ idempotent. A relation R is *continuous* if $\prec \circ R \subseteq \prec \circ R \circ \prec$.

Lemma

Let $(X, 0, \prec)$ be a set, with $\prec$ idempotent, and let $\preccurlyeq$ be another transitive relation. TFAE:

- (i) *$\preccurlyeq$ is dense and $\mathrm{id}\colon (X, \prec) \to (X, \preccurlyeq)$ is monotone continuous.*
- (ii) *$\preccurlyeq$ is continuous, dense, $\prec \subseteq \preccurlyeq$, and $\preccurlyeq = \preccurlyeq \circ \prec \circ \preccurlyeq$.*

Definition: (strong prenormal relation)

For a set $(X, 0, \prec)$ with $\prec$ idempotent, a relation $\preccurlyeq$ is *strongly prenormal* if it satisfies any of the conditions of the lemma.

Normal, closed preorders

Definition: (Closed relations)

Let $(S, 0, \prec)$ be a W-semigroup. A relation R is *closed* if whenever $(c, b) \in \prec \circ R$ for any $c \in a^{\prec}$, then $(a, b) \in \leq_{\text{ind}} \circ R$.

Definition: (Normal preorder)

Let $(S, 0, \prec)$ be a W-semigroup. A preorder $\leq$ is *prenormal* if $\prec \subseteq \leq$ and it is continuous. If $\leq$ is additively closed, then we say $\leq$ is a *normal preorder*.

Proposition

Let $(S, 0, \prec)$ be a W-semigroup. Then a preorder $\leq$ on S is normal closed if and only if $\leq$ is induced from a strongly prenormal relation.

Quotients

Let $(S, 0, \prec)$ be a W-semigroup, $\leq$ be a preorder on S. Write $S/\!\leq$ for the antisymmetrization of S by $\leq$.

Proposition

Let $(S, 0, \prec)$ be a W-semigroup, and let $\leq$ be a preorder. Then:

(i) *If $\leq$ is normal, then $(S/\!\leq, 0, \leq \circ \prec)$ is a W-semigroup and the natural map $\pi \colon S \to S/\!\leq$ is a W-morphism.*

(ii) *If $(S/\!\leq, 0, \leq \circ \prec)$ is a W-semigroup and $\pi \colon S \to S/\!\leq$ is a W-morphism, then $\leq$ is $\prec \circ \leq$-normal and $\leq \circ \prec$ is strongly $\prec$-normal.*

Remark:

This result becomes significantly simpler if $\prec$ is auxiliary for $\leq$, that is, $\leq \circ \prec = \prec \circ \leq = \prec$.

The Fundamental Theorem

Definition:

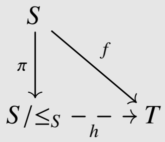
For a W-morphism $f\colon (S, 0, \prec_S) \to (T, 0, \prec_T)$, set

$$\ker(f) = \{(a,b)\colon f(a)^{\prec_T} \subseteq f(b)^{\prec_T}\} = \{(a,b)\colon f(a) \leq_{\mathrm{ind}_T} f(b)\}.$$

Clearly $\leq_{\mathrm{ind}_S} \subseteq \ker(f)$.

Theorem

Let $(S, 0, \prec_S)$ and $(T, 0, \prec_T)$ be W-semigroups, and $f\colon S \to T$ be a W-morphism. Let $\leq_S$ be a normal preorder on S and $\leq_T$ a normal closed partial order on T for which $\prec_T$ is auxiliary. Then $\exists! h\colon (S/{\leq_S}, \leq_S \circ \prec_S) \to (T, \prec_T)$ order-preserving W-morphism such that



is commutative iff $\leq_S \subseteq \ker(f)$.

Ideals

Definition:

An *ideal* of a W-semigroup $(S, 0, \prec)$ is a subsemigroup I such that $0 \in I$ and $a \prec b, b \in I \implies a \in I$. We say I is *closed* if $a^{\prec} \subseteq I$ implies $a \in I$.

For an ideal I, set $a \leq_I b$ if for any $c \prec a$, there is $d \in I$ such that $c \prec b + d$. Set $S/I = S/\leq_I$.

Proposition

Let $(S, 0, \prec)$ be a W-semigroup that satisfies (O1) (wrt $\prec$) and with $\prec \subseteq \ll$. There is a Galois correspondence

$$\{\text{Closed ideals of } S\} \longleftrightarrow \{\text{Normal closed preorders on } S\},$$

given by $I \mapsto \leq_I$ and $\leq \mapsto 0_{\leq} := \{a \in S : a \leq 0\}$.

