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Categorical Aspects of Cuntz Semigroups

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Mathematisches Institut

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Christian Schons

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Christian Schons
Hittorfstraße 6
48149 Münster

c.schons@uni-muenster.de
Matrikelnummer: 394447

Dekan:	Prof. Dr. Xiaoyi Jiang
Erstgutachter:	Dr. Hannes Thiel
Zweitgutachter:	Prof. Dr. Wilhelm Winter

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Introduction

The Cuntz semigroup is an important invariant for C*-algebras. The original version was defined by Cuntz similarly to the Murray-von Neumann semigroup using positive elements instead of projections (see [Cun78]). Thirty years later, Toms presented an example of two simple, separable, unital, nuclear C*-algebras that have the same Elliott invariant and even the same real rank and stable rank, but still are not isomorphic. This was proved by showing that the Cuntz semigroups disagree (see [Tom08]). The example illustrated that the Cuntz semigroup is a powerful invariant which contains more information than for example K-theory. Since it contains a lot of information, it is in general rather difficult to compute.

A breakthrough was achieved by Coward, Elliott and Ivanescu (see [CEI08]), when they slightly changed the original definition of Cuntz and understood this Cuntz semigroup as an order-theoretic object. They obtained a functor $A \mapsto \mathrm{Cu}(A)$ that maps a C*-algebra A to its Cuntz semigroup $\mathrm{Cu}(A)$, which they considered as an object in a certain category of ordered semigroups denoted by Cu, and showed that this functor preserves sequential inductive limits.

Afterwards the category Cu was systematically studied and it turned out that it satisfies many useful properties. One of the main results is that Cu is a closed symmetric monoidal category (see [APT18b] and [APT17]).

In this thesis, I first present that the category Cu is a full, reflective subcategory of a certain category W and a full, coreflective subcategory of a category denoted by $\mathcal{Q}$. Categorical constructions in these categories are easier than in the category Cu thanks to less structure, and the setting allows to transfer them to Cu. This procedure will be illustrated by showing that Cu is complete and cocomplete, i.e. it has small limits and colimits. In addition, concrete pictures of the product and the coproduct of Cuntz semigroups are given. Furthermore, as these constructions are very important and imply that Cu is a closed symmetric monoidal category, we describe the tensor product of Cuntz semigroups and outline what the abstract bivariant Cuntz semigroup $[\![S, T]\!]$ of two given Cuntz semigroups S and T is.

In the last chapter, we focus on some new aspects. In order to understand a category, it is essential to know what the monomorphisms, epimorphisms and isomorphisms are. Regarding this, we first show that the category Cu has a generator, an object which separates the morphisms in a certain sense (see Corollary 5.20). The generator induces a faithful functor $P_c(-)\colon \mathrm{Cu} \to \mathbf{Set}$ which characterizes precisely the monomorphisms and the isomorphisms within Cu (see Theorem 5.30). A characterization of the epimorphisms turns out to be more difficult. Therefore, some special kinds of epimorphisms are considered, such as strong epimorphisms. These can be described with the help of the functor $P_c(-)$ (see Theorem 5.50). Another helpful property is that every morphism in Cu has a strong epi-mono factorization, that is, it can be factorized into a strong epimorphism followed by a monomorphism (see Proposition 5.49). Moreover, the functor $P_c(-)$ admits a left adjoint functor $G\colon \mathbf{Set} \to \mathrm{Cu}$ which means that for every set X and every Cuntz semigroup S, there is a natural bijection

$$\alpha_{S,X}\colon \mathrm{Cu}(G(X), S) \to \mathbf{Set}(X, P_c(S)).$$

The objects of the form $G(X)$ therefore are considered as free Cuntz semigroups.
Finally, we investigate what the categorical notion of subobjects means in Cu (see Theorem 5.45). For example in the categories of sets, groups and C*-algebras, the subobjects in the categorical sense correspond to subsets, subgroups and sub-C*-algebras, respectively.
The main motivation to study all these aspects was to find a Cuntz semigroup analogue of the Ext-functor for abelian groups. The last section covers our endeavour to figure out how such an Ext-functor could be defined.

1 Preliminaries

In this chapter, we some basic notions will be introduced that will be important throughout this thesis.

Definition 1.1: A *positively ordered monoid* is a commutative semigroup M with a zero element 0 and a partial order $\leq$ such that the following properties hold:

(1) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$ for all $a, b, c, d \in M$.
(2) We have $0 \leq a$ for all $a \in M$.

Let **PoM** denote the category whose objects are positively ordered monoids and whose morphisms are additive, order-preserving maps that respect the zero element. The following definition can be found in [Gie+03].

Definition 1.2: Let $(X, \leq)$ be a partially ordered set. We call a relation $\prec$ on X an *auxiliary relation* if it satisfies the following properties for all $a, b, c, d \in X$:

(1) If $a \prec b$, then $a \leq b$.
(2) If $a \leq b \prec c \leq d$, then $a \prec d$.

In the case that X is even a positively ordered monoid, we call an auxiliary relation *additive* if it is compatible with the monoid structure, in the following sense:

(3) $0 \prec a$ for all $a \in X$.
(4) If $a \prec b$ and $c \prec d$, then $a + c \prec b + d$ for all $a, b, c, d \in X$.

An auxiliary relation is a generalization of the so called *way-below relation* $\ll$, which will be defined next. It is the sequential version of the way-below relation defined in [Gie+03].

Definition 1.3: Let $(X, \leq)$ be a partially ordered set.

(a) We say that x is *way-below* y, written as $x \ll y$, if for every increasing sequence $(a_n)_{n\in\mathbb{N}}$ in X for which $\sup_n a_n$ exists and $y \leq \sup_n a_n$, we have $x \leq a_k$ for some $k \in \mathbb{N}$.
(b) If we have $x \ll x$ for some $x \in X$, we call x a *compact* element. We denote the set of all compact elements in X by X_c.

It is easy to see that the way-below relation is an auxiliary relation. Furthermore, if M is a positively ordered monoid for which the way-below relation is additive, then M_c is a submonoid of M. Note that we always have $0 \ll m$ for all $m \in M$ since 0 is the smallest element.
Next, we can define *abstract Cuntz semigroups*. This definition is due to Coward, Elliott and Ivanescu; see [CEI08].

Definition 1.4: An *abstract Cuntz semigroup* (also called a Cu-*semigroup*) is a positively ordered monoid S that satisfies the following axioms:

(O1) Every increasing sequence $(s_n)_{n\in\mathbb{N}}$ has a supremum $\sup_n s_n$ in S.
(O2) For every $s \in S$, there is a $\ll$-increasing sequence $(s_n)_{n\in\mathbb{N}}$ with $\sup_n s_n = s$.
(O3) If $s' \ll s$ and $t' \ll t$, then $s' + t' \ll s + t$ for all $s, s', t, t' \in S$.
(O4) If $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are increasing sequences in S, then

$$\sup_n(s_n + t_n) = \sup_n s_n + \sup_n t_n.$$

A map $f\colon S \to T$ between Cu-semigroups is called a Cu-*morphism* if f preserves the zero element, addition, order, the way-below relation and suprema of increasing sequences. The set of all Cu-morphisms from S to T will be denoted by $\mathrm{Cu}(S,T)$. The category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms will be called Cu.
A *generalized* Cu-*morphism* is a map between Cu-semigroups S and T that preserves the zero element, addition, order and suprema of increasing sequences, but not necessarily the way-below relation. The set of these is denoted by $\mathrm{Cu}[S,T]$.

The theory of Cu-semigroups is interesting because we can assign to every C*-algebra an object within the category Cu. We sketch how this can be done.

1.5: The Cuntz semigroup of a C*-algebra
Let A be a C*-algebra and let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable, infinite-dimensional Hilbert space. Define the following relation on $(A\otimes\mathcal{K})_+$: For $a, b \in (A\otimes\mathcal{K})_+$, we say that a is Cuntz subequivalent to b, written as $a \precsim b$, if there is a sequence $(s_n)_{n\in\mathbb{N}}$ in $A\otimes\mathcal{K}$ so that $a = \lim_{n\to\infty} s_n b s_n^*$. We then set $a \sim b$ if $a \precsim b$ and $b \precsim a$, and we say that a is Cuntz equivalent to b. One can show that $\sim$ defines an equivalence relation on $(A\otimes\mathcal{K})_+$. Let $[a]$ denote the equivalence class of a. The Cuntz semigroup of A is defined as the set of equivalence classes:

$$\mathrm{Cu}(A) := {}^{(A\otimes\mathcal{K})_+}\!\big/_{\sim}.$$

In order to define an addition on $\mathrm{Cu}(A)$, we choose a *-isomorphism $\varphi\colon M_2(\mathcal{K}) \to \mathcal{K}$, which induces a *-isomorphism $\mathrm{id}_A \otimes \varphi\colon M_2(A\otimes\mathcal{K}) \to A\otimes\mathcal{K}$. We define

$$[a] + [b] := [(\mathrm{id}_A \otimes \varphi)\big(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\big)].$$

It can be shown that this addition is well-defined and independent of the choice of φ. An order can be defined by setting $[a] \leq [b]$ if $a \precsim b$. This neither depends on the representatives of the equivalence classes and defines a partial order on $\mathrm{Cu}(A)$. The zero element in $\mathrm{Cu}(A)$ is the class of $0 \in A$. With this structure, $\mathrm{Cu}(A)$ becomes a positively ordered monoid.

In [CEI08], it was shown that the object $\mathrm{Cu}(A)$ is a Cu-semigroup in the sense of Definition 1.4. Let $\mathbf{C}^*$ denote the category whose objects are C*-algebras and whose morphisms are *-homomorphisms.

Theorem 1.6: *Let A be a C*-algebra. Then there is a functor* $\mathrm{Cu}(-)\colon \mathbf{C}^* \to \mathrm{Cu}$ *which sends a C*-algebra A to the* Cu-*semigroup* $\mathrm{Cu}(A)$ *and a *-homomorphism* $\varphi\colon A \to B$ *to the* Cu-*morphism* $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$, *given by*

$$\mathrm{Cu}(\varphi)([a]) := [\varphi \otimes \mathrm{id}_{\mathcal{K}}(a)].$$

1.7: Additional axioms
There are two more known axioms that are satisfied by every Cuntz semigroup that arises from a C*-algebra. Let S be a Cu-semigroup. We say that S satisfies (O5) if for all $a', a, b', b, c \in S$ with

$$a + b \leq c; \;\; a' \ll a \text{ and } b' \ll b,$$

there exists an $x \in S$ such that

$$a' + x \leq c \leq a + x \text{ and } b' \leq x.$$

We say that S satisfies (O6) if for all $a', a, b, c \in S$ with $a' \ll a \leq b + c$, there are elements $e, f \in S$ such that

$$a' \leq e + f; \; e \leq a, b \text{ and } f \leq a, c.$$

In was shown in [APT18b, Proposition 4.7] that (O5) is satisfied for every Cuntz semigroup of a C*-algebra. For (O6), this was shown in [Rob13, Proposition 5.1.1]. There are good reasons not to add these axioms to the definition of a Cu-semigroup since they do not always pass through important constructions (e.g. tensor products) that are possible for the category Cu defined above. In the last chapter, we will see an example of a Cu-semigroup that does not satisfy (O5) and therefore does not arise from a C*-algebra.

Remark 1.8: For the Cu-semigroup $\mathrm{Cu}(A)$ coming from a C*-algebra A, we have a very nice description of the way-below relation: For $\varepsilon > 0$, let $f_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f_\varepsilon(x) := \max\{0, x - \varepsilon\}$. We write $(a - \varepsilon)_+$ instead of $f_\varepsilon(a)$ for $a \in (A \otimes \mathcal{K})_+$. For another element $b \in (A \otimes \mathcal{K})_+$ we have $[a] \ll [b]$ in $\mathrm{Cu}(A)$ if and only if there is an $\varepsilon > 0$ such that $a \precsim (b - \varepsilon)_+$. Particularly we have $[(a - \varepsilon)_+] \ll [a]$ for all $a \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$.
With the help of this, one can show that $\sup_n [(a - \frac{1}{n})_+] = [a]$ for every $a \in (A \otimes \mathcal{K})_+$, which verifies (O2).

Examples 1.9:

(a) Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ be the extended natural numbers with the usual order and addition for $\mathbb{N}$ and $n + \infty = \infty = \infty + n$ and $n \leq \infty$ for all $n \in \overline{\mathbb{N}}$. Then we have $m \ll n$ if and only if $m \neq \infty$ and $m \leq n$. So we have $\overline{\mathbb{N}}_c = \mathbb{N}$. The main observation is that increasing sequences in $\overline{\mathbb{N}}$ either tend to ∞ or become a constant sequence with a finite value at some point. We have $\infty \not\ll \infty$ because the sequence given by $a_n = n$ for $n \in \mathbb{N}$ satisfies $\infty \leq \sup_n a_n$, but $\infty \not\leq a_n$

for all $n \in \mathbb{N}$. With these observations in mind it is easily verified that $\overline{\mathbb{N}}$ is a Cu-semigroup. Furthermore, it is well known that $\overline{\mathbb{N}}$ is the Cuntz semigroup of the compact operators $\mathcal{K}$.

(b) Consider the extended positive real line $\overline{\mathbb{P}} := [0, \infty]$ with the usual addition and order and the ∞ element behaving as in (a). Then we have that $s \ll t$ if and only if $s < t$ or $s = 0$. We have $s \not\ll s$ for all $s \neq 0$ because we can consider the sequence defined by $a_n := s - \frac{1}{kn}$ (with $k \in \mathbb{N}$ as large that $s - \frac{1}{k} \geq 0$). Then we obtain $s \leq \sup_n a_n$, but there is no $n \in \mathbb{N}$ with $s \leq a_n$. So there are no compact elements in $\overline{\mathbb{P}}$ except 0 which always is compact. The axioms (O2) and (O3) are easily verified with the above characterization of the way-below relation. To verify (O1) and (O4), we can use that for an increasing sequence $(a_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{P}}$ we have $\sup_n a_n = \lim_n a_n$, that the real numbers are complete and that the addition is continuous. The Cu-semigroup $\overline{\mathbb{P}}$ can be realized as the Cuntz semigroup of the Jacelon-Razak algebra $\mathcal{W}$ (see [Rob13]).

(c) Consider the disjoint union $\mathbb{N} \sqcup (0, \infty]$ and denote it by Z. In order to distinguish whether an element in in the first or in the second component we write x' for the element in $(0, \infty]$ with value x. Within the components, we equip Z with the usual order and addition. For $n \in \mathbb{N}$ and $x' \in (0, \infty]$, we set $n + x' = (n + x)'$ and $n \leq x'$ if and only if $n' < x'$ in $(0, \infty]$ and finally $x' \leq n$ if and only if $x' \leq n'$ in $(0, \infty]$. One can prove that this defines a Cu-semigroup. It is known that Z is the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$ (see [PT07]).

(d) The set $\mathrm{Lsc}([0, 1], \overline{\mathbb{N}})$ of lower-semicontinuous functions is a Cu-semigroup with the pointwise order and addition. This is the Cuntz-semigroup of the C*-algebra $C([0, 1])$ (see [APS11]).

We keep these examples in mind, some of them will appear later again. The next Lemma can be found in [APT17, Proposition 2.10]. This statement is the initial point for some constructions that will appear later on.

Lemma 1.10: *Let S be a* Cu*-semigroup and let $s \in S$. Then there exists a family $(s_t)_{t\in(0,1]}$ in S such that $s_1 = s$, $s_t \ll s_{t'}$ whenever $t < t'$ in $(0, 1]$, and such that $s_t = \sup_{t'<t} s_{t'}$ for every $t \in (0, 1]$.*

Remark 1.11: Let S be a Cu-semigroup. Then every increasing family $(x_t)_{t\in(0,1)}$ has a supremum. To see this, consider the sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n := 1 - \frac{1}{n}$. The supremum of the sequence $(x_{t_n})_{n\in\mathbb{N}}$ exists by (O1) and we obtain

$$\sup_{t\in(0,1)} x_t = \sup_n x_{t_n}.$$

Furthermore, if we have $a, b \in S$ with $a \ll b$, then the following property holds: Whenever we have an increasing family $(x_t)_{t\in(0,1)}$ with $b \leq \sup_{t\in(0,1)} x_t$, then there is a $t' \in (0, 1)$ with $a \leq x_{t'}$. Considering the sequence $(t_n)_{n\in\mathbb{N}}$ from above, we also deduce that a Cu-morphism $f\colon S \to T$ has the property $f(\sup_{t\in(0,1)} x_t) = \sup_{t\in(0,1)} f(x_t)$ for every increasing family $(x_t)_{t\in(0,1)}$.

2 Categorical Setup

In this chapter, we will introduce the categories W and $\mathcal{Q}$. It will be shown that the category Cu is a full, reflective subcategory of W and a full, coreflective subcategory of $\mathcal{Q}$. This setting is very useful, since categorical constructions in Cu are rather difficult, whereas they are less difficult in W and $\mathcal{Q}$. So we do them in these categories and then apply useful results from category theory to transfer them to the category Cu. The chapter mainly depends on [APT18b, Chapter 2] and [APT17, Section 4].

2.1 The category W and a reflection

The following definition can be found in [APT18a, Definition 2.5].

Definition 2.1: We call a commutative monoid S together with a transitive relation $\prec$ satisfying $0 \prec a$ for all $a \in S$ a W-semigroup if the following conditions hold:

(W1) For every $a \in S$ there is a $\prec$-increasing sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \prec a$ for every $n \in \mathbb{N}$ and for every $b \in S$ with $b \prec a$ there is a $n \in \mathbb{N}$ with $b \prec a_n$.

(W3) If $a, a', b, b' \in S$ with $a' \prec a$ and $b' \prec b$, then $a' + b' \prec a + b$.

(W4) If $a, b, c \in S$ with $a \prec b + c$, then there are $b', c' \in S$ with $b' \prec b$ and $c' \prec c$ such that $a \prec b' + c'$.

A map $f\colon S \to T$ between W-semigroups S and T is called a W-morphism if f preserves the zero element, addition, the relation $\prec$ and furthermore if $a \in S$ and $b \in T$ with $b \prec f(a)$, then there is an element $a' \in S$ with $a' \prec a$ and $b \prec f(a')$.

Remark 2.2: Let S be a Cu-semigroup.

(a) By forgetting about the partial order and taking the way-below relation as relation we easily recognize that S is a W-semigroup. If T is another Cu-semigroup and $f\colon S \to T$ is a Cu-morphism, then it is also a W-morphism when we consider S and T as W-semigroups. In order to see this, we only have to verify the last condition in the definition of a W-morphism. Let $a \in S$ and $b \in T$ with $b \ll f(a)$. Since S is a Cu-semigroup, we find a $\ll$-increasing sequence $(a_n)_{n\in\mathbb{N}}$ with $\sup_n a_n = a$. Since f is a Cu-morphism, we obtain $b \ll \sup_n f(a_n)$, so that there is a $k \in \mathbb{N}$ with $b \ll f(a_k)$. Since the $(a_n)_{n\in\mathbb{N}}$ is $\ll$-increasing, we also have $a_k \ll a$.

