Lower bounds on the radius of comparison of crossed products by countable amenable groups

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A rough outline

- Introduction.
- Radius of comparison.
- Mean dimension.
- Example: shifts.
- Cohomological dimension.
- Mean cohomological independence dimension.

Introduction

Throughout, X will be a compact metric space, G will be a countable amenable group, and T will be an action of G on X, expressed as a homomorphism $g \mapsto T_g$ from G to the homeomorphisms of X.

Conjecture

Assume that T is essentially free and minimal. Then the mean dimension $\mathrm{mdim}(T)$ and the radius of comparison $\mathrm{rc}(C^*(G, X, T))$ (both described below) are related by $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

The inequality $\mathrm{rc}(C^*(G, X, T)) \leqslant \frac{1}{2}\mathrm{mdim}(T)$ is now known for some special classes of groups G. (Work of Niu.) However, very little is known about the opposite inequality: we really only know about some special examples with $G = \mathbb{Z}$, the minimal subshifts of Giol and Kerr, and even for those we don't know that the expected lower bound is actually correct.

Introduction (continued)

Conjecture

If T is essentially free and minimal, then $\mathrm{rc}(C^*(G, X, T)) = \frac{1}{2}\mathrm{mdim}(T)$.

In this talk, we discuss an approach to getting lower bounds (not as good as the conjecture says, but at least nontrivial) for general (G, T), using rational cohomology.

We only need G to be amenable, and we don't use either minimality or any kind of freeness.

Our results show that various examples of such crossed products really are in the class described in one of the talks yesterday as "mysterious" (or even "nasty").

Reminder of the definition of the radius of comparison

Definition

Let A be a C*-algebra. For $a, b \in M_\infty(A)_+$, we write $a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^{\infty}$ in $M_\infty(A)$ such that $\lim_{n\to\infty} v_n b v_n^* = a$.

Definition

Let A be a unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$. Define $d_\tau \colon M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$.

Definition

Let A be a unital C*-algebra. For $r \in [0, \infty)$, say A has *r-comparison* if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$. Then the *radius of comparison* of A is

$$\mathrm{rc}(A) = \inf\big(\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\}\big)$$

if it exists, and ∞ otherwise.

Mean dimension 1: Open covers; order

Throughout, X is a compact metric space, and $\mathcal{U}$ and $\mathcal{V}$ are finite collections of open subsets of X. For the definition of mean dimension they will be covers of X, but for later use we do not want to require this. Rather, we do the following.

Definition

Let $Y \subset X$ be closed. An *open cover of* Y is a finite collection $\mathcal{U}$ of open subsets of X whose union contains Y.

Definition

Let $\mathcal{U}$ be a finite collection of open subsets of X (not necessarily a cover). The *order* $\mathrm{ord}(\mathcal{U})$ of $\mathcal{U}$ is the least number $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n + 2$ distinct elements of $\mathcal{U}$ is empty.

For example, a cover by disjoint open sets (as is expected in a zero dimensional space) has order zero.

Mean dimension 2: Refinement; dimension

Open cover of $Y \subset X$: finite collection $\mathcal{U}$ of open subsets of X whose union contains Y.

$\operatorname{ord}(\mathcal{U})$ is the least $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n + 2$ distinct elements of $\mathcal{U}$ is empty.

Definition

Let X be a compact metric space, let $Y \in X$ be closed. and let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of Y. Then $\mathcal{V}$ *refines* $\mathcal{U}$ (over Y) if every set in $\mathcal{V}$ is contained in some set in $\mathcal{U}$.

The union of the sets in $\mathcal{V}$ might be smaller than the union of the sets in $\mathcal{U}$, but must still contain Y.

Definition

Let X be a compact metric space, let $Y \in X$ be closed, and let $\mathcal{U}$ be a finite open cover of Y. Then $\mathcal{D}_Y(\mathcal{U})$ is the least possible order of any refinement of $\mathcal{U}$ over Y. Write $\mathcal{D}(\mathcal{U})$ if $Y = X$.