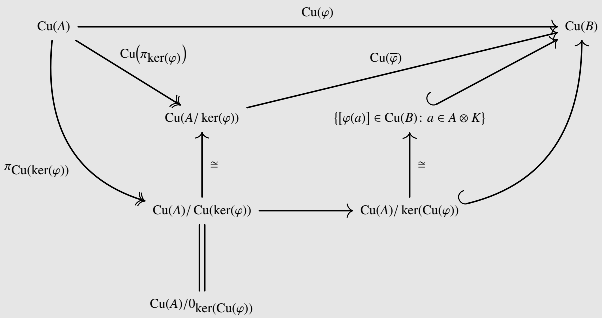
Corollary

Let S be a Cu-semigroup. The poset of closed ideals of S embeds as a full reflective subcategory of the poset of normal preorders on S.

Relation with quotients of C*-algebras

Theorem

Given C-algebras A and B and $\varphi\colon A \to B$ a *-hom, we have:*



Generating normal preorders

Definition:

Let $(S, 0, \prec)$ be a W-semigroup, and R a continous relation. Let $E_S(R) = \{\leq : \text{ any of (i)–(iii) below implies } a \leq b\}$, and $E_S^+(R) := \{\leq : \text{ any of (i)–.(iv) below implies } a \leq b\}$:

(i) $a^{\prec} \subseteq b^{\prec}$.

(ii) aRb.

(iii) For any $c \in a^{\prec}$, we have $c \leq b$.

(iv) $(a = a_1 + a_2, b = b_1 + b_2$ and $a_i \leq b_i$ for $i = 1, 2.)$

Define $\leq_R$=smallest element of $E_S(R)$ under set containment. Likewise $\leq^R$=smallest element of $E_S^+(R)$ under set containment.

Theorem

Given a W-semigroup, and a continuous relation R, then $(\leq_R)$ $\leq^R$ is the smallest (pre)normal closed preorder that contains R.

$(\leq_R)$ $\leq^R$ is called the (pre)normal preorder generated by R.

A characterization of Cuntz subequivalence

We give an elementwise characterizacion of $\leq_R$:

Theorem

For a W*-semigroup* $(S, 0, \prec)$*, a cts relation* R *and* $a, b \in S$*, tfae:*

1. $a \leq_R b$
2. *for any* $c \prec a$*, either* $c \prec b$ *or else* $\exists d_i, e_i \in S$ *such that*

$$c \prec d_1 R e_1 \prec d_2 R e_2 \prec \cdots \prec d_n R e_n \prec b.$$

Let A be a C*-algebra, and let us take the W-set $(A_+, 0, \prec_{\text{her}})$.

Corollary

The Cuntz subequivalence is the smallest preorder $\precsim$ *on* A_+ *such that, for* $a, b \in A_+$*, any of the following implies* $a \precsim b$*:*

1. $a \leq_{\text{her}} b$ *(i.e.* $a \in \text{her}(b)$*).*
2. $a = xbx^*$ *for some* $x \in A$*.*
3. *For any* $c \prec_{\text{her}} a$*, we have* $c \precsim b$*.*

Sketch of proof:

- In A_+, set $(a,b)\in R$ if $a = xbx^*$, a transitive relation.
- R is continuous: $a \prec_{\text{her}} bRc \implies \exists \varepsilon > 0, x\in A : a \prec_{\text{her}} (b-\varepsilon)_+$ and $b = x^*cx$. Set $L = \max\{1, \|x^*x\|\}$.

$$x^*\left(c - \left(c - \frac{\varepsilon}{L}\right)_+\right)x \le x^*\left(c - \left(c - \frac{\varepsilon}{L}\right)\right)x = \frac{\varepsilon}{L}x^*x \le \varepsilon\ ,$$

which implies

$$(x^*cx - \varepsilon)_+ \le \left(x^*cx - x^*\left(c - \left(c - \frac{\varepsilon}{L}\right)_+\right)x\right)_+ = \left(x^*\left(c - \frac{\varepsilon}{L}\right)_+ x\right)_+ \le x^*\left(c - \frac{\varepsilon}{L}\right)_+ x\ .$$

Hence, setting $b' = x^*\left(c - \frac{\varepsilon}{L}\right)_+ x$ and $c' = \left(c - \frac{\varepsilon}{L}\right)_+$, we have $a \prec_{\text{her}} b'$, $(b', c') \in R$, and $c' \prec_{\text{her}} c$.