(b) We can express the partial order in S via the way-below relation as follows: For $a \in S$, set $a^{\ll} := \{x \in S \mid x \ll a\}$. Let $a, b \in S$. We have $a \leq b$ if and only if $a^{\ll} \subseteq b^{\ll}$. The forward direction is obvious; for the backward implication let $(a_n)_{n\in\mathbb{N}}$ be a $\ll$-increasing sequence with $\sup_n a_n = a$. Because of $a_n \in a^{\ll}$ for all $n \in \mathbb{N}$ we obtain $a_n \in b^{\ll}$ for all $n \in \mathbb{N}$, which in particular means $a_n \leq b$ for all $n \in \mathbb{N}$ and finally leads to $a = \sup_n a_n \leq b$.

Lemma 2.3: *Let* $\iota\colon \mathrm{Cu} \to \mathrm{W}$ *be the inclusion functor, which maps a* Cu*-semigroup* S *to the* W*-semigroup* $(S, \ll)$ *and a* Cu*-morphism to the same morphism considered as a* W*-morphism. Then* ι *embeds* Cu *as a full subcategory into* W.

Proof: We only need to show that a W-morphism between Cu-semigroups is already a Cu-morphism. So, let S and T be Cu-semigroups and $f\colon S \to T$ be a W-morphism. We first show that f preserves the partial order. Let $a, b \in S$ with $a \le b$. By Remark 2.2 (b), in order to show $f(a) \le f(b)$ it is equivalent to show $f(a)^{\ll} \subseteq f(b)^{\ll}$. Let $x \ll f(a)$. Since f is a W-morphism, we find $a' \in S$ with $a' \ll a$ and $x \ll f(a')$. As we have $a \le b$, we also obtain $a' \ll b$. Since f preserves the way below relation, we obtain $x \ll f(a') \ll f(b)$, which implies $x \in f(b)^{\ll}$.
It remains to show that f preserves suprema of increasing sequences - the rest follows immediately from the definition of a W-morphism. Let $(s_n)_{n\in\mathbb{N}}$ be an increasing sequence in S. As f is order preserving, we have $\sup_n f(s_n) \le f(\sup_n s_n)$. To show the inverse inequality, we set $a := f(\sup_n s_n)$ and $b := \sup_n f(s_n)$ and show $a^{\ll} \subseteq b^{\ll}$. Let $x \in a^{\ll}$. Since f is a W-morphism, there is an element $x' \in S$ with $x' \ll \sup_n s_n$ and $x \ll f(x')$. We find a $k \in \mathbb{N}$ such that $x' \le s_k$. We obtain

$$x \ll f(x') \le f(s_k) \le \sup_n f(s_n),$$

which implies $x \in b^{\ll}$. □

Remark 2.4: In [APT18b] it is shown that we can assign to a given W-semigroup $(S, \prec)$ a Cu-semigroup $\gamma(S)$. Let us recall some details: Let $\bar{S}$ be the set of $\prec$-increasing sequences in S. We write $\mathbf{a}$ for a sequence $(a_n)_{n\in\mathbb{N}}$ in $\bar{S}$ and set $\mathbf{a} + \mathbf{b} := (a_k + b_k)_{k\in\mathbb{N}}$ for $\mathbf{a}, \mathbf{b} \in \bar{S}$. Furthermore, we set $\mathbf{a} \subset \mathbf{b}$ if and only if for every $k \in \mathbb{N}$ there is an $l \in \mathbb{N}$ such that $a_k \prec b_l$, which defines a preorder on $\bar{S}$. We obtain an equivalence relation on $\bar{S}$ by setting $\mathbf{a} \sim \mathbf{b}$ if and only if $\mathbf{a} \subset \mathbf{b}$ and $\mathbf{b} \subset \mathbf{a}$. Then we define

$$\gamma(S) := {}^{\bar{S}}\!/\!_{\sim}.$$

The addition on $\bar{S}$ induces an addition on $\gamma(S)$ and the preorder on $\bar{S}$ induces a partial order on $\gamma(S)$ by setting $[\mathbf{a}] \le [\mathbf{b}]$ if and only if $\mathbf{a} \subset \mathbf{b}$. Indeed, with this structure, $\gamma(S)$ becomes a Cu-semigroup (see [APT18b, Proposition 3.1.6]). There is also given a nice description of the way-below relation in $\gamma(S)$: For $\mathbf{a}, \mathbf{b} \in \gamma(S)$ we have $\mathbf{a} \ll \mathbf{b}$ if and only if there is a $k \in \mathbb{N}$ such that $a_l \prec b_k$ for all $l \in \mathbb{N}$.
Let T be another W-semigroup and $f\colon S \to T$ be a W-morphism. Then f induces a Cu-morphism $\gamma(f)\colon \gamma(S) \to \gamma(T)$ via $[(a_n)_{n\in\mathbb{N}}] \mapsto [(f(a_n))_{n\in\mathbb{N}}]$. So, γ defines a functor from W to Cu, the so called Cu-*completion functor*. We also have a W-morphism $\alpha_S\colon S \to \gamma(S)$ which sends an element $s \in S$ to the element $[(a_n)_{n\in\mathbb{N}}]$ where $(a_n)_{n\in\mathbb{N}}$ is a sequence which we get by applying (W1) to the element s. It is easy to see that the equivalence class of $(a_n)_{n\in\mathbb{N}}$ does not depend on the choice of the sequence.

Lemma 2.5: *Let* S *and* T *be* W*-semigroups. Consider the* W*-morphisms* α_S *and* α_T *from Remark 2.4.*

(1) The map α_S has dense image in the following sense: Whenever we have $b', b \in \gamma(S)$ with $b' \ll b$, then there is an element $a \in S$ with $b' \leq \alpha_S(a) \leq b$.
(2) The map α is natural in the following sense: Let $f\colon S \to T$ be a W*-morphism. Then the following diagram commutes:*

$$\begin{array}{ccc} S & \xrightarrow{\;f\;} & T \\ {\scriptstyle\alpha_S}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_T} \\ \gamma(S) & \xrightarrow{\gamma(f)} & \gamma(T). \end{array}$$

Proof: (1) Let $b', b \in \gamma(S)$ with $b' \ll b$; write $b' = [(b'_n)_{n\in\mathbb{N}}]$ and $b = [(b_n)_{n\in\mathbb{N}}]$. This means that there is a $k \in \mathbb{N}$ such that $b'_l \prec b_k$ for all $l \in \mathbb{N}$. Then by the definitions of $\leq$ in $\gamma(S)$ and α_S, we easily see that $b' \leq \alpha_S(b_k) \leq b$.
(2) Let $s \in S$. On the one hand, we have $\alpha_T(f(s)) = [(t_n)_{n\in\mathbb{N}}]$, where $(t_n)_{n\in\mathbb{N}}$ is a sequence satisfying (W1) for the element $f(s) \in T$. On the other hand, we have $\gamma(f)(\alpha_S(s)) = [(f(s_n))_{n\in\mathbb{N}}]$, where $(s_n)_{n\in\mathbb{N}}$ is a sequence satisfying (W1) for the element $s \in S$. We show now that the sequence $(f(s_n))_{n\in\mathbb{N}}$ also satisfies (W1) for the element $f(s) \in T$: We have $s_n \prec s$ for all $n \in \mathbb{N}$. As f preserves the $\prec$-relation, we obtain $f(s_n) \prec f(s)$ for all $n \in \mathbb{N}$. Next, let $a \prec f(s)$. Since f is a W-morphism, we get an element $s' \in S$ with $s' \prec s$ and $a \prec f(s')$. We find $n \in \mathbb{N}$ with $s' \prec s_n$ and deduce $a \prec f(s') \prec f(s_n)$ and finally $a \prec f(s_n)$. This shows that $(f(s_n))_{n\in\mathbb{N}}$ satisfies (W1) for $f(s)$. Since the equivalence class in $\gamma(S)$ is independent of the choice of the sequence satisfying (W1), we finally obtain $[(t_n)_{n\in\mathbb{N}}] = [(f(s_n))_{n\in\mathbb{N}}]$ which shows that the diagram commutes. □

Remark 2.6: The first statement of Lemma 2.5 implies that every element in $\gamma(S)$ can be written as the supremum of a sequence contained in the image of α_S.
Let M be a positively ordered monoid. Then $(M, \leq)$ is a W-semigroup and the γ-functor can be applied to M. In this case, we obtain $\alpha_M(M) = \gamma(M)_c$ because the compact elements in $\gamma(M)$ are represented as the classes of constant sequences in M. Identifying M with the constant sequences in M, we get that every element in $\gamma(M)$ can be written as the supremum of a sequence in M.

Lemma 2.7: *Let S be a* Cu*-semigroup. Then the map $\alpha_S\colon S \to \gamma(S)$ is an order-isomorphism.*

Proof: Let $s, t \in S$ such that $\alpha_S(s) \leq \alpha_S(t)$. Since S is a Cu-semigroup, we find $\ll$-increasing sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ with $\sup_n s_n = s$ and $\sup_n t_n = t$. We have $[(s_n)_{n\in\mathbb{N}}] = \alpha_S(s) \leq \alpha_S(t) = [(t_n)_{n\in\mathbb{N}}]$. Let $k \in \mathbb{N}$. Then we find $l \in \mathbb{N}$ such that $s_k \ll t_l$ and deduce $s_k \leq \sup_l t_l = t$. Therefore, we obtain $s = \sup_k s_k \leq t$. This shows that α_S is an order-embedding.
In order to show surjectivity of α_S, let $a \in \gamma(S)$. Let $(a_n)_{n\in\mathbb{N}}$ be a $\ll$-increasing sequence in S such that $a = [(a_n)_{n\in\mathbb{N}}]$. As S is a Cu-semigroup, $x := \sup_n a_n$ exists in S and we have $\alpha_S(x) = a$ by the definition of α_S. □

Lemma 2.8: *Let S be a* W*-semigroup, T be a* Cu*-semigroup and $g\colon S \to T$ be a W-morphism. Then there is a* Cu*-morphism $\bar{g}\colon \gamma(S) \to T$ such that $g = \bar{g} \circ \alpha_S$.*

Proof: Since T is a Cu-semigroup, the map α_T is an order-isomorphism by Lemma 2.7. Set $\bar{g} := \alpha_T^{-1} \circ \gamma(g)$. This clearly defines a Cu-morphism. Using Lemma 2.5 (2) at the second step, we have $\bar{g} \circ \alpha_S = \alpha_T^{-1} \circ \gamma(g) \circ \alpha_S = g$ as desired. □

Theorem 2.9: *The category* Cu *is a full, reflective subcategory of the category* W*. More precisely, the functor $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ is left adjoint to the inclusion functor $\iota\colon \mathrm{Cu} \to \mathrm{W}$ from Lemma 2.3, i.e. there is a natural bijection*

$$\mathrm{Cu}(\gamma(S), T) \cong \mathrm{W}(S, \iota(T)),$$

which sends a Cu*-morphism $f\colon \gamma(S) \to T$ to the* W*-morphism $f \circ \alpha_S$, where α_S is the* W*-morphism from Remark 2.4.*

Proof: In order to show injectivity, let $f, g \in \mathrm{Cu}(\gamma(S), T)$ with $f \circ \alpha_S = g \circ \alpha_S$. Let $s \in \gamma(S)$. Since $\gamma(S)$ is a Cu-semigroup, we can choose a $\ll$-increasing sequence $(s_n)_{n\in\mathbb{N}}$ in $\gamma(S)$ with $\sup_n s_n = s$. Because α_S has dense image (see 2.5 (1)), for every $n \in \mathbb{N}$, we can choose $s'_n \in S$ such that $s_n \leq \alpha(s'_n) \leq s_{n+1}$. We then have $\sup_n \alpha(s'_n) = \sup_n s_n = s$ and using that f and g preserve suprema of increasing sequences, we obtain

$$f(s) = f(\sup_n \alpha_S(s'_n)) = \sup_n f(\alpha_S(s'_n)) = \sup_n g(\alpha_S(s'_n)) = g(\sup_n \alpha_S(s'_n)) = g(s),$$

which shows $f = g$.
Surjectivity directly follows from Lemma 2.8. The bijection is natural because the map α is natural (see 2.5). □

2.2 The category $\mathcal{P}$ and the path-construction

In this chapter, we introduce the path-construction that was developed in [APT17]. We first recall the definition of the category $\mathcal{P}$, whose objects are semigroups with very little structure.

Definition 2.10: A pair $(S, \prec)$, where S is a commutative monoid and $\prec$ is a transitive relation on S is called a $\mathcal{P}$-semigroup if the following conditions hold:
(1) We have $0 \prec s$ for all $s \in S$.
(2) If $s', s, t', t \in S$ with $s' \prec s$ and $t' \prec t$, then $s' + t' \prec s + t$.
A $\mathcal{P}$-morphism $f\colon S \to T$ between $\mathcal{P}$-semigroups S and T is a monoid morphism that preserves the relation $\prec$. We denote the category whose objects are $\mathcal{P}$-semigroups and whose morphisms are $\mathcal{P}$-morphisms by $\mathcal{P}$.

Definition 2.11: Let S be a $\mathcal{P}$-semigroup. A *path* in S is a map $f\colon (0,1) \to S$ with the property that for $\lambda, \lambda' \in (0,1)$ with $\lambda < \lambda'$, we have $f(\lambda) \prec f(\lambda')$. We denote the set of paths in S by $P(S)$. Let $f, g \in P(S)$. We define an addition on $P(S)$ by setting $(f+g)(\lambda) := f(\lambda) + g(\lambda)$ for $\lambda \in (0,1)$ and a relation by setting $f \precsim g$ if and only if for every $\lambda \in (0,1)$ there is a $\mu \in (0,1)$ such that $f(\lambda) \prec g(\mu)$. We write $f \sim g$ if and only if $f \precsim g$ and $g \precsim f$.

Lemma 2.12: *Let S be a $\mathcal{P}$-semigroup. With the addition from 2.11, $P(S)$ becomes a commutative monoid. Furthermore, the relation $\precsim$ on $P(S)$ is reflexive, transitive and satisfies the following properties:*

(1) For every $f \in P(S)$, we have $0 \precsim f$.

(2) If $f, f', g, g' \in P(S)$ with $f' \precsim f$ and $g' \precsim g$, then $f' + g' \precsim f + g$.

The relation $\sim$ defines an equivalence relation on $P(S)$.

Proof: We only verify statement (2). Let $f, f', g, g' \in P(S)$ with $f' \precsim f$ and $g' \precsim g$ and $\lambda \in (0,1)$. We find $\mu_1, \mu_2 \in (0,1)$ such that $f'(\lambda) \prec f(\mu_1)$ and $g'(\lambda) \prec g(\mu_2)$. By choosing $\mu > \max\{\mu_1, \mu_2\}$ and using that S is a $\mathcal{P}$-semigroup, we obtain

$$(f' + g')(\lambda) \prec (f+g)(\mu).$$

It is clear that $\sim$ defines an equivalence relation since it is just the antisymmetrization of the relation $\precsim$, which is already reflexive and transitive. □

Definition 2.13: Let S be a $\mathcal{P}$-semigroup. We set

$$\tau(S) := {}^{P(S)}\!/\!_{\sim}.$$

Given $f, g \in P(S)$, we define an addition and an order on $\tau(S)$ by setting

$$[f] + [g] := [f+g] \text{ and } [f] \leq [g] \text{ if and only if } f \precsim g \text{ in } P(S).$$

Using Lemma 2.12, we easily see that $\tau(S)$ is a positively ordered monoid. In [APT17], it is shown that $\tau(S)$ is even a Cu-semigroup for every $\mathcal{P}$-semigroup S. Since $\mathcal{P}$-semigroups carry very little structure, the path construction is a helpful tool to create Cu-semigroups. We also have a useful description of the way-below relation on $\tau(S)$, see [APT17, Lemma 3.16].

Lemma 2.14: *Let S be a $\mathcal{P}$-semigroup and $f, f' \in P(S)$. Then we have $[f'] \ll [f]$ in $\tau(S)$ if and only if there is a $\mu \in (0,1)$ such that $f'(\lambda) \prec f(\mu)$ for all $\lambda \in (0,1)$.*

Remark 2.15: Let S and T be $\mathcal{P}$-semigroups and $\varphi\colon S \to T$ be a $\mathcal{P}$-morphism. We obtain a well-defined morphism $\tau(\varphi)\colon \tau(S) \to \tau(T)$ which is given by $[f] \mapsto [\varphi \circ f]$. In [APT17], it is shown that $\tau(\varphi)$ indeed is a Cu-morphism. Of course, we have $\tau(\mathrm{id}_S) = \mathrm{id}_{\tau(S)}$. We also easily see that for another $\mathcal{P}$-semigroup R and a morphism $\psi\colon T \to R$ we have $\tau(\psi \circ \varphi) = \tau(\psi) \circ \tau(\varphi)$. We sum up the statements above in the next lemma.

Lemma 2.16: *The τ construction defines a functor $\tau\colon \mathcal{P} \to \mathrm{Cu}$ by sending a $\mathcal{P}$-semigroup S to the* Cu*-semigroup $\tau(S)$ and a $\mathcal{P}$-morphism $\varphi\colon S \to T$ between $\mathcal{P}$-semigrousps S and T to the* Cu*-morphism $\tau(\varphi)\colon \tau(S) \to \tau(T)$ which is given by $\tau(\varphi)([f]) := [\varphi \circ f]$.*

2.3 The category $\mathcal{Q}$ and a coreflection

In this chapter, we will introduce the category $\mathcal{Q}$ and we will show that the functor τ is a coreflection.

Definition 2.17: A $\mathcal{Q}$-semigroup is a positively ordered monoid S with an additive auxiliary relation $\prec$ such that the following conditions hold:
(O1) For every increasing sequence $(a_n)_{n\in\mathbb{N}}$ in S the supremum $\sup_n a_n$ exists in S.
(O4) If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are increasing sequences in S, then we have

$$\sup_n(a_n + b_n) = \sup_n a_n + \sup b_n.$$

If S and T are $\mathcal{Q}$ semigroups, a map $f\colon S \to T$ will be called a $\mathcal{Q}$-morphism if f preserves the zero element, addition, order, the auxiliary relation and suprema of increasing sequences. We denote the set of $\mathcal{Q}$-morphisms from S to T by $\mathcal{Q}(S,T)$. A generalized $\mathcal{Q}$-morphism is a map $g\colon S \to T$ that preserves the zero element, addition, order and suprema of increasing sequences. The set of generalized $\mathcal{Q}$-morphisms from S to T is denoted by $\mathcal{Q}[S,T]$.