Mean dimension 3: Joins and images

Definition

Let X be a compact metric space, and let $\mathcal{U}$ and $\mathcal{V}$ be finite collections of open subsets of X (not necessarily covers). Then the *join* $\mathcal{U} \vee \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is

$$\mathcal{U} \vee \mathcal{V} = \{ U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V} \}.$$

If $\mathcal{U}$ covers $Y \subset X$ and $\mathcal{V}$ covers $Z \subset X$, then $\mathcal{U} \vee \mathcal{V}$ covers $Y \cap Z$.

Definition

Let X be a compact metric space, let $\mathcal{U}$ be a finite collections of open subsets of X, and let $h \colon X \to X$ be a homeomorphism. We define

$$h(\mathcal{U}) = \{ h(U) : U \in \mathcal{U} \}.$$

If $\mathcal{U}$ covers $Y \subset X$ then $h(\mathcal{U})$ covers $h(Y)$.

Inverse images under continuous maps are defined similarly.

Mean dimension 4: Definition

Now we consider only open covers of X.

Recall: $\operatorname{ord}(\mathcal{U})$ is the least $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n+2$ distinct elements of $\mathcal{U}$ is empty; $\mathcal{D}(\mathcal{U})$ is the least order of any refinement of $\mathcal{U}$; $\mathcal{U} \vee \mathcal{V} = \{ U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V} \}$; and $h(\mathcal{U})$ is taken setwise.

Definition

Let X be a compact metric space and let $h \colon X \to X$ be a homeomorphism. Denote by $\operatorname{Cov}(X)$ the set of finite open covers of X. Then the *mean dimension* of h is

$$\operatorname{mdim}(h) = \sup_{\mathcal{U} \in \operatorname{Cov}(X)} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}.$$

(One needs to check that the limit exists.)

In fact, mean dimension makes sense for an action T of an arbitrary countable amenable group, using Følner sets in place of intervals in $\mathbb{Z}$. We omit the details, but call it $\operatorname{mdim}(T)$ and refer to it later.

Mean dimension 5: The shift

$$\mathrm{mdim}(h) = \sup_{\mathcal{U} \in \mathrm{Cov}(X)} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}.$$

The expression $\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)$ tells you how much of the dimension of X one sees starting with $\mathcal{U}$ and applying h a total of n times. So we are looking at the "linear rate of growth of dimension with iteration of h".

The basic example is the shift on K^G for a compact metric space K, given, for $x = (x_h)_{h \in G} \in K^G$, by $T_h(x)_g = x_{h^{-1}g}$. If K is a finite complex, then one can check that $\mathrm{mdim}(T) = \dim(K)$. The simplest case is to take $\mathcal{U}$ to be the inverse image of an open cover of K under one of the projection maps $K^G \to K$. This depends on $\dim(K^n) = n \dim(K)$.

The shift is of course not free and not minimal. For example, it has fixed points.

Mean dimension 6: Minimal subshifts

The shift on K^G is $T_h(x)_g = x_{h^{-1}g}$. Not free and not minimal.

The known minimal examples nonzero mean dimension (first construction by Lindenstrauss and Weiss; most general by Dou) are subshifts. Here is a brief description of the idea, for $G = \mathbb{Z}$. Let K be a finite complex.

Start with $K^{\mathbb{Z}}$. Choose a "thin" arithmetic progression, and restrict to these sequences whose n-th coordinate is a particular point for each n in the chosen progression. Take the union of the translates of this under $\mathbb{Z}$.

Repeat this process with a much "thinner" arithmetic progression and a suitable choice of points. Do this infinitely often, and take the decreasing intersection X of the subsets one gets.

This construction gives a density parameter ρ, a subset $J \subset G$ with density ρ (what is left after deleting all the arithmetic progressions used), and a point $z \in K^G$, such that if $y \in K^G$ and $y_g = z_g$ for all $g \in G \setminus J$, then $y \in X$. In particular, there is (with abuse of notation) a copy of K^J in X.

The resulting subshift has mean dimension $\rho \dim(K)$.