Sketch of proof cont'd

- R also satisfies: $R\circ \prec_{\mathrm{her}} \circ R \subseteq \precsim_{\mathrm{her}} \circ R \circ \precsim_{\mathrm{her}}$.
  To see this: If aRb, $b \prec_{\mathrm{her}} c$, and cRd, choose
  $x, y \in A : a = xbx^*, c = ydy^*$, and let $\varepsilon > 0 : b \in \mathrm{her}((c-\varepsilon)_+)$.
  Let f be cts s. t. $f(c)$ is a unit for $(c-\varepsilon)_+$. Then
  $f(c)bf(c) = bf(c) = b$, hence $b \leq Nc$ for some N. Thus
  $xbx^* \in \mathrm{her}(xcx^*)$. This implies $a \precsim_{\mathrm{her}} xcx^*$, with $xcx^* = xyd(xy)^*$, i.e.,
  xcx^*Rd. Thus $(a, d) \in \precsim_{\mathrm{her}} \circ R \circ \precsim_{\mathrm{her}}$.
- Now assume $a \precsim_{\mathrm{Cu}} b$ and let $c \prec_{\mathrm{her}} a$. Then $c \in \mathrm{her}(a - \frac{\varepsilon}{2})_+)$ for some
  $\varepsilon > 0$. Choose $x \in A, \delta > 0$ s.t. $\|(a-\varepsilon/2)_+ - x(b-\delta)_+x^*\| < \varepsilon/2$.
  There is $y \in A$ s.t. $(a-\varepsilon)_+ = y(b-\delta)_+y^*$. Thus
  $c \prec_{\mathrm{her}} (a-\varepsilon)_+R(b-\delta)_+ \prec_{\mathrm{her}} b$.
- The converse is similar.

Dynamical semigroups

Definition:

Let G be a group. A G-*action* on a W-semigroup $(S, 0, \prec)$ is a map $\alpha\colon G \times S \to S$, $(g, a) \mapsto ga$, such that

1. $a \prec b \implies ga \prec gb$.
2. $g(a + b) = ga + gb$.
3. $ea = a$.
4. $(gh)a = g(ha)$.

We will call W-semigroup with a G-action a $G - \mathrm{W}$-semigroup.

Denote by $\approx_\alpha$ the relation: $a \approx_\alpha b$ if $a = gb$ for $g \in G$. Note that $\approx_\alpha$ is cts: If $a \prec b = gc$, then let $\tilde{a}$ be such that $a \prec \tilde{a} \prec b$. Clearly, $a \prec \tilde{a}$, $\tilde{a} \approx_\alpha g^{-1}\tilde{a}$, and $g^{-1}\tilde{a} \prec c$.

Let $\leq^\alpha$ be the normal closed preorder generated by $\approx_\alpha$. Denote $S/G = S/\leq^\alpha$.

Elementwise characterization of $\leq^{\alpha}$

Theorem

For a $G-\mathrm{W}$-semigroup S, and $a,b\in S$, tfae:

- (i) $a\leq^{\alpha} b$.
- (ii) *For any $c\prec a$, either $c\prec b$ or there are $g_{ij}\in G$ and $d_{ij}\in S$ such that*

$$c\prec\sum_{j=1}^{n}d_{1j},\ \sum_{j=1}^{n}g_{1j}d_{1j}\prec\sum_{j=1}^{n}d_{2j},\ldots,$$
$$\sum_{j=1}^{n}g_{m-1,j}d_{m-1,j}\prec\sum_{j=1}^{n}d_{mj},\ \sum_{j=1}^{n}g_{mj}d_{mj}\prec b.$$

Corollary

For a C-dynamical system (A,G,α), order on $W(A)/G$ is induced from smallest preorder $\lesssim$ on $M_{\infty}(A)_{+}$ s. t. any of (i)–(iv) implies $a\lesssim b$.*

- (i) $a\lesssim_{\mathrm{Cu}} b$.
- (ii) $a=\alpha_{g}(b)$.
- (iii) *For any $c\lesssim(a-\varepsilon)_{+}$ for some ε, have $c\lesssim b$.*
- (iv) $a=a_1\oplus a_2$, $b=b_1\oplus b_2$ and $a_i\lesssim b_i$.

Universal Property of the Dynamical Semigroup

Theorem

Let $(S, +, 0, \prec_S, G, \alpha)$ be a $G - \mathrm{W}$-semigroup and let $(T, +, 0_T, \leq_T, \prec_T)$ be a W-semigroup. Then for any invariant W-morphism $f\colon S \to T$, there is a unique W-morphism $f_G\colon S/G \to T$ such that $f = f_G \circ \pi_G$, i.e., the following diagram commutes:

$$\begin{array}{ccc} S & & \\ {\scriptstyle \pi_G}\downarrow & \searrow^{f} & \\ S/G & \xrightarrow[f_G]{} & T \end{array}$$

Moreover, the pair $(S/G, \pi_G)$ is the unique pair of a W-semigroup and a W-morphism that satisfies the above universal property.