Remark 2.18: We have a functor $\iota\colon \mathrm{Cu} \to \mathcal{Q}$ which maps a Cu-semigroup S to S with the way-below relation as auxiliary relation. Since this relation is additive and S as a Cu-semigroup satisfies (O1) and (O4), S becomes a $\mathcal{Q}$-semigroup. A Cu-morphism $f\colon S \to T$ is sent to f considered as a $\mathcal{Q}$-morphism from $(S,\ll)$ to $(T,\ll)$. Of course, ι defines a functor.

Lemma 2.19: *The functor $\iota\colon \mathrm{Cu} \to \mathcal{Q}$ embeds* Cu *as a full subcategory into $\mathcal{Q}$.*

Proof: We only have to verify that if we have Cu-semigroups S and T and a $\mathcal{Q}$-morphism $f\colon (S,\ll) \to (T,\ll)$, then $f\colon S \to T$ is a Cu-morphism, which is clear by the definition of a $\mathcal{Q}$-morphism. □

Because of the existence of suprema of increasing sequences in $\mathcal{Q}$-semigroups, the following definition makes sense.

Definition 2.20: Let S be a $\mathcal{Q}$-semigroup and let $f \in P(S)$. We set

$$f(1) := \sup_{\lambda\in(0,1)} f(\lambda)$$

and refer to $f(1)$ as the *endpoint* of f.

The next lemma can be found in [APT17, Proposition 4.6].

Lemma 2.21: *Let S be a $\mathcal{Q}$-semigroup and let $f, g \in P(S)$. Then the following statements hold:*
(1) We have $(f+g)(1) = f(1) + g(1)$ in S.
(2) If $f \precsim g$ in $P(S)$, then $f(1) \leq g(1)$ in S.
(3) If $[f] \ll [g]$ in $\tau(S)$, then $f(1) \prec g(1)$ in S.

(4) If $([f_n])_{n\in\mathbb{N}}$ *is an increasing sequence in* $\tau(S)$ *with* $\sup_n[f_n] = [f]$*, then we have* $f(1) = \sup_n f_n(1)$ *in* S*.*

Definition and Lemma 2.22: Let S be a $\mathcal{Q}$-semigroup and let $\varphi_S\colon \tau(S) \to S$ be the map given by $\varphi_S([f]) := f(1)$. Then φ_S is a $\mathcal{Q}$-morphism. We call φ_S the *endpoint map.*

Proof: We immediately get from Lemma 2.21 that φ_S is a well-defined $\mathcal{Q}$-morphism. □

Lemma 2.23: *Let* S *be a* Cu*-semigroup, considered as a* $\mathcal{Q}$*-semigroup* $(S, \ll)$*. Then the endpoint map* $\varphi_S\colon \tau(S) \to S$ *is an order-isomorphism.*

Proof: Let $[f], [g] \in \tau(S)$ with $\varphi_S([f]) \le \varphi_S([g])$. This means $\sup_\lambda f(\lambda) \le \sup_\lambda g(\lambda)$. Let $\lambda_0 \in (0,1)$ and choose $\lambda_1 > \lambda_0$ in $(0,1)$. Then we have

$$f(\lambda_0) \ll f(\lambda_1) \le \sup_\lambda f(\lambda) \le \sup_\lambda g(\lambda).$$

Then we find $\mu \in (0,1)$ with $f(\lambda_0) \le f(\mu)$. By choosing $\mu' > \mu$ in $(0,1)$ we deduce $f(\lambda_0) \ll g(\mu')$. We have shown $f \precsim g$, which means $[f] \le [g]$ in $\tau(S)$ and therefore φ_S is an order-embedding.
In order to show that φ_S is surjective, let $s \in S$. By Lemma 1.10, we can choose a path $(s_\lambda)_{\lambda\in(0,1)}$ that is $\ll$-increasing with $\sup_\lambda s_\lambda = s$. So the map $f\colon (0,1) \to S$ given by $f(\lambda) := s_\lambda$ defines an element in $P(S)$ and we obtain

$$\varphi_S([f]) = f(1) = \sup_\lambda s_\lambda = s.$$ □

Lemma 2.24: *(1) The endpoint map is natural in the following sense: Let* S *and* T *be* $\mathcal{Q}$*-semigroups and* $\alpha\colon S \to T$ *be a* $\mathcal{Q}$*-morphism. Then the following diagram commutes:*

$$\begin{array}{ccc} \tau(S) & \xrightarrow{\tau(\alpha)} & \tau(T) \\ \downarrow{\scriptstyle \varphi_S} & & \downarrow{\scriptstyle \varphi_T} \\ S & \xrightarrow{\alpha} & T \end{array}$$

(2) Let R *be a* Cu*-semigroup and* S *be a* $\mathcal{Q}$*-semigroup. Then for every* $\mathcal{Q}$*-morphism* $\beta\colon R \to S$ *exists a* Cu*-morphism* $\bar\beta\colon R \to \tau(S)$ *such that* $\beta = \varphi_S \circ \bar\beta$*.*

Proof: (1) Let $[f] \in \tau(S)$. Using that α is a $\mathcal{Q}$-morphism at the third step, we obtain

$$(\varphi_T \circ \tau(\alpha))([f]) = \varphi_T([\alpha \circ f]) = \sup_\lambda \alpha(f(\lambda)) = \alpha(\sup_\lambda f(\lambda)) = (\alpha \circ \varphi_S)([f]).$$

(2) Since R is a Cu-semigroup, the endpoint map φ_R is an order-isomorphism. Set $\bar\beta := \tau(\beta) \circ \varphi_R^{-1}$, which defines a Cu-morphism. Using the commutativity of the diagram in (1) at the second step, we have $\varphi_S \circ \bar\beta = \varphi_S \circ \tau(\beta) \circ \varphi_R^{-1} = \beta$. □

Theorem 2.25: *The category* Cu *is a full, coreflective subcategory of $\mathcal{Q}$. The functor $\tau\colon \mathcal{Q} \to \mathrm{Cu}$ is right adjoint to the inclusion functor $\iota\colon \mathrm{Cu} \to \mathcal{Q}$. More precisely, for every* Cu*-semigroup S and every $\mathcal{Q}$-semigroup T there is a natural bijection*

$$\mathrm{Cu}(S, \tau(T)) \cong \mathcal{Q}(\iota(S), T).$$

which sends a Cu*-morphism $\alpha\colon S \to \tau(T)$ to the $\mathcal{Q}$-morphism $\varphi_T \circ \alpha$.*

Proof: At first, we show that the map is injective. Let $\alpha_1, \alpha_2 \in \mathrm{Cu}(S, \tau(T))$ such that $\varphi_T \circ \alpha_1 = \varphi_T \circ \alpha_2$. Let $s \in S$. Since S is a Cu-semigroup, we can find a $\ll$-increasing sequence with $\sup_n s_s = s$. Let $n \in \mathbb{N}$ and choose $f_n, g_n, g \in \tau(T)$ such that $\alpha_1(s_n) = [f_n], \alpha_2(s_n) = [g_n]$ and $\alpha_2(s) = [g]$. Since α_2 preserves the way-below relation, we have $[g_n] \ll [g]$ in $\tau(T)$, which means that there is a $\mu \in (0,1)$ such that $g_n(\lambda) \prec g(\mu)$ for all $\lambda \in (0,1)$. We deduce $g_n(1) = \sup_\lambda g_n(\lambda) \leq g(\mu)$ and obtain

$$f_n(\lambda) \leq f_n(1) = \varphi_T(\alpha_1(s_n)) = \varphi_T(\alpha_2(s_n)) = g_n(1) \leq g(\mu).$$

Choose $\mu' \in (0,1)$ with $\mu' > \mu$. As g is $\prec$-increasing, we have $g(\mu) \prec g(\mu')$ and therefore $f_n(\lambda) \prec g(\mu')$. We have shown $f_n \precsim g$ which exactly says $\alpha_1(s_n) \leq \alpha_2(s)$. Since α_1 preserves suprema of increasing sequences, we deduce

$$\alpha_1(s) = \alpha_1(\sup_n s_n) = \sup_n \alpha_1(s_n) \leq \alpha_2(s),$$

which shows $\alpha_1(s) \leq \alpha_2(s)$ for all $s \in S$. With the same argument we can show $\alpha_2(s) \leq \alpha_1(s)$ for all $s \in S$ and obtain $\alpha_1 = \alpha_2$.
To show that the map is surjective, let $\beta \in \mathcal{Q}(\iota(S), T)$. By (2) of the previous lemma, we obtain $\bar{\beta} \in \mathrm{Cu}(S, \tau(T))$ with $\psi_{S,T}(\bar{\beta}) = \varphi_T \circ \bar{\beta} = \beta$.
The naturality of the bijection follows directly from the naturality of the endpoint map. □

There is a strong relationship between the reflection functor $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ and the functor $\tau\colon \mathcal{P} \to \mathrm{Cu}$. Observe that every W-semigroup is in particular a $\mathcal{P}$-semigroup and that every W-morphism is a $\mathcal{P}$-morphism. Therefore, we obtain an inclusion functor $\kappa\colon \mathrm{W} \to \mathcal{P}$.

Lemma 2.26: *The functors $\tau \circ \kappa$ and γ are naturally equivalent, that is, there is a natural transformation $\Phi\colon \tau \circ \kappa \to \gamma$ such that for every* W*-semigroup S, the Cu-morphism $\Phi_S\colon \tau(\kappa(S)) \to \gamma(S)$ is a natural* Cu*-isomorphism.*

Proof: The proof can be found in [APT18a, Section 1]. □

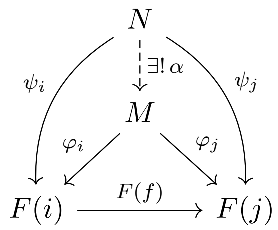
3 Limits and Colimits

In this chaper, it will be shown that the category Cu is complete and cocomplete. It is based on [APT18a]. First, we show that Cu has limits. Let us first recall the definition of a limit in an arbitrary category.

3.1 Completeness

Definition 3.1: Let I be a small category, $\mathscr{C}$ be a category and $F\colon I \to \mathscr{C}$ be a functor. A *cone* over F is an object N of $\mathscr{C}$ together with morphisms $\psi_i\colon N \to F(i)$ for every $i \in I$ such that for any morphism $f\colon i \to j$ we have $F(f) \circ \psi_i = \psi_j$. We write $(N, (\psi_i)_{i\in I})$ for this cone over F.

Definition 3.2: Let I be a small category, $\mathscr{C}$ be a category and $F\colon I \to \mathscr{C}$ be a functor. A *limit* of F is a cone $(M, (\varphi_i)_{i\in I})$ over F which satisfies the following universal property: For any other cone $(N, (\psi_i)_{i\in I})$ over F there is a unique morphism $\alpha\colon N \to M$ such that $\varphi_i \circ \alpha = \psi_i$ for all $i \in I$. We also write $\mathscr{C}\text{-}\varprojlim F$ for the limit of F.



Because of the universal property, the limit is unique up to a canonical isomorphism.

Definition and Lemma 3.3: Let I be a small category and $F\colon I \to \mathcal{Q}$ be a functor. Set $S_i := F(i)$ for an object $i \in I$ and

$$S := \{(s_i)_{i\in I} \mid s_i \in S_i \text{ for all } i \in I,\ F(f)(s_i) = s_j \text{ for } f\colon i \to j \text{ in } I\} \subseteq \textbf{Set-}\prod_{i\in I} S_i.$$

Then S equipped with the componentwise addition, order and auxiliary relation becomes a $\mathcal{Q}$-semigroup.

Proof: The zero element of S is the tupel where s_i is the zero element in S_i for each $i \in I$.
Addition is well-defined since the morphisms $F(f)$ are additive. As all the S_i are positively ordered monoids, it is also easy to verify that S becomes a positively

ordered monoid. It is also easy to see that the relation $\prec$ given by $(s_i)_{i\in I} \prec (t_i)_{i\in I}$ if and only if $s_i \prec t_i$ in S_i for all $i \in I$ defines an additive auxiliary relation on S.
In order to show that S satisfies (O1), let $((s_i^n)_{i\in I})_{n\in\mathbb{N}}$ be an increasing sequence in S. By definition of the order, we have for every $i \in I$ that the sequence $(s_i^n)_{n\in\mathbb{N}}$ is increasing in S_i. Since S_i is a $\mathcal{Q}$-semigroup and therefore fulfills (O1), the element $s_i := \sup_n s_i^n$ exists. Because the $F(f)$ are compatible with suprema of increasing sequences, we obtain:

$$F(f)(s_i) = F(f)(\sup_n s_i^n) = \sup_n F(f)(s_i^n) = \sup_n s_j^n = s_j.$$

So the element $s := (s_i)_{i\in I}$ indeed lies in S. It now can easily be verified that s is the supremum of the sequence $((s_i^n)_{i\in I})_{n\in\mathbb{N}}$.
As addition is definded componentwise and suprema are the componentwise suprema and because S_i satisfies (O4) for every $i \in I$, we also easily obtain that S safisfies (O4). □

Note that considered as a set, the set S is equal to the limit of the family $(S_i)_{i\in I}$ in the category **Set**.

Lemma 3.4: *The category $\mathcal{Q}$ is complete, that is, it has small limits. More precisely, if I is a small category and $F\colon I \to \mathcal{Q}$ a functor, then $(S, (\pi_i)_{i\in I})$ is the limit of F, where S is defined as in 3.3 and the maps $\pi_j\colon S \to S_j$ are given by $\pi_j((s_i)_{i\in I}) := s_j$ for $j \in I$.*

Proof: Since addition, order and auxiliary relation are defined componentwise and since the supremum of an increasing sequence in S is also the componentwise supremum, we easily see that the π_i define $\mathcal{Q}$-morphisms.
By the definition of S, it is also clear that for a morphism $f\colon i \to j$ in I, we have $F(f) \circ \pi_i = \pi_j$. Thus, $(S, (\pi_i)_{i\in I})$ is a cone over F. It remains to show that this cone has the universal property. Let $(T, (q_i)_{i\in I})$ be another cone over F. We define $\alpha\colon T \to S$ by setting $\alpha(t) = (q_i(t))_{i\in I}$. Then we obviously have $\pi_i \circ \alpha = q_i$ for every $i \in I$. It is also clear that α is unique with this property since for $s', s \in S$, we have $s' = s$ if and only if $\pi_i(s) = \pi_i(s')$ for all $i \in I$. Finally, to see that α is a $\mathcal{Q}$-morphism we only have to use that the relations in S are defined componentwise and that the q_i are $\mathcal{Q}$-morphisms for every $i \in I$. □

Theorem 3.5: *The category* Cu *is complete. More precisely, if I is a small category and $F\colon I \to \mathrm{Cu}$ is a functor, let $(S, (\pi_i)_{i\in I})$ be the limit of $\iota \circ F$ in $\mathcal{Q}$. Then $(\tau(S), (\psi_i)_{i\in I})$ is the limit of F in* Cu*, where ψ_i is given by the composition of $\tau(\pi_i)$ with the endpoint map $\varphi_{F(i)}$,i.e. $\psi_i\colon \tau(S) \xrightarrow[\tau(\pi_i)]{} \tau(F(i)) \xrightarrow[\varphi_{F(i)}]{\cong} F(i)$. So we have*

$$\mathrm{Cu}\text{-}\varprojlim F = \tau(\mathcal{Q}\text{-}\varprojlim(\iota \circ F)).$$

Proof: We have shown that the category $\mathcal{Q}$ is complete and that Cu is a full, coreflective subcategory of $\mathcal{Q}$. It is a basic result from category theory that a full, coreflective subcategory of a complete category is complete (see e.g. [Bor94]); and that the limit is given as presented in the statement. □

3.2 Products, pullbacks and equalizers

We will state now some important categorical constructions and will show how these are special cases of limits. As a consequence, these constructed objects exist in complete categories.

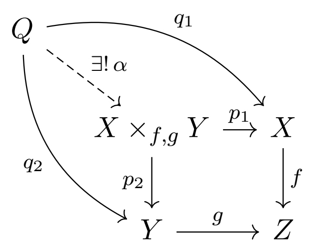
Definition 3.6: Let I be a set and $(X_i)_{i\in I}$ a family of objects in a category $\mathscr{C}$. The product of this family is an object X together with morphisms $\pi_i\colon X \to X_i$ for $i \in I$ such that the following universal property holds: For any other object D in $\mathscr{C}$ with morphisms $f_i\colon D \to X_i$ for all $i \in I$, there is a unique morphism $f\colon D \to X$ such that $\pi_i \circ f = f_i$ for $i \in I$. We also write $\mathscr{C}\text{-}\prod_{i\in I} X_i$ for X and $\prod_{i\in I} f_i$ for the morphism f.

Lemma 3.7: *Let $\mathscr{C}$ be a locally small category that has products and let $(Y_i)_{i\in I}$ be a family of objects in $\mathscr{C}$. Then for every object X in $\mathscr{C}$ we have a canonical bijection*

$$\mathscr{C}(X, \mathscr{C}\text{-}\prod_{i\in I} Y_i) \cong \mathbf{Set}\text{-}\prod_{i\in I} \mathscr{C}(X, Y_i).$$

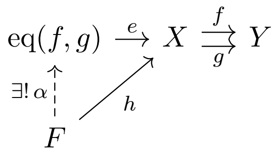
Proof: We map $f\colon X \to \mathscr{C}\text{-}\prod_{i\in I} Y_i$ to the family of morphisms $(\pi_i \circ f)_{i\in I}$ and a family of morphisms $(f_i\colon X \to Y_i)_{i\in I}$ to the morphism $\prod_{i\in I} f_i$. By the universal property of the product, these mappings are inverse to each other. □

Definition 3.8: Let $\mathscr{C}$ be a category, let X, Y and Z be objects in $\mathscr{C}$ and $f\colon X \to Z$ and $g\colon Y \to Z$ be morphisms. The *pullback* along f and g is an object P together with morphisms $p_1\colon P \to X$ and $p_2\colon P \to Y$ such that $f \circ p_1 = g \circ p_2$ and such that the following universal property holds: For every object Q together with morphisms $q_1\colon Q \to X$ and $q_2\colon Q \to Y$ with $f \circ q_1 = g \circ q_2$ there is a unique morphism $\alpha\colon Q \to P$ such that $p_1 \circ \alpha = q_1$ and $p_2 \circ \alpha = q_2$. We also write $X \times_{f,g} Y$ for the pullback along f and g.