Cohomological dimension

Definition

Let X be a compact metric space and let R be an abelian group. The *cohomological dimension of* X *with respect to* R is

$$\dim_R(X) = \sup\big(\{n \in \mathbb{Z}_{\geqslant 0} \colon \text{there is a closed subset } Y \subset X \text{ such that } \check{H}^n(X, Y; R) \neq 0\}\big).$$

To see that this isn't off by one, let $X = B^n$, the closed unit ball in $\mathbb{R}^n$, let $Y = S^{n-1} = \partial B^n$, and consider the long exact sequence

$$\cdots \longrightarrow \check{H}^{n-1}(B^n; R) \longrightarrow \check{H}^{n-1}(S^{n-1}; R) \longrightarrow \check{H}^n(B^n, S^{n-1}; R) \longrightarrow \check{H}^n(B^n; R) \longrightarrow \cdots .$$

The first and last groups shown are zero, and $\check{H}^{n-1}(S^{n-1}; R) \cong R$, so $\check{H}^n(B^n, S^{n-1}; R) \cong R$.

It is known that $\dim_R(X) \leqslant \dim_{\mathbb{Z}}(X) \leqslant \dim(X)$.

Cohomological dimension (continued)

$$\dim_R(X) = \sup\big(\{n \in \mathbb{Z}_{\geqslant 0} : \text{there is a closed subset } Y \subset X \text{ such that } \check{H}^n(X, Y; R) \neq 0\}\big).$$

It is known that $\dim_R(X) \leqslant \dim_{\mathbb{Z}}(X) \leqslant \dim(X)$. If X is a finite complex, then $\dim_R(X) = \dim(X)$ for any $R \neq 0$. If X is finite dimensional, then $\dim_{\mathbb{Z}}(X) = \dim(X)$. But $\dim_{\mathbb{Z}}(X) = 3$ and $\dim(X) = \infty$ can happen, and $\dim_{\mathbb{Q}}(X) < \dim_{\mathbb{Z}}(X)$ can happen.

Theorem (Elliott-Niu)

Let X be a compact metric space. Then

$$\mathrm{rc}(C(X)) \geqslant \begin{cases} \dfrac{\dim_{\mathbb{Q}}(X) - 1}{2} - 1 & \dim_{\mathbb{Q}}(X) \text{ is odd} \\ \dfrac{\dim_{\mathbb{Q}}(X)}{2} - 2 & \dim_{\mathbb{Q}}(X) \text{ is even} \\ \infty & \dim_{\mathbb{Q}}(X) = \infty. \end{cases}$$

Cohomological dimension (continued)

Theorem (Elliott-Niu; roughly stated)

Let X be a compact metric space. Then $\mathrm{rc}(C(X))$ is at least about $\frac{1}{2}\dim_{\mathbb{Q}}(X) - 2$.

For any compact metric X, it is known that $\mathrm{rc}(C(X)) \leqslant \frac{1}{2}(\dim(X) - 1)$ (unless the right hand side is negative, in which case $\mathrm{rc}(C(X)) = 0$).

For finite complexes, $\dim(X) = \dim_{\mathbb{Q}}(X)$, so Elliott and Niu give lower bounds for $\mathrm{rc}(C(X))$ in terms of $\dim(X)$. For general X, there are no known nontrivial lower bounds for $\mathrm{rc}(C(X))$ in terms of $\dim(X)$.

Heuristically, the idea of the proof is to construct complex vector bundles over a suitable closed subset $Y \subset X$ with bad comparison properties, interpret them as projections in $M_n(C(Y))$ for sufficiently large n, and extend to positive elements in $M_n(C(X))$. The choice of $\dim_{\mathbb{Q}}(X)$ rather than $\dim_{\mathbb{Z}}(X)$ (which is sometimes bigger) is dictated by the fact that the Chern character takes values in $\check{H}^*(X; \mathbb{Q})$, not in $\check{H}^*(X; \mathbb{Z})$. One can't use K-theory instead, because it is only $\mathbb{Z}/2\mathbb{Z}$-graded.

Mean cohomological independence dimension

Definition

Let G be a countable abelian group, let X be a compact metric space, and let T be an action of G on X. Let $k \in 2\mathbb{Z}$. For $d \in \mathbb{R}$ we say that $\mathrm{mcid}_k(T;\mathbb{Q}) \geqslant d$ if the following happens.