Topological dynamical systems

X cpct Hausdorff space, G a group acting on X via α. Then G acts on $W(C(X))$ making it a G – W-semigroup.
Let $S = \mathrm{Lsc}(X, \mathbb{N})$, the semigroup of lsc fns $f\colon X \to \mathbb{N}$, with p/w addition and order. Define $f \prec g \iff \limsup_{y\to x} f(y) \leq f(x)$. (Equivalently, $f \ll g$ in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$.) Then $(S, 0, \prec, \leq)$ is also a W-semigroup satisfying (W1)–(W4).
If $\dim X \leq 1$, one has $W(C(X)) \cong \mathrm{Lsc}(X, \mathbb{N})$ (proved by Robert (2013)).

Proposition

The restriction of the rank map $r\colon W(C(X)) \to \mathrm{Lsc}(X, \mathbb{N})$, $r([a])(x) = \mathrm{rank}(a(x))$ to $W_1(C(X)) := \{[a]\colon a \in C(X)_+\}$ is an order-isomorphism.

Connection with dynamical subequivalence

Definition: (Kerr, based on ideas by Winter)

For $A, B \subseteq X$, write $A \precsim^{\alpha} B$ if $\forall A' \subseteq A$ cpct, $\exists \mathcal{U}$ finite collection of open sets covering A', $\{g_U \in G\}_{U \in \mathcal{U}}$ such that $\{g_U U : U \in \mathcal{U}\}$ is a disjoint collection. (We say that A is *dynamically below* B.)

Proposition

For $a, b \in C(X)_+$, tfae:

1. $[a] \leq^{\alpha} [b]$ *in* $W(C(X))$.
2. $[a] \leq^{\alpha} [b]$ *in* $\mathrm{Lsc}(X, \mathbb{N})$.
3. $\mathrm{supp}_0(a) \precsim^{\alpha} \mathrm{supp}_0(b)$.

The (generalised) type semigroup for higher dimensional spaces

Proposed definition: replace $C(X,\mathbb{N})$ with $\mathrm{Lsc}(X,\mathbb{N})$, a G – W-semigroup.

Definition:

For a compact Hausdorff space X, we define the *type semigroup* as $\mathrm{Lsc}(X,\mathbb{N})/G$.

Theorem (Abstract characterization of $\mathrm{Lsc}(X,G)$)

S abelian monoid generated by open subsets of X, with $0=\emptyset$. Let $\precsim$=strongest preorder that preserves addition and satisfies:

- (i) *$U \precsim V$ iff $U \precsim^{\alpha} V$.*
- (ii) *$U+V \sim (U\cup V)+(U\cap V)$, where $\sim = \precsim$ & $\succsim$.*

Write $\mathrm{Lsc}_G(X,\mathbb{N}) = S/\sim$, equipped with obvious addition and order induced by $\precsim$. Let $\prec$ be the strongest transitive reln that is auxiliary and s.t. $[U]\prec[V]$ if $\overline{U}\subseteq V$. Then $\mathrm{Lsc}_G(X,\mathbb{N})\cong \mathrm{Lsc}(X,\mathbb{N})/G$.

Theorem

$\mathrm{Lsc}_G(X,\mathbb{N})$ *satisfies the universal property: For any* W*-semigroup* T *and any* T*-valued regular Borel measure* μ *on* X*, i.e. satisfying*

(i) $\mu(\emptyset) = 0$.

(ii) $\mu(U) \leq \mu(V)$ *if* $U \subseteq V$.

(iii) $\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$.

(iv) *If* $a \prec \mu(U)$*, there is* V *open with* $\overline{V} \subseteq U$ *such that* $a \prec \mu(V)$.

if μ *is* G*-invariant in the sense that* $\mu(gU) = \mu(U)$ *for* $g \in G$*,* U *open in* X*, then* $\exists!\mu_G\colon \mathrm{Lsc}_G(X,\mathbb{N}) \to T$ *such that* $\mu_G([U]) = \mu(U)$ *for any open subset* U.

Application: Almost finiteness and the small boundary property for $\mathbb{Z}^d$-actions

Theorem (Kerr-Szabó, 2020)

A free minimal action α on a compact space X by an amenable group is almost finite iff it has the small boundary property and dynamical strict comparison.

The Small Boundary Property for an action of G on X generalizes the condition that $\dim(X) < \infty$. If, further, G has subexponential growth, Kerr-Szabó showed that dynamical strict comparison holds, and thus they verified that such an action is almost finite.

Theorem

Free minimal actions of $\mathbb{Z}^d$ on compact metric spaces are almost finite iff they have the small boundary property.

(This uses the fact that, in this setting, the generalized type semigroup is almost unperforated, that is, it has dynamical strict comparison.)

Thank You!!