Definition 3.9: Let $\mathscr{C}$ be a category, X and Y be objects in $\mathscr{C}$ and $f, g\colon X \to Y$ be morphisms. An object E together with a morphism $e\colon E \to X$ such that $f \circ e = g \circ e$ is called *equalizer* of f and g if the following universal property is satisfied: For every object F with a morphism $h\colon F \to X$ so that $f \circ h = g \circ h$ there is a unique

morphism $\alpha\colon F \to E$ with $e \circ \alpha = h$. We also denote the equalizer of f and g by $\mathrm{eq}(f, g)$.



Remark 3.10: Products, pullbacks and as well equalizers are special limits. If you take as the small category I just a set I equipped only with the identity morphisms, then the limit of a functor $F\colon I \to \mathscr{C}$ is precisely the product of the family $(F(i))_{i\in I}$ in the category $\mathscr{C}$. With regard to pullbacks, you can take for I a set with three elements, say $I = \{a, b, c\}$, and equip it except identities with morphisms $f\colon a \to c$ and $g\colon b \to c$. Then the limit of a functor $F\colon I \to \mathscr{C}$ is the pullback along $F(f)$ and $F(g)$ in $\mathscr{C}$. Analogously, one can obtain the equalizer as a limit.

Example 3.11: In the category $\mathcal{Q}$, the product is just given by the cartesian product together with the componentwise addition, order and auxiliary relation. Let $f\colon S_0 \to T$ and $g\colon S_1 \to T$ be $\mathcal{Q}$-morphisms. Then the pullback along f and g is given by $S_0 \times_{f,g} S_1 = \{(a, b) \in S_0 \times S_1 \mid f(a) = g(b)\}$.
For a parallel pair of $\mathcal{Q}$-morphisms $f, g\colon S \to T$, the equalizer is given by the $\mathcal{Q}$-semigroup $\mathrm{eq}(f, g) = \{s \in S \mid f(s) = g(s)\}$.

Since the category Cu is complete, Cu has in particular products, pullbacks and equalizers.

Corollary 3.12:

(1) Let $(S_i)_{i\in I}$ be a family of Cu*-semigroups, then*

$$\mathrm{Cu}\text{-}\prod_{i\in I} S_i = \tau\big(\mathcal{Q}\text{-}\prod_{i\in I} S_i\big).$$

(2) Let $f\colon S_0 \to T$ and $g\colon S_1 \to T$ be Cu*-morphisms. Then the pullback along f and g in* Cu *is given by $\tau\big(S_0 \times_{f,g} S_1\big)$, where $S_0 \times_{f,g} S_1$ is the pullback along f and g in $\mathcal{Q}$.*

(3) Let $f, g\colon S \to T$ be Cu*-morphisms. Then* $\mathrm{Cu}\text{-}\mathrm{eq}(f, g) = \tau\big(\mathcal{Q}\text{-}\mathrm{eq}(f, g)\big)$.

Remark 3.13: Let I be a finite set and $(S_i)_{i\in I}$ be a family of Cu-semigroups. Then the $\mathcal{Q}$-semigroup $\mathcal{Q}\text{-}\prod_{i\in I} S_i$ with the componentwise way-below realtion is already a Cu-semigroup. Therefore, the endpoint map induces an isomorphism

$$\mathrm{Cu}\text{-}\prod_{i\in I} S_i \cong \mathcal{Q}\text{-}\prod_{i\in I} S_i.$$

To see this, we show that the way-below relation induced by the componentwise partial order agrees with the componentwise way-below relation:

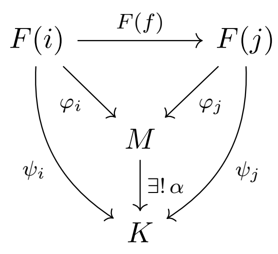
Let $(s_i')_{i\in I}, (s_i)_{i\in I} \in \mathcal{Q}\text{-}\prod_{i\in I} S_i$ such that $s_i' \ll s_i$ for all $i \in I$. Let $((x_n^{(i)})_{i\in I})_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{Q}\text{-}\prod_{i\in I} S_i$ such that $s_i \leq \sup_n x_n^{(i)}$ for every $i \in I$. For every $i \in I$, we find a $n_i \in \mathbb{N}$ such that $s_i' \leq x_{n_i}^{(i)}$. By setting $N := \max\{n_i \mid i \in I\}$ we obtain $s_i' \leq x_N^{(i)}$ for all $i \in I$ which means $(s_i')_{i\in I} \leq (x_N^{(i)})_{i\in I}$ and shows $(s_i')_{i\in I} \ll (s_i)_{i\in I}$. The inverse implication is also easy and even true if I is infinite. We used that I is finite when we defined the maximum above. If I is infinite, the endpoint map need not be injective:
Consider the case $I = \mathbb{N}$ and $S_i = [0,\infty]$ for all $i \in \mathbb{N}$. Let $f, g\colon (0,1) \to \mathcal{Q}\text{-}\prod_{i\in\mathbb{N}}[0,\infty]$ be defined by $f(t) = (t^i)_{i\in\mathbb{N}}$ and $g(t) = (t)_{i\in\mathbb{N}}$. Then $[f]$ and $[g]$ define elements in $\mathrm{Cu}\text{-}\prod_{i\in\mathbb{N}}[0,\infty]$ and it is easy to see that $f(1) = g(1)$, but $[f] \neq [g]$. In particular, $\mathcal{Q}\text{-}\prod_{i\in\mathbb{N}}[0,\infty]$ with the componentwise way-below relation is not a Cu-semigroup and the products in Cu and in $\mathcal{Q}$ differ.

3.3 Cocompleteness

In this section, it will be shown that the category Cu is cocomplete, that is, it has small colimits.

Definition 3.14: Let I be a small category, $\mathscr{C}$ be a category and $F\colon I \to \mathscr{C}$ be a functor. A *cocone* over F is an object K together with a family $(\psi_i)_{i\in I}$ of morphisms $\psi_i\colon F(i) \to K$ such that for every morphism $f\colon i \to j$ in I we have $\psi_j \circ F(f) = \psi_i$.

Definition 3.15: Let I be a small category, $\mathscr{C}$ be a category and $F\colon I \to \mathscr{C}$ be a functor. A *colimit* of F is a cocone $(M, (\varphi_i)_{i\in I})$ over F such that for any other cocone $(K, (\psi_i)_{i\in I})$ over F there exists a unique morphism $\alpha\colon M \to K$ such that $\alpha \circ \varphi_i = \psi_i$ for every $i \in I$. We also write $\mathscr{C}\text{-}\varinjlim F$ for the colimit of F.



The strategy to prove that Cu is cocomplete is analogous to the strategy for the proof of completeness. First, we want to show that the category W is cocomplete and then we want to make use of the fact that Cu is a reflective subcategory of W. In contrast to the proof of completeness, we are not able to give a description of the colimit in W which is as easy as that of the limit in $\mathcal{Q}$. Instead, we use the well-known fact that a category is cocomplete if and only if it has coproducts and coequalizers (see for example [Mac71]). In the following, we will just sketch all this and omit the details. The details can be found in [APT18a]. We can directly show that the category W has inductive limits, which are special colimits. Let us recall the definitions of an inductive limit and an inductive system.

Definition 3.16: An inductive system in a category $\mathscr{C}$ consists of a directed set I, a family of objects $(X_i)_{i\in I}$ and a family of morphisms $\varphi_{i,j}\colon X_i \to X_j$ for all $i \leq j \in I$ such that $\varphi_{i,i} = \mathrm{id}_{X_i}$ for all $i \in I$ and $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k}$ for all $i \leq j \leq k \in I$. An *inductive limit* of such an inductive system is an object X together with morphisms $\varphi_{i,\infty}\colon X_i \to X$ for every $i \in I$ such that $\varphi_{j,\infty} \circ \varphi_{i,j} = \varphi_{i,\infty}$ for all $i \leq j \in I$ and such that the following universal property holds: For any other object Y together with morphisms $g_i\colon X_i \to Y$ for every $i \in I$ such that $g_j \circ \varphi_{i,j} = g_i$ for all $i \leq j \in I$ there is a unique morphism $\alpha\colon X \to Y$ such that $g_i = \alpha \circ \varphi_{i,\infty}$.

Remark 3.17: In the category of sets, the inductive limit of an inductive system $((X_i)_{i\in I}, (\varphi_{i,j})_{i\leq j\in I})$ is given as follows: Let $X := \bigsqcup_{i\in I} X_i$ be the disjoint union of the X_i. For $x_i \in X_i, x_j \in X_j$ we set $x_i \sim x_j$ if there is a $k \geq i, j$ such that $\varphi_{i,k}(x_i) = \varphi_{j,k}(x_j)$ in X_k. This defines an equivalence relation and the set $\tilde{X} := X/\sim$ together with the canonical maps $\varphi_{i,\infty}\colon X_i \to \tilde{X}$ forms the inductive limit is the category of sets. In categories where the objects are just sets with additional algebraic structure, one can consider the inductive limit in **Set** and define the algebraic structure canonically on the inductive limit. In this way we obtain the inductive limit in the category W.

Lemma 3.18: *The category* W *has inductive limits. More precisely, if we have an inductive system* $\big((S_i)_{i\in I}, (\varphi_{i,j})_{i\leq j\in I}\big)$ *of* W*-semigroups, its inductive limit is given by* $S := \bigsqcup_{i\in I} S_i/\sim$*, where* $\sim$ *is defined as in Remark 3.17, with the addition defined by* $[a] + [b] := [\varphi_{i,k}(a) + \varphi_{j,k}(b)]$ *for* $a \in S_i, b \in S_j$ *and any* $k \geq i, j$ *and the transitive relation given by* $[a] \prec [b]$ *if there is a* $k \geq i, j$ *such that* $\varphi_{i,k}(a) \prec \varphi_{j,k}(b)$ *in* S_k*.*

Proof: The proof can be found in [APT18b, Theorem 2.1.10]. □

We have seen that the product is a special case of a limit. Similarly, we can see that the coproduct is a special case of a colimit.

Definition 3.19: Let I be a set, $\mathscr{C}$ be a category and $(X_i)_{i\in I}$ a family of objects in $\mathscr{C}$. An object X together with morphisms $j_i\colon X_i \to X$ is called the *coproduct* of the family $(X_i)_{i\in I}$ if the following universal property holds: For any other object D in $\mathscr{C}$ with morphisms $f_i\colon X_i \to D$ there is a unique morphism $f\colon X \to D$ such that $f \circ j_i = f_i$. We also denote the coproduct by $\mathscr{C}\text{-}\coprod_{i\in I} X_i$ and the morphism f by $\coprod_{i\in I} f_i$.

Lemma 3.20: *Let* $\mathscr{C}$ *be a locally small category that has coproducts and let* $(X_i)_{i\in I}$ *be a family of objects in* $\mathscr{C}$*. Then for every object* Y *in* $\mathscr{C}$*, we have a bijection*

$$\mathscr{C}(\mathscr{C}\text{-}\coprod_{i\in I} X_i, Y) \cong \mathbf{Set}\text{-}\prod_{i\in I} \mathscr{C}(X_i, Y).$$

Proof: We assign a morphism $f\colon \mathscr{C}\text{-}\coprod_{i\in I} X_i \to Y$ to the family of morphisms $(f \circ j_i)_{i\in I}$ and map a family of morphisms $(f_i\colon X_i \to Y)_{i\in I}$ to the morphism $\coprod_{i\in I} f_i$. Using the universal property of the coproduct, we deduce that these mappings are inverse to each other. □

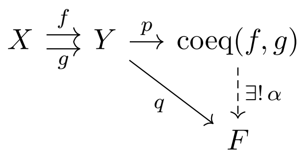
The fact that W has inductive limits can now be used to show that W has coproducts.

Lemma 3.21: *The category* W *has coproducts.*

Proof: It is easy to see that the category W has finite coproducts: it is just the coproduct of the W-semigroups considered as sets endowed with the componentwise addition and binary relation. Using the basic result from category theory which says that categories have coproducts if they have finite coproducts and inductive limits proves the claim. □

Next, we state what the coequalizers of W look like and first recall the definition of coequalizers.

Definition 3.22: Let $\mathscr{C}$ be a category and let $f, g\colon X \to Y$ be morphisms. An object C together with a morphism $p\colon Y \to C$ such that $f \circ p = g \circ p$ is called the *coequalizer* of f and g if the following universal property is satisfied: For every object F with a morphism $q\colon Y \to F$ such that $f \circ q = g \circ q$ there is a unique morphism $\alpha\colon C \to F$ such that $\alpha \circ p = q$. We also denote the coequalizer of f and g by $\mathrm{coeq}(f, g)$.



Remark 3.23: Let S and T be W-semigroups and let $f, g\colon S \to T$ be W-morphisms. For $x, y \in T$ we define $x \sim_0 y$ if there are elements $z \in T$ and $a, b \in S$ such that $x = z + f(a) + g(b)$ and $y = z + f(b) + g(a)$. Let $\sim$ be the transitive closure of $\sim_0$. One can show that $\sim$ is an equivalence relation on T that is compatible with the addition.
Let $x, y \in T$. We set $x \precsim_0 y$ if there are a and b in T such that $x \sim a \prec b \sim y$. Let $\precsim$ be the transitive closure of $\precsim_0$. One can show that $\precsim$ is a transitive relation that is compatible with the addition.

Lemma 3.24: *The category* W *has coequalizers. More precisely, for* W*-morphisms* $f, g\colon S \to T$*, the coequalizer of* f *and* g *is given by* $\big(T/{\sim}, \prec\big)$*, where* $\sim$ *is the equivalence relation from 3.23 and where* $\prec$ *is given by* $[a] \prec [b]$ *if* $a \precsim b$ *for* $a, b \in T$*.*

Proof: The proof can be found in [APT18a, Chapter 2]. □

Lemma 3.25: *The category* W *is cocomplete.*

Proof: We have seen that W has coproducts and coequalizers. In general, a category is cocomplete if and only if it has coproducts and coequalizers. □

Theorem 3.26: *The category* Cu *is cocomplete. Let* I *is a small category and* $F\colon I \to \mathrm{Cu}$ *be a functor. Let* $(K, (\psi_i)_{i\in I})$ *be the colimit of* $\iota \circ F$ *in* W*. Then*

$(\gamma(K), (\varphi_i)_{i\in I})$ is the colimit of F in Cu*, where the maps φ_i are given by the composition: $F(i) \xrightarrow[\alpha_{F(i)}]{\cong} \gamma(F(i)) \xrightarrow[\gamma(\psi_i)]{} \gamma(M)$. The map $\alpha_{F(i)}$ is the order isomorphism from Lemma 2.7. In short, we have*

$$\mathrm{Cu}\text{-}\varinjlim F = \gamma\big(\mathrm{W}\text{-}\varinjlim(\iota \circ F)\big).$$

Proof: Since the category W is cocomplete and Cu is a reflective subcategory of W with reflector γ, we get from category theory that we get the colimit in Cu by applying γ to the colimit in W (see e.g. [Bor94, Propodition 3.5.3]). □

Remark 3.27: We can also express the colimit in Cu in terms of the τ-functor. Since the functors γ and $\tau \circ \kappa$ are naturally equivalent (see Lemma 2.26), where $\kappa\colon \mathrm{W} \to \mathcal{P}$ is the inclusion functor, we also have

$$\mathrm{Cu}\text{-}\varinjlim F = (\tau \circ \kappa)\big(\mathrm{W}\text{-}\varinjlim(\iota \circ F)\big).$$

3.4 Coproducts

Since the coproduct is a special case of a colimit, the category Cu in particular has coproducts. The aim of this last section of the chapter is to give a precise description of the coproduct in Cu that will be helpful later in this thesis. First, we consider the category $\mathcal{P}$ and describe the coproduct there.

Remark 3.28: Let $(S_i)_{i\in I}$ be a family of $\mathcal{P}$-semigroups. Define

$$S := \{(s_i)_{i\in I} \mid s_i \in S_i \text{ for all } i \in I,\ s_i \neq 0 \text{ for at most finitely many } s_i\}.$$

Let $k_i\colon S_i \to S$ be the canonical inclusion, that is, $k_i(s) = (\delta_{i,j}\, s)_{j\in I}$. It can easily be verified that $(S, (k_i)_{i\in I})$ is the coproduct of the family $(S_i)_{i\in I}$ in $\mathcal{P}$. If $(f_i\colon S_i \to T)_{i\in I}$ is a family of $\mathcal{P}$-morphisms, the morphism we obtain by the universal property is given by $\coprod_{i\in I} f_i\colon \mathcal{P}\text{-}\coprod_{i\in I} S_i \to T$ with $\big(\coprod_{i\in I} f_i\big)((s_i)_{i\in I}) = \sum_{i\in I} f_i(s_i)$.

Lemma 3.29: *Let $\kappa\colon \mathrm{W} \to \mathcal{P}$ be the inclusion functor and let $(S_i)_{i\in I}$ be a family of* W*-semigroups. Then we have*

$$\kappa\big(\mathrm{W}\text{-}\coprod_{i\in I} S_i\big) \cong \mathcal{P}\text{-}\coprod_{i\in I} S_i.$$

Proof: See [APT18a, Chapter 4]. □

Corollary 3.30: *Let $(S_i)_{i\in I}$ be a family of* Cu*-semigroups. Then we have*

$$\mathrm{Cu}\text{-}\coprod_{i\in I} S_i \cong \tau\big(\mathcal{P}\text{-}\coprod_{i\in I} S_i\big).$$

Proof: We immediately get this by Remark 3.27 and Lemma 3.29. □

Remark 3.31: Let $(S_i)_{i\in I}$ be a family of Cu-semigroups and let $x \in \text{Cu-}\coprod_{i\in I} S_i$. By Corollary 3.30, we can find a path $x'\colon (0,1) \to \mathcal{P}\text{-}\coprod_{i\in I} S_i$ such that $x = [x']$.
The morphisms belonging to the coproduct in Cu are given as follows:
Let $k_i\colon S_i \to \mathcal{P}\text{-}\coprod_{i\in I} S_i$ be the canonical inclusions defined in Remark 3.28. The maps $j_i\colon S_i \to \text{Cu-}\coprod_{i\in I} S_i$ are given by $j_i = \tau(k_i) \circ \varphi_{S_i}^{-1}$, where $\varphi_{S_i}\colon \tau(S_i) \to S_i$ is the endpoint map.
Let T be another Cu-semigroup and $f_i\colon S_i \to T$ for $i \in I$ be a family of Cu-morphisms. By the universal property of the coproduct in $\mathcal{P}$, we get $f := \coprod_{i\in I} f_i\colon \mathcal{P}\text{-}\coprod_{i\in I} S_i \to T$ such that $f \circ k_i = f_i$. One can verify that the map $g := \varphi_T \circ \tau(f)\colon \text{Cu-}\coprod_{i\in I} S_i \to T$ is the unique Cu-morphism that shows the universal property of the coproduct in Cu. The equation $g \circ j_i = f_i$ is verified using the naturality of the endpoint map (see 2.24).