For every $\varepsilon > 0$ there are a closed subset $Y \subset X$, a finite collection $\mathcal{U}$ of open subsets of X which covers Y, such that $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$, and $\eta \in \check{H}^k(Y;\mathcal{U};\mathbb{Q})$ (Čech classes using $\mathcal{U}$) such that for every finite subset $G_0 \subset G$ and every $\delta > 0$ there are a (G_0, δ)-invariant nonempty finite set $F \subset G$ and a subset $F_0 \subset F$ for which:

1. The cup product of $T_g^*(\eta)$, over all $g \in F_0$, which makes sense as an element of $\check{H}^{k \cdot \mathrm{card}(F_0)}\left(\bigcap_{g \in F_0} T_g^{-1}(Y);\mathbb{Q}\right)$, is nonzero.
2. $\dfrac{k \cdot \mathrm{card}(F_0)}{\mathrm{card}(F)} > d - \varepsilon.$

(One can use other rings in place of $\mathbb{Q}$.)

Why subsets of X and the Følner set?

The condition:

For every $\varepsilon > 0$ there are a closed subset $Y \subset X$, an open cover $\mathcal{U}$ of Y, with $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$, and $\eta \in \check{H}^k(Y; \mathcal{U}; \mathbb{Q})$ such that for every finite $G_0 \subset G$ and every $\delta > 0$ there are a (G_0, δ)-invariant nonempty finite $F \subset G$ and $F_0 \subset F$ for which:

1. The cup product of $T_g^*(\eta)$, over all $g \in F_0$, is nonzero.
2. $\dfrac{k \cdot \mathrm{card}(F_0)}{\mathrm{card}(F)} > d - \varepsilon$.

Recall that our collections of open sets need not cover X. Then $\mathcal{D}_Y(\mathcal{U})$ is the least order of a refinement of $\mathcal{U}$ over Y. We need nontrivial cohomology, so, as in the definition of cohmological dimension, we need to use closed subsets $Y \subset X$. (For example, maybe $X = [0,1]^G$.)

Unlike for the full shift, we can't expect the cup product of $T_g^*(\eta)$, over all $g \in F$, to be nonzero. We can only require this to happen for a subset $F_0 \subset F$, preferably of large "density". The left hand side in (2) is density times dimension in cohomology. (This causes unexpected trouble later.)

Subshifts

The condition for $\mathrm{mcid}_k(T; \mathbb{Q}) \geqslant d$:

For every $\varepsilon > 0$ there are a closed subset $Y \subset X$, an open cover $\mathcal{U}$ of Y, with $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$, and $\eta \in \check{H}^k(Y; \mathcal{U}; \mathbb{Q})$ such that for every finite $G_0 \subset G$ and every $\delta > 0$ there are a (G_0, δ)-invariant nonempty finite $F \subset G$ and $F_0 \subset F$ for which:

1. The cup product of $T_g^*(\eta)$, over all $g \in F_0$, is nonzero.
2. $\dfrac{k \cdot \mathrm{card}(F_0)}{\mathrm{card}(F)} > d - \varepsilon$.

The construction of minimal subshifts gives a density parameter ρ, a subset $J \subset G$ with density ρ, and a point $z \in K^G$, such that if $y \in K^G$ and $y_g = z_g$ for all $g \in G \setminus J$, then $y \in X$. In particular, there is (with abuse of notation) a copy of K^J in X.

In the conditions for $\mathrm{mcid}_k(T; \mathbb{Q}) \geqslant d$ for this kind of subshift:

- The set Y will be the inverse image of a copy of S^k in K under one of the coordinate projections $K^G \to K$.
- Given F, the set F_0 will be $F \cap J$.

Why rational coefficients?

For every $\varepsilon > 0$ there are a closed subset $Y \subset X$, an open cover $\mathcal{U}$ of Y, with $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$, and $\eta \in \check{H}^k(Y; \mathcal{U}; \mathbb{Q})$ such that for every finite $G_0 \subset G$ and every $\delta > 0$ there are a (G_0, δ)-invariant nonempty finite $F \subset G$ and $F_0 \subset F$ for which:

1. The cup product of $T_g^*(\eta)$, over all $g \in F_0$, is nonzero.
2. $\dfrac{k \cdot \mathrm{card}(F_0)}{\mathrm{card}(F)} > d - \varepsilon$.

We need rational coefficients (just like in Elliott-Niu) to get from a cohomology class to a vector bundle (plus technical details). The Chern character is an isomorphism from rational K-theory to rational cohomology. To make it additive instead of multiplicative, one needs complicated rational polynomials in the Chern classes.

