OPEN PROBLEMS ABOUT CUNTZ SEMIGROUPS

Abstract. A list of problems proposed during the workshop *Cuntz semigroups*, July 12-16, 2021, at the University of Münster.

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Questions about the range.

(1) Characterize which topological cones arise as $T(A)$ or $\mathrm{QT}(A)$ for certain classes of C*-algebras A, in particular for exact C*-algebras of stable rank one. A specific subquestion: If A is an exact C*-algebra, does there exist a nuclear C*-algebra B such that $T(A) \cong T(B)$?

Here, $T(A)$ and $\mathrm{QT}(A)$ denote the topological cones of lower-semicontinuous $[0, \infty]$-valued traces (2-quasitraces) on a C*-algebra A, [ERS11]. In [MR20], Moodie and Robert characterize $T(A)$ for exact C*-algebras of real rank zero and stable rank one, which turn out to be the same cones realized by AF-algebras.

(2) Given a separable, exact C*-algebra A of real rank zero and stable rank one, is there an AF-algebra B such that $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$?

(3) Given a simple, separable, exact C*-algebra A, is there a nuclear C*-algebra B such that $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$?

(4) What is $\mathrm{Cu}(A)$ for a Villadsen algebra?

Relations between dimension theories.

(1) Can stable rank one for a C*-algebra A be characterized in terms of $\mathrm{Cu}(A)$?

If A has stable rank one, then $\mathrm{Cu}(A)$ is weakly cancellative and satisfies Riesz refinement, [APRT18]. What are sufficient conditions for stable rank one?

If A has real rank zero, then stable rank one corresponds to cancellation in $V(A)$, which can be detected in $\mathrm{Cu}(A)$. Geffen and Winter announced that a simple, stably finite C*-algebra that is pure (the Cuntz semigroup is almost unperforated and almost divisible) has stable rank one. Could one use the covering dimension $\dim(\mathrm{Cu}(A))$ from [TV21]?

(2) What are the possible dimensions of Cuntz semigroups of simple C*-algebras?

There are examples of value 0 (real rank zero), and 1 (e.g. the Jiang-Su algebra). This is related to the computation of the nuclear dimension of simple C*-algebras, which can only take the values $0, 1$ and ∞ by [CET+21].

Subquestion: What is $\dim \mathrm{Cu}(A)$ for A Villadsen, or for A finite but not stably finite (Rørdam's examples)?

(3) In [Niu14], Niu shows that every AH-algebra A with diagonal maps satisfies $\mathrm{rc}(A) \leqslant \frac{1}{2}\,\mathrm{mdim}(A)$, where $\mathrm{mdim}(A)$ denotes the mean dimension introduced by Niu. Find useful conditions under which this bound is sharp.

(4) What is $\mathrm{rc}(C(X))$ when X is 'bad', e.g. in the sense that $\dim_{\mathbb{Q}}(X) \neq \dim(X)$, [EN13]. Can it be much smaller than $\frac{1}{2}\dim(X)$?

(5) Can one obtain an upper bound for $\mathrm{rc}(C(X,A))$ in terms of $\mathrm{rc}(C(X))$ and $\mathrm{rc}(A)$, [AAV20]? Specifically, do we have

$$\mathrm{rc}(C(X,A)) \leqslant \tfrac{1}{2}\dim(X) + \mathrm{rc}(A) + 1?$$

(6) Is there a relation between $\mathrm{rc}(\mathrm{Cu}(A))$ and $\dim(\mathrm{Cu}(A))$? Specifically, do we have $\mathrm{rc}(\mathrm{Cu}(A)) \leqslant \frac{1}{2}\dim(\mathrm{Cu}(A))$? This holds for commutative C*-algebras, since $\dim(\mathrm{Cu}(X)) = \dim(X)$, [TV21].

(7) Does there exist a simple, unital AH-algebra A with $\mathrm{rc}(A) < \frac{1}{2}\mathrm{drr}(A)$?

One always has $\mathrm{rc}(A) \leqslant \frac{1}{2}\mathrm{drr}(A)$, [Tom06].

(8) Develop a theory where the radius of comparison is given as a function: Given a unital C*-algebra A, with Choquet simplex $\mathrm{QT}_1(A)$ of normalized 2-quasitraces, and a function $r\colon \mathrm{QT}_1(A) \to [0,\infty)$, let us say that A has r-comparison if for all $x,y \in \mathrm{Cu}(A)$, if $\widehat{x} + r < \widehat{y}$ in $\mathrm{LAff}(\mathrm{QT}_1(A))$ then $x \leqslant y$.

Given two functions r_1, r_2 such that A has r_1- and r_2-comparison, does it follow that A has r-comparison for a function $r \leqslant r_1, r_2$?

Crossed products.

(1) Let A be a simple, unital, nuclear, separable C*-algebra and assume that $\mathrm{rc}(A) < \infty$. Let α be an automorphims, say, properly outer (+some version of the Rokhlin property). Is it true that $\mathrm{rc}(A \rtimes_\alpha \mathbb{Z}) = 0$?

During the workshop, it has (almost) been proved that pointwise outerness is not good enough here.