4 Tensor Products and Abstract Bivariant Cuntz Semigroups

In this chapter, we will show that the category Cu is a closed symmetric monoidal category. This is one of the main results of [APT17]. We first state what it means for a category to be a closed symmetric monoidal category.

4.1: Closed symmetric monoidal categories
A *monoidal* category $\mathcal{V}$ consists of a category $\mathcal{V}_0$, a functor $\otimes\colon \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$, an object I of $\mathcal{V}_0$ and natural isomorphisms $a_{XYZ}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X\colon I \otimes X \to X$ and $r_X\colon X \otimes I \to X$ for all objects X, Y and Z of $\mathcal{V}_0$ that satisfy certain coherence axioms. We call $\otimes$ the *tensor product* and I the *unit object*. If we assume that the category $\mathcal{V}_0$ is locally small, we obtain a functor $V\colon \mathcal{V}_0 \to \mathbf{Set}$, given by $X \mapsto \mathcal{V}_0(I, X)$.
A monoidal category $\mathcal{V}$ is said to be *symmetric* if for all objects X, Y in $\mathcal{V}_0$ there is a natural isomorphism $c_{XY}\colon X \otimes Y \to Y \otimes X$ that as well satisfies certain axioms.
A monoidal category $\mathcal{V}$ is called *closed* if the functor $- \otimes Y\colon \mathcal{V}_0 \to \mathcal{V}_0$ has a right adjoint $[Y, -]\colon \mathcal{V}_0 \to \mathcal{V}_0$ for every object Y of $\mathcal{V}_0$, i.e. for all objects X, Y, Z of $\mathcal{V}_0$ there is a natural bijection

$$\mathcal{V}_0(X \otimes Y, Z) \cong \mathcal{V}_0(X, [Y, Z]). \tag{4.1}$$

Setting $X = I$ and using the natural isomorphism $l_Y\colon I \otimes Y \to Y$, we obtain a natural bijection

$$\mathcal{V}_0(I, [Y, Z]) \cong \mathcal{V}_0(I \otimes Y, Z) \cong \mathcal{V}_0(Y, Z).$$

The object $[Y, Z]$ is also called the *internal hom* object of Y and Z.
Setting $Y = I$ in (4.1) and using the natural isomorphism $r_X\colon X \otimes I \to X$, we get natural isomorphisms $\mathcal{V}_0(X, Z) \cong \mathcal{V}_0(X, [I, Z])$ for all objects X, Y, Z in $\mathcal{V}_0$, which implies $Z \cong [I, Z]$ for all Z in $\mathcal{V}_0$.
Replacing X by $W \otimes X$ in (4.1) and using the natural isomorphism

$$a_{WXY}\colon (W \otimes X) \otimes Y \to W \otimes (X \otimes Y),$$

we get for all W, X, Y, Z in $\mathcal{V}_0$ natural bijections

$$\begin{aligned}\mathcal{V}_0(W, [X, [Y, Z]]) \cong \mathcal{V}_0(W \otimes X, [Y, Z]) &\cong \mathcal{V}_0((W \otimes X) \otimes Y, Z)\\ &\cong \mathcal{V}_0(W \otimes (X \otimes Y), Z)\\ &\cong \mathcal{V}_0(W, [X \otimes Y, Z]),\end{aligned}$$

which implies that there is a natural isomorphism

$$[X,[Y,Z]] \cong [X \otimes Y, Z] \text{ for all } X,Y,Z \text{ in } \mathcal{V}_0.$$

For examples and more details, see [Kel05, Chapter 1]. In this chapter, we will make these things concrete for the category Cu. We make no difference here between the monoidal category $\mathcal{V}$ and the underlying category $\mathcal{V}_0$.

4.1 Tensor products

Let $\mathscr{C}$ be a locally small category that has products. The tensor product $\otimes$ in $\mathscr{C}$ which belongs to the definition of a monoidal category is often associated to the bimorphisms in the sense that there is a natural bijection

$$\mathrm{Bi}\mathscr{C}(X \times Y, Z) \cong \mathscr{C}(X \otimes Y, Z) \text{ for all } X,Y \text{ and } Z \text{ in } \mathscr{C}.$$

In order to show that Cu has tensor products, it is again very useful to have the reflection functor $\gamma\colon \mathrm{W} \to \mathrm{Cu}$. The strategy used in [APT18b] roughly works as follows: First, a tensor product in **PoM**, the category of positively ordered monoids, is constructed. For W-semigroups, one can equip the tensor product in **PoM** with a suitable relation such that it becomes a W-semigroup. Finally, for Cu-semigroups S and T, their tensor product in Cu is given by $S \otimes_{\mathrm{Cu}} T = \gamma(S \otimes_{\mathrm{W}} T)$. It turns out that with the right notion of Cu-bimorphisms, this tensor product has the desired universal property.

Definition 4.2: Let M,N and P be positively ordered monoids. We call a map $\varphi\colon M \times N \to P$ a PoM-*bimorphism* if the map $\varphi(m,-)\colon N \to P, n \mapsto \varphi(m,n)$ is a PoM-morphism for each $m \in M$ and if the map $\varphi(-,n)\colon M \to P, m \mapsto \varphi(m,n)$ is a PoM-morphism for each $n \in N$.

The following definition can be found in [APT18b, Definition 6.3.1].

Definition 4.3: Let S,T and P be Cu-semigroups. A map $\varphi\colon S \times T \to P$ is called a Cu-*bimorphism*, if it is a PoM-bimorphism and if the following conditions hold:

(a) We have $\sup_k \varphi(a_k,b_k) = \varphi(\sup_k a_k, \sup_k b_k)$ for all increasing sequences $(a_k)_{k\in\mathbb{N}}$ in S and $(b_k)_{k\in\mathbb{N}}$ in T.

(b) If $a',a \in S$ and $b',b \in T$ satisfy $a' \ll a$ and $b' \ll b$, then $\varphi(a',b') \ll \varphi(a,b)$.

We denote the set of Cu-bimorphisms from $S \times T$ to P by $\mathrm{BiCu}(S \times T, P)$.

The following theorem can be found in [APT18b, Theorem 6.3.3]. We can equip both $\mathrm{Cu}(S,T)$ and $\mathrm{BiCu}(S,T)$ with the pointwise order and addition, such that these sets become positively ordered monoids.

Theorem 4.4: *For* Cu-*semigroups* S *and* T *there exists a* Cu-*semigroup* $S \otimes_{\mathrm{Cu}} T$ *and a* Cu-*bimorphism* $\omega\colon S \times T \to S \otimes_{\mathrm{Cu}} T$ *such that for every* Cu-*semigroup* P *the following properties hold:*

(a) *For every* Cu*-bimorphism* $\varphi\colon S \times T \to P$ *there is a unique* Cu*-morphism* $\overline{\varphi}\colon S \otimes_{\mathrm{Cu}} T \to P$ *such that* $\varphi = \overline{\varphi} \circ \omega$.

(b) *If* $f, g\colon S \otimes T \to P$ *are* Cu*-morphisms, then* $f \leq g$ *if and only if* $f \circ \omega \leq g \circ \omega$.

Therefore, the map that sends a Cu*-morphism* $\alpha\colon S \otimes_{\mathrm{Cu}} T \to P$ *to the* Cu*-bimorphism* $\alpha \circ \omega$ *defines a natural bijection*

$$\mathrm{Cu}(S \otimes_{\mathrm{Cu}} T, P) \cong \mathrm{BiCu}(S \times T, P).$$

Remark 4.5: Let S, S', T and T' be Cu-semigroups. We write $s \otimes t$ instead of $\omega(s,t)$. Let $f\colon S \to T$ and $g\colon S' \to T'$ be Cu-morphisms. Then the map

$$\omega \circ (f \times g)\colon S \times S' \to T \otimes_{\mathrm{Cu}} T'$$

which is given by $(s, s') \mapsto f(s) \otimes g(s')$ is a Cu-bimorphism. By Theroem 4.4 (a), we get a unique morphism $f \otimes g\colon S \otimes_{\mathrm{Cu}} S' \to T \otimes_{\mathrm{Cu}} T'$ such that $(f \otimes g)(s \otimes s') = f(s) \otimes g(s')$ for all $s \in S$ and $s' \in S'$. This yields a functor $\otimes\colon \mathrm{Cu} \times \mathrm{Cu} \to \mathrm{Cu}$.
In [APT18b, Corollary 6.3.6] it is shown that $\otimes_{\mathrm{Cu}}$ is assosiative and that we have a natural isomorphism $S \otimes_{\mathrm{Cu}} \overline{\mathbb{N}} \cong S$ for every Cu-semigroup S. This isomorphism is given by $s \otimes n \mapsto ns$ for $s \in S$ and $n \in \mathbb{N}$ and the element $s \otimes \infty$ with $s \in S$ is mapped to $\sup_n ns$. Moreover, we have for all Cu-semigroups S and T that $S \otimes_{\mathrm{Cu}} T \cong T \otimes_{\mathrm{Cu}} S$. From this, we can deduce that Cu is a symmetric monoidal category.

4.2 The internal hom functor

Next, we want to show that the monoidal category Cu is closed. In order to show this, we need to find a functor $[\![-,-]\!]\colon \mathrm{Cu} \times \mathrm{Cu} \to \mathrm{Cu}$ that is right adjoint to the tensor product. Again, the coreflection functor $\tau\colon \mathcal{Q} \to \mathrm{Cu}$ is very useful. The strategy is to construct such a functor in the category $\mathcal{Q}$ and transfer this with the coreflection functor to the category Cu. We sum up the main steps on the way to the internal hom functor in Cu, following along [APT17, Chapter 4f].
Analogously to Cu-bimorphisms, we can define $\mathcal{Q}$-bimorphisms. The next definition can be found in [APT17, Definition 4.19].

Definition 4.6: Let S, T and P be $\mathcal{Q}$-semigroups and let $\varphi\colon S \times T \to P$ be a PoM-bimorphism. We call φ a *$\mathcal{Q}$-bimorphism* if the following conditions are satisfied:

(a) We have $\sup_k \varphi(a_k, b_k) = \varphi(\sup_k a_k, \sup_k b_k)$ for all increasing sequences $(a_k)_{k\in\mathbb{N}}$ in S and $(b_k)_{k\in\mathbb{N}}$ in T.

(b) If we have $a', a \in S$ and $b', b \in T$ with $a' \prec a$ and $b' \prec b$, then $\varphi(a', b') \prec \varphi(a, b)$.

It is easy to see that if S, T and P are Cu-semigroups, then a map $\varphi\colon S \times T \to P$ is a Cu-bimorphism if and only if it is a $\mathcal{Q}$-bimorphism when we consider S, T and P as $\mathcal{Q}$-semigroups. Thus, we have

$$\mathrm{BiCu}(S \times T, P) \cong \mathrm{Bi}\mathcal{Q}(S \times T, P).$$

Remark 4.7: Let S and T be $\mathcal{Q}$-semigroups. We can equip the set of generalized $\mathcal{Q}$-morphisms $\mathcal{Q}[S,T]$ with a relation by setting $f \prec g$ if $f(s') \prec g(s)$ for all $s', s \in S$ with $s' \prec s$. Then $\prec$ defines an auxiliary relation on $\mathcal{Q}[S,T]$ and the generalized Cu-morphisms become a $\mathcal{Q}$-semigroup. Note that the set $\mathcal{Q}(S,T)$ is not in general a $\mathcal{Q}$-semigroup since the supremum of $\mathcal{Q}$-morphisms need not preserve the auxiliary relation.
When we assume S and T to be Cu-semigroups, then we have $\mathcal{Q}[S,T] = \mathrm{Cu}[S,T]$ and $(\mathrm{Cu}[S,T], \prec)$ is a $\mathcal{Q}$-semigroup. In this case, the relation $\prec$ is given by $f \prec g$ if and only if $f(s') \ll g(s)$ for all $s', s \in S$ with $s' \ll s$. In particular, we have $f \prec f$ if and only if f is a Cu-morphism.
Let $\varphi\colon S \times T \to P$ be a $\mathcal{Q}$-bimorphism. Then for every $s \in S$, we get a generalized $\mathcal{Q}$-morphism $\varphi_s\colon T \to P$ by setting $\varphi_s(t) := \varphi(s,t)$. Furthermore, if $s, s' \in S$ with $s' \prec s$, then $\varphi_{s'} \prec \varphi_s$ in $\mathcal{Q}[T,P]$. We obtain a $\mathcal{Q}$-morphism $\tilde{\varphi}\colon S \to \mathcal{Q}[T,P]$ which is given by $\tilde{\varphi}(s) = \varphi_s$.

The following statement is [APT17, Theorem 4.22].

Theorem 4.8: *Let S, T and P be* Cu*-semigroups. Then we have a natural bijection*

$$\mathrm{Bi}\mathcal{Q}(S \times T, P) \cong \mathcal{Q}(S, \mathcal{Q}[T,P]).$$

The map assigns a Cu*-bimorphism $\varphi\colon S \times T \to P$ to the $\mathcal{Q}$-morphism $\tilde{\varphi}$ from Remark 4.7.*

Remark 4.9: Let T and T' be $\mathcal{Q}$-semigroups. For every $\mathcal{Q}$-semigroup S, we obtain a covariant functor $\mathcal{Q}[S,-]\colon \mathcal{Q} \to \mathcal{Q}$ which maps a $\mathcal{Q}$-morphism $f\colon T \to T'$ to the $\mathcal{Q}$-morphism $f_*\colon \mathcal{Q}[S,T] \to \mathcal{Q}[S,T']$ given by $f_*(\alpha) = f \circ \alpha$.
Let S and S' be $\mathcal{Q}$-semigroups. Analogously, for every $\mathcal{Q}$-semigroup T we obtain a contravariant functor $\mathcal{Q}[-,T]\colon \mathcal{Q} \to \mathcal{Q}$ that assigns a $\mathcal{Q}$-morphism $g\colon S \to S'$ to the $\mathcal{Q}$-morphism $g^*\colon \mathcal{Q}[S',T] \to \mathcal{Q}[S,T]$ given by $g^*(\alpha) = \alpha \circ g$. It is straightforward to show that f_* and g^* indeed are $\mathcal{Q}$-morphisms.

Now, we can define the internal hom functor for the category Cu.

Definition 4.10: Let S and T be Cu-semigroups. Then we define

$$[\![S,T]\!] := \tau(\mathrm{Cu}[S,T], \prec).$$

We call $[\![S,T]\!]$ the *internal hom* of S and T or the *abstract bivariant Cuntz semigroup* of S and T.

Remark 4.11: For every Cu-semigroup S, we obtain a covariant functor

$$[\![S,-]\!]\colon \mathrm{Cu} \to \mathrm{Cu}$$

which is the composition of the functors $\iota\colon \mathrm{Cu} \to \mathcal{Q}$, $\mathcal{Q}[S,-]\colon \mathcal{Q} \to \mathcal{Q}$ and $\tau\colon \mathcal{Q} \to \mathrm{Cu}$. This functor maps a Cu-semigroup T to the Cu-semigroup $[\![S,T]\!]$ and a Cu-morphism

$f: T \to T'$ is mapped to the Cu-morphism $\tau(f_*)\colon [\![S,T]\!] \to [\![S,T']\!]$, where f_* is the $\mathcal{Q}$-morphism from Remark 4.9.
Analogously, for every Cu-semigroup T, we obtain a contravariant functor $[\![-,T]\!]$ which is the composition of the functors $\iota\colon \mathrm{Cu} \to \mathcal{Q}$, the functor $\mathcal{Q}[-,T]\colon \mathcal{Q} \to \mathcal{Q}$ and the functor $\tau\colon \mathcal{Q} \to \mathrm{Cu}$.

Finally, we can show that for every Cu-semigroup S, the functor $[\![S,-]\!]$ is right adjoint to the functor $-\otimes S$. We can consider the endpoint map $\varphi_{\mathrm{Cu}[S,T]}\colon [\![S,T]\!] \to \mathrm{Cu}[S,T]$ defined in 2.22. For this special case, we also write $\sigma_{S,T}$ instead of $\varphi_{\mathrm{Cu}[S,T]}$. The next theorem is [APT17, Theorem 5.10].

Theorem 4.12: *Let S, T and P be* Cu*-semigroups. Then there is a natural bijection*

$$\mathrm{Cu}(S, [\![T,P]\!]) \cong \mathrm{Cu}(S \otimes_{\mathrm{Cu}} T, P).$$

In particular, Cu *is a closed symmetric monoidal category.*

Proof: Using the definition of the internal hom and that $\mathcal{Q}[S,T] = \mathrm{Cu}(S,T)$ at the first step, the bijection from Theorem 2.25 at the second step and that from Theorem 4.8 at the last step, we obtain:

$$\mathrm{Cu}(S, [\![T,P]\!]) = \mathrm{Cu}(S, \tau(\mathcal{Q}[S,T])) \cong \mathcal{Q}(S, \mathcal{Q}[T,P]) \cong \mathrm{Bi}\mathcal{Q}(S \times T, P).$$

Finally, we have that $\mathrm{Bi}\mathcal{Q}(S \times T, P) \cong \mathrm{BiCu}(S \times T, P)$ and by Theorem 4.4, we also have $\mathrm{BiCu}(S \times T, P) \cong \mathrm{Cu}(S \otimes_{\mathrm{Cu}} T, P)$. Since all these bijections are natural, the composition of them gives the desired statement. □

Remark 4.13: Looking up the definitions of the bijections from the last proof, we can say how the bijection $\mathrm{Cu}(S, [\![T,P]\!]) \cong \mathrm{Cu}(S \otimes_{\mathrm{Cu}} T, P)$ is given (see also [APT17, p. 5.10]):
A Cu-morphism $\alpha\colon S \to [\![T,P]\!]$ is mapped to the Cu-morphism $\tilde{\alpha}\colon S \otimes_{\mathrm{Cu}} T \to P$ is given by $\tilde{\alpha}(s \otimes t) = \sigma_{T,P}(\alpha(s))(t)$.

Definition 4.14: Let S and T be Cu-semigroups.