We need our vector bundle to have no Chern classes in degree greater than k. (Reason later if time.) Having $\mathcal{D}_Y(\mathcal{U}) \in \{k, k+1\}$ does this: all higher even cohomology from this cover is zero.

Lower bounds from $\mathrm{mcid}(T)$

Theorem

Let G be a countable amenable group, let X be a compact metrizable space, and let T be an action of G on X. Let k be an even integer. Then $\mathrm{rc}(C^*(G, X, T)) \geqslant \mathrm{mcid}_k(T; \mathbb{Q}) - 1 - k/2$.

For Dou's subshifts, this gives the following.

Corollary

Let G be a countable amenable group, let Z be a polyhedron, and let $\rho \in (0, 1)$. Let (X, T) be Dou's minimal subshift of the shift on Z^G with $\mathrm{mdim}(T) = \dim(Z)\rho$. Then

$$\mathrm{rc}(C^*(G, X, T)) > \frac{1}{2}\mathrm{mdim}(T)\left(1 - \frac{1-\rho}{\rho}\right) - 2.$$

If ρ is close to 1, this lower bound is nearly $\frac{1}{2}\mathrm{mdim}(T) - 2$. It says nothing if $\rho \leqslant \frac{1}{2}$.

Symmetric mean cohomological independence dimension

Definition

Let X be a compact metrizable space, let G be a countable amenable group, and let T be an action of G on X. Let $k \in 2\mathbb{Z}$. Then $\mathrm{smcid}_k(T;\mathbb{Q})$ is the largest $d \in [0, \infty)$ such that the following happens. There are a finite open cover $\mathcal{U}$ of X such that $\mathcal{D}_X(\mathcal{U}) \in \{k, k+1\}$ and $\eta \in \check{H}^k(X;\mathcal{U};\mathbb{Q})$ (Čech classes using $\mathcal{U}$), such that for every finite subset $G_0 \subset G$ and every $\varepsilon > 0$ there are a (G_0, ε)-invariant nonempty finite set $F \subset G$ and $r \in \{0, 1, 2, \ldots, \mathrm{card}(F)\}$ for which:

1. The r-th elementary symmetric polynomial $\sigma_r\big((T_g^*(\eta))_{g \in F}\big)$ is nonzero.
2. $\dfrac{kr}{\mathrm{card}(F)} > d - \varepsilon$.

Compare what we require to be nonzero: $\sigma_r\big((T_g^*(\eta))_{g \in F}\big) \in \check{H}^{kr}(X;\mathbb{Q})$ now, and $\smile_{g \in F_0} T_g^*(\eta)\big|_{\bigcap_{h \in F_0} T_h^{-1}(Y)} \in \check{H}^{k \cdot \mathrm{card}(F_0)}\big(\bigcap_{g \in F_0} T_g^{-1}(Y);\mathbb{Q}\big)$ before.

Symmetric version vs. original version

Compare what we require to be nonzero:

$$\sigma_r\big((T_g^*(\eta))_{g \in F} \big) \in \check{H}^{kr}(X; \mathbb{Q})$$

now, and

$$\smile_{g \in F_0} T_g^*(\eta)|_{\bigcap_{h \in F_0} T_h^{-1}(Y)} \in \check{H}^{k \cdot \mathrm{card}(F_0)} \left(\bigcap_{g \in F_0} T_g^{-1}(Y); \mathbb{Q} \right)$$

before.

One should think $r = \mathrm{card}(F_0)$ and $Y = X$. In the new version, we avoid the loss that comes from not knowing what happens with the coordinates in $F \setminus F_0$. However, in the new version, if $Y \neq X$ then we must take

$$\sigma_r\big((T_g^*(\eta))_{g \in F} \big) \in \check{H}^{kr} \left(\bigcap_{g \in F} T_g^{-1}(Y); \mathbb{Q} \right).$$

If the action is minimal, $\bigcap_{g \in G} T_g^{-1}(Y) = \varnothing$, so there is $F \subset G$ finite such that $\bigcap_{g \in F} T_g^{-1}(Y) = \varnothing$.

Lower bounds from $\mathrm{smcid}(T)$

Theorem

Let G be a countable amenable group, let X be a compact metrizable space, and let T be an action of G on X. Let k be an even integer. Then $\mathrm{rc}(C^*(G, X, T)) \geqslant \frac{1}{2}\mathrm{smcid}_k(T; \mathbb{Q}) - 1$.