(2) Let A be a simple, stably finite, unital, separable C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group on A. Does it follow that

$$\frac{\mathrm{rc}(A)}{|G|} \leqslant \mathrm{rc}(A \rtimes_\alpha G) \leqslant \mathrm{rc}(A)?$$

The two bounds occur in the following 'extreme' cases: Let A_0 be any C*-algebra, and let $A = A_0$ with α the trivial action. Then $A \rtimes_\alpha G \cong A \otimes C^*(G)$, and it follows that $\mathrm{rc}(A \rtimes_\alpha G) = \mathrm{rc}(A)$. In the second case, let $A = C(G, A_0)$ with α the translation action. Then $A \rtimes_\alpha G \cong M_{|G|}(A_0)$, and it follows that $\mathrm{rc}(A \rtimes_\alpha G) = \frac{\mathrm{rc}(A)}{|G|}$.

If A is nuclear, $\mathcal{Z}$-stable and $\mathrm{T}(A)$ is a Bauer simplex with finite-dimensional extreme boundary, then $A \rtimes_\alpha G$ is $\mathcal{Z}$-stable, [GH18].

(3) A specific case of the above question is: Let A be a unital, stably finite C*-algebra, and let $\alpha\colon G \curvearrowright A$ be a finite group action. Assume that α has the Rokhlin property. Does it follow that $\mathrm{rc}(A \rtimes G) = \frac{1}{|G|}\mathrm{rc}(A)$?

The inequality $\mathrm{rc}(A \rtimes G) \leqslant \frac{1}{|G|}\mathrm{rc}(A)$ is known to hold assuming that α has the weak tracial Rokhlin property, [AVGP19]. An equivalent question is if $\mathrm{rc}(A^\alpha) = \mathrm{rc}(A)$, where A^α denotes the fixed-point algebra. This question also makes sense if G is compact.

The following two problems indicate why question (2) above might be hard.

(4) (long open) If A is a simple C*-algebra of stable rank one, and $G \curvearrowright A$ is a finite group action, does it follow that $A \rtimes G$ has stable rank one?

Without simplicity this is known to be false, [Bla90, Example 8.2.1]. The analogous question for real rank zero is known have a negative answer: Elliott constructed an example of a simple AF-algebra with an action of $\mathbb{Z}/2\mathbb{Z}$ such that $A \rtimes \mathbb{Z}/2\mathbb{Z}$ has real rank one, [Ell93, Example 9].

(5) If A is a non-simple, purely infinite C*-algebra (in the sense of Kichberg-Rørdam, [KR00]), and $G \curvearrowright A$ is a finite group action, does it follow that $A \rtimes G$ is purely infinite?

The answer is 'Yes' if A also has the ideal property and $G = \mathbb{Z}/2\mathbb{Z}$ (then $A \rtimes G$ also has the ideal property and is purely infinite), but it is already not known if A has the ideal property and $G = \mathbb{Z}/3\mathbb{Z}$, [PP15, Theorem 6.8(4)].

(6) Given $\lambda \in \mathbb{R} \backslash \mathbb{Q}$, consider the quasifree flow $\sigma_\lambda \colon \mathbb{R} \curvearrowright \mathcal{O}_2$ given by $\sigma_\lambda^t(s_1) = e^{2\pi i t} s_1$ and $\sigma_\lambda^t(s_2) = e^{2\pi \lambda i t} s_2$. Is the crossed product $\mathcal{O}_2 \rtimes_{\sigma_\lambda} \mathbb{R}$ always $\mathcal{Z}$-stable?

It is known that $\mathcal{O}_2 \rtimes_{\sigma_\lambda} \mathbb{R}$ is always simple, nuclear, stable, satisfies the UCT and has vanishing K-theory. If $\lambda < 0$, then $\mathcal{O}_2 \rtimes_{\sigma_\lambda} \mathbb{R}$ is purely infinite and consequently is isomorphic to $\mathcal{O}_2 \otimes \mathcal{K}$; while if $\lambda > 0$, then $\mathcal{O} \rtimes_{\sigma_\lambda} \mathbb{R}$ is stably projectionless, [Kis80, KK96]. For a dense G_δ-subset of positive, irrational numbers λ, it is known that $\mathcal{O}_2 \rtimes_{\sigma_\lambda} \mathbb{R}$ is isomorphic to $W \otimes \mathcal{K}$, the stabilization of the Jacelon-Razac algebra, [Rob12].

Further questions.

(1) A trace τ on a C*-algebra induces a 'trace-kernel' in A, or $\ell_\infty(A)$, or an ultrapower A_ω, and similarly in their Cuntz semigroups. Characterize which ideals with well-behaved quotients arise this way, without referring to traces. The goal is to generalize some of the theory of trace-kernels to the traceless setting.

(2) Can one measure $d_U(\varphi, \psi)$ in terms of the Cuntz semigroup (taking also K_1^{alg} into account) for *-homomorphisms $\varphi, \psi \colon C(\mathbb{T}) \to \mathcal{Z}$ (or more general A)? Here

$$d_u(\varphi, \psi) = \inf_{u \in U(A)} \sup_{f \in \mathrm{Lip}_1} \|\varphi(f) - u\psi(f)u^*\|.$$

This works under the assumption that A has real rank zero, stable rank one, weakly unperforated K-theory, and finitely many extremal traces, [JSV21, Theorem 4.11].

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