(1) The *unit map* $\mathrm{d}_{S,T}\colon S \to [\![T, S \otimes_{\mathrm{Cu}} T]\!]$ is defined as the Cu-morphism that corresponds to the identity map on $S \otimes_{\mathrm{Cu}} T$ under the isomorphism

$$\mathrm{Cu}(S, [\![T, S \otimes_{\mathrm{Cu}} T]\!]) \cong \mathrm{Cu}(S \otimes_{\mathrm{Cu}} T, S \otimes_{\mathrm{Cu}} T).$$

(2) The *counit map* is defined as the Cu-morphism $\mathrm{e}_{S,T}\colon [\![S,T]\!] \otimes_{\mathrm{Cu}} S \to T$ that corresponds to the identity on $[\![S,T]\!]$ under the isomorphism

$$\mathrm{Cu}([\![S,T]\!] \otimes_{\mathrm{Cu}} S, T) \cong \mathrm{Cu}([\![S,T]\!], [\![S,T]\!]).$$

Lemma 4.15: *Let S and T be* Cu*-semigroups. Then we have a natural bijection*

$$[\![S,T]\!]_c \cong \mathrm{Cu}(S,T).$$

Proof: For every Cu-semigroup R, we have a natural bijection $\mathrm{Cu}(\overline{\mathbb{N}}, R) \cong R_c$ which maps a Cu-morphism $f\colon \overline{\mathbb{N}} \to R$ to the compact element $f(1)$. Using this at the first step, applying Theorem 4.12 and using the natural isomorphism $\overline{\mathbb{N}} \otimes_{\mathrm{Cu}} S \cong S$, we obtain

$$[\![S,T]\!]_c \cong \mathrm{Cu}(\overline{\mathbb{N}}, [\![S,T]\!]) \cong \mathrm{Cu}(\overline{\mathbb{N}} \otimes_{\mathrm{Cu}} S, T) \cong \mathrm{Cu}(S,T). \qquad \square$$

It is easy to verify that under the bijection from the last lemma, a Cu-morphism $f\colon S \to T$ corresponds to the class of the constant path with value f.

Lemma 4.16: *Let S be a* Cu*-semigroup. Then we have a natural isomorphism*

$$[\![\overline{\mathbb{N}}, S]\!] \cong S.$$

Proof: It is easy to see that the map $\mathrm{ev}_1\colon \mathrm{Cu}[\overline{\mathbb{N}}, S] \to S$ given by $\mathrm{ev}_1(f) = f(1)$ is an isomorphism of $\mathcal{Q}$ semigroups. Since S is a Cu-semigroup, we deduce that $\mathrm{Cu}[\overline{\mathbb{N}}, S]$ is a Cu-semigroup as well. Therefore, the endpoint-map $\sigma_{\overline{\mathbb{N}},S}\colon [\![\overline{\mathbb{N}}, S]\!] \to \mathrm{Cu}[\overline{\mathbb{N}}, S]$ is an isomorphism (see Lemma 2.23) and we obtain

$$[\![\overline{\mathbb{N}}, S]\!] = \tau(\mathrm{Cu}[\overline{\mathbb{N}}, S]) \cong \mathrm{Cu}[\overline{\mathbb{N}}, S] \cong S. \qquad \square$$

The following theorem states that the tensor product and the internal hom functor behave well with regard to limits and colimits (see also [APT18a, Section 2]).

Theorem 4.17: *Let I be a small category and $F\colon I \to \mathrm{Cu}$ be a functor.*

(1) For every Cu*-semigroup T, we have*

$$\left(\mathrm{Cu}\text{-}\varinjlim F\right) \otimes_{\mathrm{Cu}} T \cong \mathrm{Cu}\text{-}\varinjlim \left(F(-) \otimes T\right).$$

(2) For every Cu*-semigroup T, we have*

$$[\![T, \mathrm{Cu}\text{-}\varprojlim F]\!] \cong \mathrm{Cu}\text{-}\varprojlim \left([\![T, F(-)]\!]\right).$$

(3) For every Cu*-semigroup T and every family of* Cu*-semigroups $(S_j)_{j\in J}$, we have*

$$[\![\mathrm{Cu}\text{-}\coprod_{j\in J} S_j, T]\!] \cong \mathrm{Cu}\text{-}\prod_{j\in J} [\![S_j, T]\!].$$

Proof: For every Cu-semigroup T, the functor $-\otimes T\colon \mathrm{Cu} \to \mathrm{Cu}$ is left-adjoint to the functor $[\![T, -]\!]\colon \mathrm{Cu} \to \mathrm{Cu}$. It is a basic result from category theory that left-adjoint functors preserve colimits, whereas right-adjoint functors preserve limits. This implies (1) and (2).

For (3), we use the reflection γ and the coreflection τ. By Theorem 3.5, since τ is a coreflection and by Lemma 3.7, we have for another Cu-semigroup P natural bijections

$$\begin{aligned}\mathrm{Cu}(P, \mathrm{Cu}\text{-}\prod_{i\in I}[\![S_i,T]\!]) &\cong \mathrm{Cu}(P, \tau(\mathcal{Q}\text{-}\prod_{i\in I}[\![S_i,T]\!]))\\ &\cong \mathcal{Q}(P, \mathcal{Q}\text{-}\prod_{i\in I}[\![S_i,T]\!]) \cong \textbf{Set-}\prod_{i\in I}\mathcal{Q}(P,[\![S_i,T]\!]).\end{aligned}$$

Furthermore, for every $i \in I$, using that Cu is a full subcategory of W and $\mathcal{Q}$ and by Theorem 4.12, we have natural bijections

$$\mathcal{Q}(P,[\![S_i,T]\!]) \cong \mathrm{Cu}(P,[\![S_i,T]\!]) \cong \mathrm{Cu}(P\otimes_{\mathrm{Cu}} S_i, T) \cong \mathrm{W}(P\otimes_{\mathrm{Cu}} S_i, T).$$

Using Lemma 3.20, that γ is a reflection, Theorem 3.26 and statement (1) from above, we obtain natural bijections

$$\begin{aligned}\textbf{Set-}\prod_{i\in I}\mathrm{W}(P\otimes_{\mathrm{Cu}} S_i,T) &\cong \mathrm{W}(\mathrm{W}\text{-}\coprod_{i\in I}(P\otimes_{\mathrm{Cu}} S_i,T) \cong \mathrm{Cu}(\gamma\big(\mathrm{W}\text{-}\coprod_{i\in I}(P\otimes_{\mathrm{Cu}} S_i)\big),T)\\ &\cong \mathrm{Cu}(\mathrm{Cu}\text{-}\coprod_{i\in I}(P\otimes_{\mathrm{Cu}} S_i),T) \cong \mathrm{Cu}(P\otimes_{\mathrm{Cu}}\big(\mathrm{Cu}\text{-}\coprod_{i\in I}S_i\big),T)\\ &\cong \mathrm{Cu}(P,[\![\mathrm{Cu}\text{-}\coprod_{i\in I}S_i,T]\!]).\end{aligned}$$

Combinig these results yields a natural bijection

$$\mathrm{Cu}(P,\mathrm{Cu}\text{-}\prod_{i\in I}[\![S_i,T]\!]) \cong \mathrm{Cu}(P,[\![\mathrm{Cu}\text{-}\coprod_{i\in I}S_i,T]\!])$$

for every Cu-semigroup P. This implies $\mathrm{Cu}\text{-}\prod_{i\in I}[\![S_i,T]\!] \cong [\![\mathrm{Cu}\text{-}\coprod_{i\in I}S_i,T]\!]$. □

4.3 The category Cu enriched over itself

In the category of abelian groups, we luckily have that for abelian groups G and H, the set $\mathrm{Hom}(G,H)$ is again an abelian group. Unfortunately, in the category Cu, the set of Cu-morphisms between two Cu-semigroups is in general not a Cu-semigroup. And here enriched category theory comes into play.

It is a basic result from category theory that every closed symmetric monoidal category is enriched over itself. For the category Cu, this in particular means that we get a category where the objects are Cu-semigroups and where the set of morphism between Cu-semigroups S and T is given by $[\![S,T]\!]$, which again is a Cu-semigroup. We present here how this works. For the general theory of enriched categories, see [Kel05].

We first define the composition of these morphisms. The statements presented in this section can be found in [APT17, Paragraph 6.3].

Definition 4.18: Let S, T and P be Cu-semigroups. We define the *composition* *product*

$$\circ\colon [\![T,P]\!] \otimes_{\mathrm{Cu}} [\![S,T]\!] \to [\![S,P]\!]$$

as the Cu-morphism that corresponds under the isomorphism

$$\mathrm{Cu}([\![T,P]\!] \otimes_{\mathrm{Cu}} [\![S,T]\!], [\![S,P]\!]) \cong \mathrm{Cu}([\![T,P]\!] \otimes_{\mathrm{Cu}} [\![S,T]\!] \otimes_{\mathrm{Cu}} S, P)$$

to the Cu-morphism $\mathrm{e}_{T,P} \circ (\mathrm{id}_{[\![T,P]\!]} \otimes \mathrm{e}_{S,T})$.

We get a nice formula for the composition product. With this formula, we can easily observe that the composition product satisfies the axioms that are required for a category.

Theorem 4.19: *Let S, T and P be* Cu*-semigroups and let $f \in P(\mathrm{Cu}[S,T])$ and $g \in P(\mathrm{Cu}[T,P])$ be paths. For every $\lambda \in (0,1)$, the morphism $g(\lambda) \circ f(\lambda)\colon S \to P$ is a generalized* Cu*-morphism and the map $h\colon (0,1) \to \mathrm{Cu}[S,P]$ which is given by $h(\lambda) = g(\lambda) \circ f(\lambda)$ defines a path in $\mathrm{Cu}[S,P]$. Furthermore, for the composition product, we get*

$$[g] \circ [f] = [h].$$

Proof: The proof can be found in [APT17, Proposition 6.18]. □

Recall that we have $\mathrm{Cu}(S,T) \subseteq [\![S,T]\!]$ when we identify a Cu-morphism $f\colon S \to T$ with the class of the constant path in $\mathrm{Cu}[S,T]$ with value f.

Corollary 4.20: *Let S, T, P and Q be* Cu*-semigroups.*

(1) For all $x \in [\![S,T]\!], y \in [\![T,P]\!]$ and $z \in [\![P,Q]\!]$ we have

$$(z \circ y) \circ x = z \circ (y \circ z).$$

(2) Let $\mathrm{id}_S \in \mathrm{Cu}(S,S)$ and $\mathrm{id}_T \in \mathrm{Cu}(T,T)$ be the identity morphisms. Then for every $x \in [\![S,T]\!]$, we have

$$\mathrm{id}_T \circ x = x = x \circ \mathrm{id}_S .$$

In particular, we obtain a category whose objects are Cu*-semigroups and where the set of morphisms between two* Cu*-semigroups S and T is given by $[\![S,T]\!]$.*

Remark 4.21: Let us denote this category by $\mathrm{Cu}_{\mathrm{int}}$. It would be a natural next step to study the properties of this category. It would be nice if this category had the same limit and colimit as Cu. But this is not true, even the product and coproduct, if these exist in $\mathrm{Cu}_{\mathrm{int}}$, is not the same as in Cu. To see this, assume that the product and the coproduct in $\mathrm{Cu}_{\mathrm{int}}$ are the same as in Cu. Using Lemma 4.16 at the first and at the last step, Theorem 4.17 at the second step and Lemma 3.7 at the third step, for every family $(S_i)_{i\in I}$ of Cu-semigroups, we obtain

$$\mathrm{Cu}\text{-}\prod_{i\in I} S_i \cong \mathrm{Cu}\text{-}\prod_{i\in I} [\![\overline{\mathbb{N}}, S_i]\!] \cong [\![\overline{\mathbb{N}}, \mathrm{Cu}\text{-}\prod_{i\in I} S_i]\!] \cong \mathbf{Set}\text{-}\prod_{i\in I} [\![\overline{\mathbb{N}}, S_i]\!] \cong \mathbf{Set}\text{-}\prod_{i\in I} S_i.$$

Analogously, using 4.16 at the first and at the last step, 4.17 at the second step and Lemma 3.20 at the third step, for every Cu-semigroup T, we obtain

$$\text{Cu-}\prod_{i\in I} T \cong \text{Cu-}\prod_{i\in I} [\![\overline{\mathbb{N}}, T]\!] \cong [\![\text{Cu-}\coprod_{i\in I} \overline{\mathbb{N}}, T]\!] \cong \mathbf{Set}\text{-}\prod_{i\in I} [\![\overline{\mathbb{N}}, T]\!] \cong \mathbf{Set}\text{-}\prod_{i\in I} T.$$

But we have already seen in Remark 3.13 that for an infinite index set I, the product in Cu and the product in **Set** are not the same considered as sets. Therefore, the product and the coproduct in $\mathrm{Cu}_{\mathrm{int}}$, if these exist, must be different from those in Cu.

For every Cu-semigroup S, we have $[\![\overline{\mathbb{N}}, S]\!] \cong S$. This looks promising with regard to the search for a free functor on $\mathrm{Cu}_{\mathrm{int}}$. Let $U\colon \mathrm{Cu}_{\mathrm{int}} \to \mathbf{Set}$ be the functor which maps a Cu-semigroup S to its underlying set and an element $x \in [\![S, T]\!]$ to the map $\sigma_{S,T}(x)$ considered as a map between $U(S)$ and $U(T)$. We think of U as of a forgetful functor. A functor $F\colon \mathbf{Set} \to \mathrm{Cu}_{\mathrm{int}}$ would deserve the name *free functor* if it was left adjoint to the functor U. A candidate for F is the functor which maps a set X to the Cu-semigroup Cu-$\coprod_X \overline{\mathbb{N}}$. But unfortunately, as seen in the previous remark, for an infnite set X and a Cu-semigroup T, we can have

$$[\![\text{Cu-}\coprod_{X} \overline{\mathbb{N}}, T]\!] \ncong \mathbf{Set}\text{-}\prod_{X} T \cong \mathbf{Set}(X, T),$$

which shows that F is not left-adjoint to U. So the functor F is not the right candidate and we could not find a better one for $\mathrm{Cu}_{\mathrm{int}}$. Therefore, in the last chapter, we will search for a free functor in the original category Cu.

5 Categorical Algebra in Cu

In this chapter, we show that Cu has a generator. This generator induces a faithful functor $P_c(-)\colon \mathrm{Cu} \to \mathbf{Set}$. In section 5.3, we show that Cu has free objects in a certain sense. Moreover, in the following section, we characterize the monomorphisms and the isomorphisms in Cu with the help of the functor $P_c(-)$. Finally, we present a notion of subobjects within the category Cu.

5.1 Generators, monomorphisms and epimorphisms

Definition 5.1: Let $\mathscr{C}$ be a locally small category. An object $G \in \mathscr{C}$ is called a *generator* if the functor $\mathscr{C}(G,-)\colon \mathscr{C} \to \mathbf{Set}$ is faithful, that is, whenever $f, g \in \mathscr{C}(X,Y)$ are morphisms with $f \neq g$, there is an $\alpha \in \mathscr{C}(G,A)$ such that $f \circ \alpha \neq g \circ \alpha$.

If there is a faithful functor $\mathscr{C} \to \mathbf{Set}$, then $\mathscr{C}$ is called a *concrete* category and the functor is called a *concretization*. Therefore, every category that has a generator is in particular a concrete category.

Example 5.2: (a) The integers $\mathbb{Z}$ are a generator for the category of groups:
Let H and K be groups and let $f, g\colon H \to K$ be group homomorphisms with $f \neq g$. Let $h \in H$ with $f(h) \neq g(h)$. Then define $\alpha\colon \mathbb{Z} \to H$ by $\alpha(m) = h^m$. Then we obtain $f(\alpha(1)) = f(h) \neq g(h) = g(\alpha(1))$, which implies $f \circ \alpha \neq g \circ \alpha$.
(b) For the category of C*-algebras, $C_0((0,1])$ is a generator: The C*-algebra $C_0((0,1])$ is isomorphic to the universal C*-algebra generated by a positive contraction. The element corresponding to the generating element is $\mathrm{id}_{(0,1]}$. Therefore, for every C*-algebra A we have a bijection *-Hom$(C_0((0,1]),A) \cong A_{+,1}$ given by $\alpha \mapsto \alpha(\mathrm{id}_{(0,1]})$, where $A_{+,1}$ is the set of positive elements within the unit ball of A. Let A, B be C*-algebras and $\varphi, \psi\colon A \to B$ be *-homomorphisms with $\varphi \neq \psi$. Then there is an $a \in A_{+,1}$ with $\varphi(a) \neq \psi(a)$. By the above observation we find a *-homomorphism $\alpha\colon C_0((0,1]) \to A$ with $\alpha(\mathrm{id}_{(0,1]}) = a$. Then we obtain of course $f \circ \alpha \neq g \circ \alpha$.

In general, a generator of a category is not unique. For example, $\mathbb{Z} \times \mathbb{Z}$ is also a generator for **Grp** and the universal C*-algebra generated by a contraction would be another generator for $\mathbf{C}^*$.

Definition 5.3: Let $\mathscr{C}$ be a category and $f\colon X \to Y$ be a morphism in $\mathscr{C}$.

(a) We call f a *monomorphism* if for every object Z in $\mathscr{C}$ and every pair of morphisms $g, h\colon Z \to X$ we have that $f \circ g = f \circ h$ implies $f = h$. We write $f\colon X \hookrightarrow Y$ if f is a monomorphsim.

(b) We call f an *epimorphism* if for every object Z in $\mathscr{C}$ and every pair of morphisms $g, h\colon Y \to Z$ we have that $g \circ f = h \circ f$ implies $f = h$. We write $f\colon X \twoheadrightarrow Y$ if f is an epimorphism.

Remark 5.4: For the category **Set**, the monomorphisms are exactly the injective maps and the epimorphisms are precisely the surjective maps. Let $\mathscr{C}$ be a concrete category with concretization functor $F\colon \mathscr{C} \to \mathbf{Set}$ and let $f\colon X \to Y$ be a morphism. It is always true that if $F(f)$ is injective, then f is a monomorphism: Let Z be another object and $g, h\colon Z \to X$ such that $f \circ g = f \circ h$. This implies $F(f) \circ F(g) = F(f) \circ F(h)$. As $F(f)$ is injective, we get $F(g) = F(h)$ and because F is faithful, we finally get $g = h$. Analogously, if $F(f)$ is surjective, then f is an epimorphism.
The converse implications do not always hold, but if the the concretization is induced by a generator, it is also true that if f is a monomorphism, then $F(f)$ is injective.

Lemma 5.5: *Let $\mathscr{C}$ be a locally small category with a generator G. Then a morphism $f\colon X \to Y$ is a monomorphism if and only if the map $f_*\colon \mathscr{C}(G, X) \to \mathscr{C}(G, Y)$, given by $f_*(\beta) = f \circ \beta$, is injective.*

Proof: Let f be a monomorphism and $\beta_1, \beta_2 \in \mathscr{C}(G, X)$ such that $f_*(\beta_1) = f_*(\beta_2)$, which means $f \circ \beta_1 = f \circ \beta_2$. By the definition of a monomorphism, we directly get $\beta_1 = \beta_2$.
The converse implication holds since the functor $\mathscr{C}(G, -)$ is faithful (see Remark 5.4). □

With the help of this lemma, we can now easily prove that in the category of C*-algebras and the category of groups, the monomorphisms are exactly the injective maps.