For Dou's subshifts, this gives the following.

Corollary

Let k be a strictly positive even integer, let Z be a k-dimensional polyhedron such that $\check{H}^k(Z; \mathbb{Q}) \neq 0$, let G be a countable amenable group, and let $\rho \in (0, 1)$. Let (X, T) be Dou's minimal subshift of the shift on Z^G with $\mathrm{mdim}(T) = \dim(Z)\rho$. Then

$$\mathrm{rc}(C^*(G, X, T)) > \frac{1}{2}\mathrm{mdim}(T) - 1.$$

Part 1 of the argument: analysis

The following lemma (stated for projections instead of positive elements for simplicity) gets noncomparison in the crossed product from noncomparison of direct sums of group translates in matrices over $C(X)$.

The cohomology conditions give information about noncomparison of direct sums of group translates in matrices over $C(X)$.

Lemma

Let T be an action of G on a connected compact metric space X. Let $\alpha\colon G \to \mathrm{Aut}(C(X))$ be the corresponding action on $C(X)$. Set $A = C^*(G, X, T)$. Let $n \in \mathbb{Z}_{>0}$. Let $p, q \in M_n(C(X))$ be projections, suppose that q is a constant projection, and suppose that $p \precsim_A q$. Then for every $\varepsilon > 0$ there are nonempty finite subsets $P, Q \subset G$ such that $\mathrm{card}(Q) < (1 + \varepsilon)\mathrm{card}(P)$ and

$$\bigoplus_{g \in P} (\mathrm{id}_{M_n} \otimes \alpha_g^{-1})(p) \precsim_{C(X)} \bigoplus_{g \in Q} q.$$

Part 2: direct sums of group translates

For simplicity, consider the shift on $(S^k)^G$ with k even. For a complex vector bundle E over a connected space X, recall the Chern classes $c_j(E) \in \check{H}^{2j}(X; \mathbb{Z})$ and the total Chern class $c(E) = 1 + c_1(E) + \cdots + c_r(E)$, with $r = \operatorname{rank}(E)$. Further recall the sum formula (cup product on the right): $c(E \oplus F) = c(E)c(F)$, and that if F is trivial then $c(F) = 1$. Let ε be a generator of $\check{H}^k(S^k; \mathbb{Z})$. There is a vector bundle E_0 over S^k with (complex) rank $k/2$ such that $c(E_0) = 1 + m\varepsilon$ with $m \neq 0$.

Over $(S^k)^n$ take the direct sum E of the pullbacks of E_0 by the projections to the factors. The ring $\check{H}^*((S^k)^n; \mathbb{Z})$ is commutative and generated by the corresponding pullbacks $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of ε, with the relation $\varepsilon_l^2 = 0$ for all l. So $c(E) = \prod_{l=1}^n (1 + m\varepsilon_l)$. If $E \oplus F$ is trivial, then $c(F) \prod_{l=1}^n (1 + m\varepsilon_l) = 1$, which implies that $c(F) = \prod_{l=1}^n (1 - m\varepsilon_l)$. Since this has a nonzero term $(-m)^n \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n$ in degree kn, it follows that $\operatorname{rank}(F) \geqslant nk/2$.

Radius of comparison for the shift

In the lemma, take p to be the projection on the pullback of E_0 by the projection map to one of the factors. Take q to be a trivial projection. The lemma gives P and Q such that $\mathrm{card}(Q) < (1+\varepsilon)\mathrm{card}(P)$ and

$$\bigoplus_{g \in P} (\mathrm{id}_{M_n} \otimes \alpha_g^{-1})(p) \precsim_{C(X)} \bigoplus_{g \in Q} q.$$

So, by the previous slide at the first step,

$$k\mathrm{card}(P) \leqslant \mathrm{rank}(q)\mathrm{card}(Q) < (1+\varepsilon)\mathrm{rank}(q)\mathrm{card}(P).$$

Since $\varepsilon > 0$ is arbitrary, we get $\mathrm{rank}(q) \geqslant k$. That is, if we take a trivial projection q of rank less than k, the rank $k/2$ projection p is not Murray-von-Neumann subequivalent to q in the crossed product.