Example 5.6: (a) Let H and K be groups and $f\colon H \to K$ be a group homomorphism. Applying Lemma 5.5 and using that $\mathbb{Z}$ is a genarator, we obtain that f is a momomorphism if and only if $f_*\colon \mathrm{Hom}(\mathbb{Z}, H) \to \mathrm{Hom}(\mathbb{Z}, K)$ is injective. Furthermore, we have $\mathrm{Hom}(\mathbb{Z}, H) \cong H$ and $\mathrm{Hom}(\mathbb{Z}, K) \cong K$. Under these isomorphisms, the map f_* corresponds to f and therefore, we deduce that f is a monomorphism if and only if f is injective.
(b) Let A and B be C*-algebras and $\varphi\colon A \to B$ be a *-homomorphism. With Lemma 5.5, we get that the morphism φ is a monomorphism if and only if the map $\varphi_*\colon$ *-$\mathrm{Hom}(C_0((0,1]), A) \to$ *-$\mathrm{Hom}(C_0((0,1]), B)$ is injective. Using that $C_0((0,1])$ is the universal C*-algebra generated by a positive contraction, we get bijections *-$\mathrm{Hom}(C_0((0,1]), A) \cong A_{+,1}$ and *-$\mathrm{Hom}(C_0((0,1]), B) \cong B_{+,1}$. Under these bijections, φ_* corresponds to the restriction $\varphi\big|_{A_{+,1}}$. Since $\varphi\big|_{A_{+,1}}$ is injective if and only if φ is injective, we get that φ is a monomorphism if and only if φ is injective.

Remark 5.7: For the categories $\mathbf{C}^*$ and **Grp**, it is also true that every epimorphism is surjective (see [HN95] for $\mathbf{C}^*$ and [EM65] for **Grp**). Therefore, these categories are *balanced*, which means that every morphism that is a monomorphism and an epimorphism is already an isomorphism. This follows here by the fact that the inverse of a bijective group homomorphism is automatically a group homomorphism and that the same holds for bijective *-homomorphisms.

For the next definition, see [Bor94, Definition 4.5.3 and Corollary 4.5.11].

Definition 5.8: Let $\mathscr{C}$ be a locally small, finitely complete category with coproducts. A generator G is called a *strong generator* if the functor $\mathscr{C}(G,-)\colon \mathscr{C} \to \mathbf{Set}$ reflects isomorphisms, that is, if $f\colon X \to Y$ is a morphism such that the induced map f_* is bijective, then f is an isomorphism.

In the category **Grp**, the generator $\mathbb{Z}$ is a strong generator because every bijective group homomorphism is an isomorphism. The generator $C_0((0,1]))$ is also a strong generator for the category $\mathbf{C}^*$ since a *-homomorphism $\varphi\colon A \to B$ is an isomorphism if $\varphi\big|_{A_{+,1}}$ is bijective.

5.2 A generator for Cu

We have seen in the previous section that a generator is helpful to describe the monomorphisms within the category. In this section, we will present a generator for the category Cu and investigate the functor induced by this generator.

Remark 5.9: The first candidate for a generator, in analogy to the generator $\mathbb{Z}$ for the category of groups, would be $\overline{\mathbb{N}}$. But $\overline{\mathbb{N}}$ is not a generator for the category **Cu**. Let S be a Cu-semigroup. Since $1 \in \overline{\mathbb{N}}$ is compact, since Cu-morphisms preserve compact elements and as a Cu-morphism $\varphi\colon \overline{\mathbb{N}} \to S$ is completely determined by $\varphi(1)$, we obtain that $\mathrm{Cu}(\overline{\mathbb{N}}, S) \cong S_c$.
Consider now the Cu-semigroup $\overline{\mathbb{P}} = [0,\infty]$ and the Cu-morphisms $\mathrm{id}_{\overline{\mathbb{P}}}$ and $g\colon \overline{\mathbb{P}} \to \overline{\mathbb{P}}$ given by $g(x) = 2x$. Since the only compact element in $\overline{\mathbb{P}}$ is 0, we have $\mathrm{id}_{\overline{\mathbb{P}}} \circ \alpha = g \circ \alpha$ for every $\alpha \in \mathrm{Cu}(\overline{\mathbb{N}}, \overline{\mathbb{P}})$. This shows that the functor $\mathrm{Cu}(\overline{\mathbb{N}}, -)$ is not faithful.

Definition and Lemma 5.10: The set

$$\mathbb{G} := \big\{ f \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \mid f \text{ increasing},\ f(0) = 0 \big\}$$

with the order and addition inherited from $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ is a positively ordered monoid and fulfills the axioms (O1) and (O4).

Proof: Using that $\mathrm{Lsc}([0,1])$ is a positively ordered monoid, it is easy to see that $\mathbb{G}$ is one.
To prove (O1), consider an increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{G}$. Then the supremum in the Cu-semigroup $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ exists, and it is given by the pointwise supremum $f(t) := \sup_n f_n(t)$ for $t \in [0,1]$. Of course, we have $f(0) = 0$ since all f_n are in $\mathbb{G}$. We also have for every $n \in \mathbb{N}$ that $f_n(t) \le f_n(t')$ whenever $t \le t'$, which implies $\sup_n f_n(t) \le \sup_n f_n(t')$ for $t \le t'$ and finally $f \in \mathbb{G}$.
Axiom (O4) is also true for $\mathbb{G}$ since $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ satisfies (O4) and addition and suprema coincide in $\mathbb{G}$ and $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$. □

In the following part, we show that $\mathbb{G}$ is a Cu-semigroup. Therefore, it is helpful to understand the way-below relation in $\mathbb{G}$.

Remark 5.11: There are some useful elements in $\mathbb{G}$ that make things easier, namely the characteristic functions $\chi_{(t,1]}$ with $t \in [0,1)$. Let $f \in \mathbb{G}$. Since f is lower semicontinuous, for every $n \in \mathbb{N}$ the set $\{f > n\} := \{t \in [0,1] \mid f(t) > n\}$ is open in $[0,1]$. Since f is increasing, it has the form $(t,1]$ for some $t \in [0,1)$. We can consider two cases now.
Case 1: The image of f consists of only finitely many elements $0 = k_0 < k_1 < \cdots < k_l$. We obtain for $1 \leq i \leq l$ that

$$\{t \in [0,1] \mid f(t) = k_i\} = \{f > k_{i-1}\} \setminus \{f > k_i\} = (t_i, t_{i+1}] \text{ for some } t_i, t_{i+1} \in [0,1].$$

Therefore, we have $f = \sum_{i=1}^{l} (k_i - k_{i-1}) \chi_{(t_i,1]}$. If we set $f_n := \sum_{i=1}^{l} (k_i - k_{i-1}) \chi_{(t_i+\frac{1}{n},1]}$ we have that the sequence $(f_n)_{n\in\mathbb{N}}$ is increasing and $f = \sup_n f_n$. Note that $k_l = \infty$ is possible. We use the convention $\infty - n = \infty$ for $n \in \mathbb{N}$.
Case 2: The image of f consists of infinitely many elements $0 = k_0 < k_1 < \ldots$.
With the observations above, we find elements $t_i \in [0,1]$ for $i \geq 1$ such that $f\big|_{(t_i,t_{i+1}]} = k_i$. Defining $f_n := \sum_{i=1}^{n} (k_i - k_{i-1}) \chi_{(t_i,1]}$ we obtain $f = \sup_n f_n$ and $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathbb{G}$. Observe that in this case, we always have $f(1) = \infty$.
We also can replace $\chi_{(t_i,1]}$ by $\chi_{(t_i+\frac{1}{n},1]}$ in the definition of f_n and still have $f = \sup_n f_n$.

Next, we give a nice characterization of the way-below relation in $\mathbb{G}$, which is inspired by [APS11, Proposition 5.5]. It turns out that the way-below relation in $\mathbb{G}$ can be characterized in the same way as the way-below relation in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$.

Lemma 5.12: *Let $f, g \in \mathbb{G}$. Then we have $g \ll f$ if and only if for every $t \in [0,1]$ there is an open neighbourhood U_t of t and an element $c_t \in \mathbb{N}$ such that*

$$g(s) \leq c_t \leq f(s) \text{ for all } s \in U_t.$$

Proof: Assume first $g \ll f$. Assume that f belongs to case 1 in Remark 5.11. Since we can write $f = \sup_n f_n$ with $f_n = \sum_{i=1}^{l} (k_i - k_{i-1}) \chi_{(t_i+\frac{1}{n},1]}$ with t_i and k_i as in 5.11, we find $N \in \mathbb{N}$ such that $g \leq f_N$.
Let $t \in [0,1]$. If $t \leq t_1$, we can set $U_t := [0, t_1 + \frac{1}{N})$ and obtain $g(s) \leq 0 = f_N(s) \leq f(s)$ for all $s \in U_t$, so we can set $c_t = 0$. Otherwise there is an $m \geq 1$ such that $t \in (t_m, t_{m+1}]$. Choose M large enough, such that $t \in (t_m + \frac{1}{M}, t_{m+1} + \frac{1}{M})$. Set $K := \max\{N, M\}$ and $U_t := (t_m + \frac{1}{K}, t_{m+1} + \frac{1}{K})$. Because the sequence $(f_n)_{n\in\mathbb{N}}$ is increasing, we also have $g \leq f_K$. Note that f_K is constant on U_t with value $f_K(t)$. By setting $c_t = f_K(t)$, we obtain

$$g(s) \leq c_t \leq f(s) \text{ for every } s \in U_t.$$

If f belongs to case 2 in Remark 5.11, the proof works analogously.

For the backward implication assume $(h_n)_{n\in\mathbb{N}}$ to be an increasing sequence in $\mathbb{G}$ with $\sup_n h_n \geq f$. For $t \in [0,1]$ choose an open neighbourhood U_t of t and $c_t \in \mathbb{N}$ such that $g(s) \leq c_t \leq f(s)$ for all $s \in U_t$. Then we have $c_t \leq \sup_n h_n(t)$ in $\overline{\mathbb{N}}$. Since $c_t \in \overline{\mathbb{N}}$ and $c_t \neq \infty$, this means $c_t \ll \sup_n h_n(t)$ so that we find an $N_t \in \mathbb{N}$ with $c_t \leq h_{N_t}(t)$. Because h_{N_t} is lower semicontinuous, we get an open neighbourhood V_t of t such that $c_t \leq h_{N_t}(s)$ for all $s \in V_t$. Set $W_t := U_t \cap V_t$.
If we do that for every $t \in [0,1]$, we find an open covering $(W_t)_{t\in[0,1]}$ of $[0,1]$. By using compactness of $[0,1]$, we find $W_{t_1}, \ldots, W_{t_j}$ still covering $[0,1]$. Because the sequence $(h_n)_{n\in\mathbb{N}}$ is increasing and by choosing an $M \in \mathbb{N}$ larger than $N_{t_1}, \ldots, N_{t_j}$, we obtain that $g(s) \leq h_M(s)$ for every $s \in [0,1]$, which means $g \leq h_M$ and finally we have $g \ll f$. □

Remark 5.13: Applying Lemma 5.12, we see that $\chi_{(s,1]} \ll \chi_{(t,1]}$ if and only if $s > t$.

Corollary 5.14: *The set $\mathbb{G}$ satisfies axioms (O2) and (O3) and therefore is a* Cu-*semigroup.*

Proof: We first show that $\mathbb{G}$ satisfies the axiom (O3). Let $f, f', g, g' \in \mathbb{G}$ with $f' \ll f$ and $g' \ll g$. Let $t \in [0,1]$. Using Lemma 5.12, there are open neighbourhoods U_t and V_t of t as well as c_t and d_t in $\mathbb{N}$ such that $f'(s) \leq c_t \leq f(s)$ for all $s \in U_t$ and $g'(s) \leq d_t \leq g(s)$ for all $s \in V_t$. Set $W_t := U_t \cap V_t$. Then W_t is an open neighbourhood of t and for all $s \in W_t$, we obtain

$$(f' + g')(s) = f'(s) + g'(s) \leq c_t + d_t \leq f(s) + g(s) = (f + g)(s).$$

Using 5.12 again, we can deduce that $f' + g' \ll f + g$.
To verify axiom (O2), let $f \in \mathbb{G}$. With Remark 5.13 in mind and with the help of Remark 5.11, we find a $\ll$-increasing sequence $(f_n)_{n\in\mathbb{N}}$ with $\sup_n f_n = f$. □

Remark 5.15: Since $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ is the Cuntz semigroup of the C*-algebra $C([0,1])$; see [APS11, Corollary 2.7], it is natural to ask if the Cu-semigroup $\mathbb{G}$ is also the Cuntz semigroup of some C*-algebra A. But this is not true since $\mathbb{G}$ does not satisfy the axiom (O5) which is satisfied by every Cu-semigroup coming from a C*-algebra (see 1.7). To see this, consider for example the elements $a' = b' := \chi_{(\frac{3}{4},1]}$ and $a = b := \chi_{(\frac{1}{2},1]}$. By defining $c := 2 \cdot \chi_{(0,1]}$, we get $a + b \leq c$ and $a' \ll a$ as well as $b' \ll b$. Assume there is an $x \in \mathbb{G}$ such that $a' + x \leq c \leq a + x$. In particular, we have $2 = c(\frac{1}{4}) \leq a(\frac{1}{4}) + x(\frac{1}{4}) = x(\frac{1}{4})$. As x is increasing, we also have $x(1) \geq 2$, but then $3 \leq a'(1) + x(1) \leq c(1) = 2$, which is a contradiction.

In the next part, we finally show that the Cu-semigroup $\mathbb{G}$ is a generator for **Cu**.

Definition and Lemma 5.16: Let S be a Cu-semigroup. A map $f\colon (0,1] \to S$ is called a *c-path* in S if $\lambda < \lambda'$ implies $f(\lambda) \ll f(\lambda')$ and if we have $\sup_{\lambda'<\lambda} f(\lambda') = f(\lambda)$ for every $\lambda \in (0,1]$. We denote the set of c-paths in S by $P_c(S)$. With the addition given by $(f + g)(\lambda) = f(\lambda) + g(\lambda)$ for $\lambda \in (0,1]$ and the order given by $f \leq g$ if and only if $f(\lambda) \leq g(\lambda)$ for every $\lambda \in (0,1]$, the set $P_c(S)$ becomes a positively ordered monoid.

Proof: The zero element is of course the constant map with value zero. The set $P_c(S)$ is closed under addition because in S the axioms (O3) and (O4) hold. It is easy to see that $P_c(S)$ is a positively ordered monoid by using that S is a positively ordered monoid. □

The notion *c-path* stands for *continuous path*. This is justified because a c-path f has the property $\sup_{\lambda'<\lambda} f(\lambda') = f(\lambda)$ and this is similar to continuity with regard to the Scott-topology in order theory (see [Gie+03, Chapter II]).

Remark 5.17: Consider the set $X := \big\{ \sum_{i=1}^{n} \chi_{(t_i,1]} \mid n \in \mathbb{N}, t_i \in [0,1) \big\} \subseteq \mathbb{G}$. Let $x \in X$. Observe that x can be written uniquely in the form $x = \sum_{i=1}^{n} \chi_{(t_i,1]}$ if we require $t_k \leq t_{k+1}$ for $1 \leq k \leq n-1$. Let $y = \sum_{j=1}^{m} \chi_{(s_j,1]} \in X$ also be written in this form. Then we observe:
We have $x \leq y$ if and only if $n \leq m$ and $\chi_{(t_i,1]} \leq \chi_{(s_i,1]}$ for $1 \leq i \leq n$ if and only if $n \leq m$ and $s_i \leq t_i$ for $1 \leq i \leq n$. Analogously with the use of 5.12, we have $x \ll y$ if and only if $n \leq m$ and $s_i < t_i$ for $1 \leq i \leq n$.

Lemma 5.18: *Let T be a* Cu*-semigroup and $a \in P_c(T)$. Then there is a unique* Cu*-morphism $\alpha\colon \mathbb{G} \to T$ with $\alpha(\chi_{(s,1]}) = a(1-s)$ for $s \in [0,1)$.*

Proof: At first, we consider the set X from Remark 5.17 and define $\tilde{\alpha}\colon X \cup \{0\} \to T$ via $\tilde{\alpha}\big(\sum_{i=1}^{n} \chi_{(t_i,1]}\big) := \sum_{i=1}^{n} a(1-t_i)$ and $\tilde{\alpha}(0) := 0$. We want to show now that $\tilde{\alpha}$ preserves addition, order and the way-below relation.
Since an element of X can be written uniquely up to the order as a sum of characterstic functions, we have that $\tilde{\alpha}$ is well-defined and additive. Let $f \leq g \in X$. Write $f = \sum_{i=1}^{n} \chi_{(t_i,1]}$ and $g = \sum_{j=1}^{m} \chi_{(s_j,1]}$. Using the first observation in Remark 5.17, using that a is increasing and that the order is compatible with the addition, we obtain

$$\tilde{\alpha}(f) = \tilde{\alpha}\big(\sum_{i=1}^{n} \chi_{(t_i,1]}\big) = \sum_{i=1}^{n} a(1-t_i) \leq \sum_{j=1}^{m} a(1-s_j) = \tilde{\alpha}\big(\sum_{j=1}^{m} \chi_{(s_j,1]}\big) = \tilde{\alpha}(g).$$

With the last part of 5.17 and the fact that the way-below relation is additive and a is a c-path, we analogously can conclude that $\tilde{\alpha}$ preserves the way-below relation.
We define $\alpha\colon \mathbb{G} \to T$ as follows: Given $f \in \mathbb{G}$, choose a $\ll$-increasing sequence $(f_n)_{n\in\mathbb{N}}$ in X with $\sup_n f_n = f$ (which is possible, see 5.11). Set $\alpha(f) = \sup_n \tilde{\alpha}(f_n)$. In order to show that α is well-defined choose another $\ll$-increasing sequence $(g_n)_{n\in\mathbb{N}}$ in X with $\sup_n g_n = f$. Let $n \in \mathbb{N}$. Because of $f_n \ll \sup_k g_k$, we find a $k \in \mathbb{N}$ such that $f_n \leq g_k$. Applying $\tilde{\alpha}$ leads to $\tilde{\alpha}(f_n) \leq \tilde{\alpha}(g_k)$. So for all $n \in \mathbb{N}$ we have $\tilde{\alpha}(f_n) \leq \sup_n \tilde{\alpha}(g_n)$ and since $\sup_n \tilde{\alpha}(f_n)$ is the least upper bound of the sequence $(\tilde{\alpha}(f_n))_{n\in\mathbb{N}}$, we obtain $\sup_n \tilde{\alpha}(f_n) \leq \sup_n \tilde{\alpha}(g_n)$. By switching the roles of $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$, this already shows $\sup_n \tilde{\alpha}(f_n) = \sup_n \tilde{\alpha}(g_n)$.

Obviously, α preserves the zero element. Let $f, g \in \mathbb{G}$ with $\ll$-increasing sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in X such that $f = \sup_n f_n$ and $g = \sup_n g_n$. Using that $(f_n + g_n)_{n\in\mathbb{N}}$ is $\ll$-increasing with supremum $f + g$ (because of (O3) and (O4)) at the first step, that $\tilde{\alpha}$ preserves addition at the second step and using (O4) at the third step, we have

$$\alpha(f+g) = \sup_n \tilde{\alpha}(f_n+g_n) = \sup_n(\tilde{\alpha}(f_n)+\tilde{\alpha}(g_n)) = \sup_n \tilde{\alpha}(f_n)+\sup_n \tilde{\alpha}(g_n) = \alpha(f)+\alpha(g),$$

which means that α preserves the addition. In order to show that α is order preserving, we can use the same argument we used above to show that α is well defined. Let $f, g \in \mathbb{G}$ with $f \ll g$ and $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ as above. As we have $\sup_n f_n \ll \sup_n g_n$, we find $k \in \mathbb{N}$ such that $f_n \leq g_k$ for all $n \in \mathbb{N}$. Applying $\tilde{\alpha}$ yields $\tilde{\alpha}(f_n) \leq \tilde{\alpha}(g_k)$ for all $n \in \mathbb{N}$. Using this at the second step and using that $\tilde{\alpha}$ preserves the way-below relation at the third step, we obtain

$$\alpha(f) = \sup_n \tilde{\alpha}(f_n) \leq \tilde{\alpha}(g_k) \ll \tilde{\alpha}(g_{k+1}) \leq \sup_n \tilde{\alpha}(g_n) = \alpha(g).$$

We conclude $\alpha(f) \ll \alpha(g)$.
The sequence given by $s_n := \chi_{(s+\frac{1}{n},1]}$ is $\ll$-increasing with supremum $\chi_{(s,1]}$. By definition we have $\alpha(\chi_{(s,1]}) = \sup_n \tilde{\alpha}(s_n) = \sup_n(a(1-s-\frac{1}{n})) = a(1-s)$. This shows $\alpha(\chi_{(s,1]}) = a(1-s)$ and, as we already have shown that α is additive, $\alpha\big|_X = \tilde{\alpha}$. It remains to show that α preserves suprema of increasing sequences. Let $h \in \mathbb{G}$ and $(h_n)_{n\in\mathbb{N}}$ be an increasing sequence with $\sup_n h_n = h$. Since α is order preserving, we always have $\sup_n \alpha(h_n) \leq \alpha(h)$. To show the inverse inequality, choose a $\ll$-increasing sequence $(r_n)_{n\in\mathbb{N}}$ in X with $\sup_n r_n = h$. By definition and since α extends $\tilde{\alpha}$, we have $\alpha(h) = \sup_n \tilde{\alpha}(r_n) = \sup_n \alpha(r_n)$. Given $n \in \mathbb{N}$, we have $r_n \ll \sup_k h_k$, so that we can find $k \in \mathbb{N}$ with $r_n \leq h_k$. By applying α, we can conclude that for every $n \in \mathbb{N}$, we have $\alpha(r_n) \leq \sup_n \alpha(h_n)$, which leads to $\sup_n \alpha(r_n) \leq \sup_n \alpha(h_n)$ and precicely says $\alpha(h) \leq \sup_n \alpha(h_n)$. We also have that α is unique with $\alpha(\chi_{(s,1]}) = a(1-s)$ (see the following remark). □

Remark 5.19: For $f, g \in \mathrm{Cu}(\mathbb{G}, T)$, we have $f = g$ if and only if $f(\chi_{(s,1]}) = g(\chi_{(s,1]})$ for all $s \in [0,1)$. To see this, let again $X = \left\{ \sum_{i=1}^{n} \chi_{(t_i,1]} \mid n \in \mathbb{N}, t_i \in [0,1) \right\}$. Since f and g are additive, the right side assumption implies $f\big|_X = g\big|_X$. Let $a \in \mathbb{G}$. By 5.11, we can find an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\sup_n x_n = a$. Then we obtain $f(a) = \sup_n f(x_n) = \sup_n g(x_n) = g(a)$, so $f = g$.
With the same kind of argument, we obtain $f \leq g$ if and only if $f(\chi_{(s,1]}) \leq g(\chi_{(s,1]})$ for all $s \in [0,1)$.

Corollary 5.20: *The* Cu*-semigroup* $\mathbb{G}$ *is a generator for the category* **Cu**.

Proof: Let S and T be Cu-semigroups and $f, g \in \mathrm{Cu}(S,T)$ with $f \neq g$. Then there is an element $s \in S$ with $f(s) \neq g(s)$. Using Lemma 1.10, we find a c-path $s \in P_c(S)$ with $s(1) = s$. Let $\alpha \in \mathrm{Cu}(\mathbb{G}, S)$ be the corresponding Cu-morphism with

$\alpha(\chi_{(s,1]}) = s(1-s)$ which we get by Lemma 5.19. Assuming $f \circ \alpha = g \circ \alpha$ would mean that $f(s(\lambda)) = g(s(\lambda))$ for all $\lambda \in (0,1)$ and therefore

$$f(s) = \sup_{\lambda<1} f(s(\lambda)) = \sup_{\lambda<1} g(s(\lambda)) = g(s),$$

which leads to a contradiction. Thus, we obtain $f \circ \alpha \neq g \circ \alpha$. □

Corollary 5.21: *Let S be a* Cu*-semigroup. Then the map*

$$\mu_S \colon \mathrm{Cu}(\mathbb{G}, S) \to P_c(S), h \mapsto \big(h(\chi_{(1-\lambda,1]})\big)_{\lambda\in(0,1)},$$

defines a PoM*-isomorphism. Moreover, this isomorphism is natural in the following sense: Let $f \colon S \to T$ be a* Cu*-morphism. Then the diagram*

$$\begin{array}{ccc} \mathrm{Cu}(\mathbb{G}, S) & \xrightarrow{f_*} & \mathrm{Cu}(\mathbb{G}, T) \\ {\scriptstyle\mu_S}\downarrow & & \downarrow{\scriptstyle\mu_T} \\ P_c(S) & \xrightarrow{P_c(f)} & P_c(T) \end{array}$$

commutes, where the horizontal maps are given by $P_c(f)(s) = f \circ s$ for $s \in P_c(S)$ and $f_*(\varphi) = f \circ \varphi$ for $\varphi \in \mathrm{Cu}(\mathbb{G}, S)$.

Proof: As h is a Cu-morphism, $\mu_T(h)$ is a c-path. It is straightforward to see that μ_T is a PoM-morphism.
Let $\alpha \colon P_c(T) \to \mathrm{Cu}(\mathbb{G}, T), a \mapsto \alpha(a)$, where $\alpha(a)$ is the Cu-morphism corresponding to the path a from Lemma 5.18. It is easy to verify that α is the inverse of μ_T. Next, we show that α is a PoM-morphism as well and therefore μ_S is a PoM-isomorphism. Let $a, b \in P_c(T)$. Clearly, α preserves the zero element. It is also easy to see that $\alpha(a+b)$ and $\alpha(a)+\alpha(b)$ coincide on the elements $\chi_{(s,1]}$. By Remark 5.19 they already coincide on $\mathbb{G}$, so α is additive. Using part two of 5.19, we also see that α is order preserving.
It is straightforward to verify the commutativity of the diagram. □

Remark 5.22: On the one hand, we have the functor $\mathrm{Cu}(\mathbb{G}, -) \colon \mathrm{Cu} \to \mathbf{Set}$ which sends a Cu-semigroup S to the set $\mathrm{Cu}(\mathbb{G}, S)$ and a Cu-morphism $f \colon S \to T$ to the morphism $f_* \colon \mathrm{Cu}(\mathbb{G}, S) \to \mathrm{Cu}(\mathbb{G}, T)$ given by $\alpha \mapsto f \circ \alpha$. On the other hand, we have the functor P_c which sends S to $P_c(S)$, the set of c-paths in S, and f to the morphism $P_c(f)$. The previous corollary says that these functors are naturally equivalent. Since $P_c(-)$ is a faithful functor to the category of sets, we think of it as of a forgetful functor. We refer to the functor $P_c(-)$ as the *path-functor*.

5.3 Free objects in Cu

In this section, it will be shown that the path-functor has a left adjoint.

Remark 5.23: Let X and Y be sets and $f\colon X \to Y$ be a map. Then we can define a map $\beta_f\colon \mathcal{P}\text{-}\coprod_X \mathbb{G} \to \mathcal{P}\text{-}\coprod_Y \mathbb{G}$ via $(g_x)_{x\in X} \mapsto (\sum_{x\in X} \delta_{f(x),y}\, g_x))_{y\in Y}$. It can be easily veryfied that β_f is a $\mathcal{P}$-morphism and that we have $\beta_{g\circ f} = \beta_g \circ \beta_f$ for another set Z and a map $g\colon Y \to Z$. Therefore, it induces a Cu-morphism $\tau(\alpha_f)\colon \text{Cu-}\coprod_X \mathbb{G} \to \text{Cu-}\coprod_Y \mathbb{G}$ and we have $\tau(\beta_{g\circ f}) = \tau(\beta_g) \circ \tau(\beta_f)$.

Definition 5.24: The functor $G\colon \mathbf{Set} \to \text{Cu}$ which sends a set X to the Cu-semigroup $\text{Cu-}\coprod_X \mathbb{G}$ and a map $f\colon X \to Y$ to the Cu-morphism

$$G(f) := \tau(\beta_f)\colon \text{Cu-}\coprod_X \mathbb{G} \to \text{Cu-}\coprod_Y \mathbb{G},$$

where $\tau(\beta_f)$ is the Cu-morphism from Remark 5.23 is called the *free functor* on Cu.

Because of the next theorem, the name *free functor* is justified.

Theorem 5.25: *The free functor $G\colon \mathbf{Set} \to \text{Cu}$ is left adjoint to the path functor $P_c\colon \text{Cu} \to \mathbf{Set}$, i.e. for every Cu-semigroup S and every set X there is a natural bijection*

$$\alpha_{S,X}\colon \text{Cu}(\text{Cu-}\coprod_X \mathbb{G}, S) \to \mathbf{Set}(X, P_c(S)).$$

Proof: For $x \in X$ let $j_x\colon \mathbb{G} \to \text{Cu-}\coprod_X \mathbb{G}$ be the canonical Cu-morphism that belongs to the Cu-coproduct. Let $f\colon \text{Cu-}\coprod_X \mathbb{G} \to S$ be a Cu-morphism. We set $\alpha_{S,X}(f)(x)(\lambda) := f \circ j_x(\chi_{(1-\lambda,1]})$. Then $\alpha_{S,X}$ is a bijection since it is the composition of the following bijections:

$$\text{Cu}(\text{Cu-}\coprod_X \mathbb{G}, S) \xrightarrow{\varphi} \mathbf{Set}\text{-}\prod_X \text{Cu}(\mathbb{G}, S) \xrightarrow{\mu_S} \mathbf{Set}\text{-}\prod_X P_c(S) \xrightarrow{\psi} \mathbf{Set}(X, P_c(S)),$$

where φ is the bijection from Lemma 3.20, μ_S is the isomorphism from 5.21 and ψ is the canonical bijection that sends an element $(g_x)_{x\in X}$ to the map which is given by $x \mapsto g_x$. In oder to show the naturality of the bijection, we have to verify that for every Cu-morphism $h\colon S \to T$ and every map between sets $f\colon Y \to X$ the following diagram commutes:

$$\begin{array}{ccc} \text{Cu}(\text{Cu-}\coprod_X \mathbb{G}, S) & \xrightarrow{\alpha_{S,X}} & \mathbf{Set}(X, P_c(S)) \\ {\scriptstyle G(f)^*\circ h_*}\big\downarrow & & \big\downarrow{\scriptstyle f^*\circ P(h)_*} \\ \text{Cu}(\text{Cu-}\coprod_Y \mathbb{G}, T) & \xrightarrow{\alpha_{T,Y}} & \mathbf{Set}(Y, P_c(T)). \end{array}$$

Let $\beta \in \text{Cu}(\text{Cu-}\coprod_X \mathbb{G}, S)$. We have to show that $P_c(h) \circ \alpha_{S,X}(\beta) \circ f$ is equal to $\alpha_{T,Y}(h \circ \beta \circ G(f))$ in $\text{Set}(Y, P_c(T))$. Spelling out the definitions of the morphisms, for $y \in Y$, we on the one hand get

$$(P_c(h) \circ \alpha_{S,X}(\beta) \circ f)(y)(\lambda) = (h \circ \beta \circ j_{f(y)})(\chi_{(1-\lambda,1]}) \text{ for } \lambda \in (0,1],$$

and on the other hand we get

$$\alpha_{T,Y}(h \circ \beta \circ G(f))(y)(\lambda) = (h \circ \beta \circ G(f) \circ j_y)(\chi_{(1-\lambda,1]}) \text{ for } \lambda \in (0,1].$$

So we only have to show $G(f) \circ j_y = j_{f(y)}$ for every $y \in Y$. We have $G(f) = \tau(\beta_f)$, $j_y = \tau(k_y) \circ \varphi_{\mathbb{G}}^{-1}$ and $j_{f(y)} = \tau(k_{f(y)}) \circ \varphi_{\mathbb{G}}^{-1}$, where k_y and $k_{f(y)}$ are the maps belonging to the coproduct in $\mathcal{P}$ and $\varphi_{\mathbb{G}}$ is the endpoint map (see Remark 3.31). So it is sufficient to verify $\beta_f \circ k_y = k_{f(y)}$ on the level of $\mathcal{P}$-semigroups. For $g \in \mathbb{G}$ we obtain

$$\beta_f \circ k_y(g) = \beta_f\big((\delta_{y,y'}\, g)_{y' \in Y}\big) = \Big(\sum_{y' \in Y} \delta_{f(y'),x}\, \delta_{y,y'}\, g\Big)_{x \in X} = (\delta_{f(y),x}\, g)_{x \in X} = k_{f(y)}.$$

□

Remark 5.26: We denote the unit of the adjunction by η and the counit by ε. The unit is the natural transformation $\eta\colon \mathrm{Id}_{\mathbf{Set}} \to P_c \circ G$ given as follows: For a set X, the map $\eta_X\colon X \to P_c(\mathrm{Cu}\text{-}\coprod_X \mathbb{G})$ is the map that corresponds under the bijection above to the identity morphism $\mathrm{Cu}\text{-}\coprod_X \mathbb{G} \to \mathrm{Cu}\text{-}\coprod_X \mathbb{G}$. Analogously, the counit is the natural transformation $\varepsilon\colon G \circ P_c \to \mathrm{Id}_{\mathrm{Cu}}$ which assigns to a Cu-semigroup S the Cu-morphism $\varepsilon_S\colon \mathrm{Cu}\text{-}\coprod_{P_c(S)} \mathbb{G} \to S$ that corresponds to $\mathrm{id}_{P_c(S)}$.

Remark 5.27: Theorem 5.25 says that the objects of the form $\mathrm{Cu}\text{-}\coprod_X \mathbb{G}$ for a set X are free objects in the category Cu.
In other categories, this is very similar. For example in the category of groups, the coproduct of a family $(H_x)_{x\in X}$ is given by the free product of groups $*_X H_x$ and a generator is given by $\mathbb{Z}$. The functor induced by the generator is the usual forgetful functor and for another group H, we have a natural bijection

$$\mathrm{Hom}(*_X\mathbb{Z}, H) \cong \mathbf{Set}(X, H).$$

Within the category of C*-algebras, a generator is given by $C_0((0,1])$, the functor induced by this generator assings to a C*-algebra A the set $A_{+,1}$ and the coproduct is given by the free product of C*-algebras. For another C*-algebra B, we have a natural bijection

$$^*\text{-}\mathrm{Hom}(*_X C_0((0,1]), B) \cong \mathbf{Set}(X, B_{+,1}).$$

5.4 Characterization of monomorphisms and isomorphisms in Cu

We will show now that the path functor P_c reflects isomorphisms. Therefore, we will need the following lemma.

Lemma 5.28: *Let S be a* Cu*-semigroup and $x, y \in P_c(S)$ be c-paths with $x(1) = y(1)$. Then there is a c-path $z \in P_c(S)$ with $z(1) = x(1)$ which has the following property: There is a sequence $0 = a_0' < a_0 \le b_0' < b_0 \le a_1' < \dots$ in $(0,1)$ that converges to 1 such that $z\big|_{(a_k',a_k]} = x\big|_{(a_k',a_k]}$ and $z\big|_{(b_k',b_k]} = y\big|_{(b_k',b_k]}$ for all $k \in \mathbb{N}$.*

Proof: We will inductively construct a sequence $0 = s_0' < s_0 < t_0' < t_0 < s_1' < \dots$ in $(0,1)$ such that $s_i' \ge \frac{i}{i+1}$ and such that $x(s_i) \ll y(t_i')$, $y(t_i) \ll x(s_{i+1}')$ for $i \in \mathbb{N}$.

For $i = 0$ we set $s'_0 := 0$ and $s_0 := \frac{1}{2}$. We have $x(s_0) \ll \sup_{\lambda<1} y(\lambda)$, so we can find $\lambda_0 \in (0,1)$ with $x(s_0) \leq y(\lambda_0)$. Choose $t'_0 \in (0,1)$ such that $t'_0 > \max\{\lambda, s_0\}$. Then we have $x(s_0) \ll y(t'_0)$. Next, choose $t_0 \in (0,1)$ with $t_0 > t'_0$.
If s'_i, s_i, t'_i and t_i with the above property are already constructed for $i \leq n$, we will define the elements with index $n+1$ as follows:
Since $y(t_n) \ll \sup_{\lambda<1} x(\lambda)$, we find $\lambda_0 \in (0,1)$ such that $y(t_n) \leq x(\lambda_0)$. Choose $s'_{n+1} \in (0,1)$ such that $s'_{n+1} > \max\left\{\frac{n+1}{n+2}, \lambda_0\right\}$. Then we have $y(t_n) \ll x(s'_{n+1})$. For s_{n+1} choose some element in $(0,1)$ larger than s'_{n+1} and define t'_{i+1} and t_{i+1} analogously to the case $i = 0$.
Now, we define $\tilde{z}\colon \bigcup_{i\in\mathbb{N}}(s'_i, s_i] \cup (t'_i, t_i] \cup \{1\} \to S$ by

$$\tilde{h}(t) := \begin{cases} x(t), & \text{if } t \in (s'_i, s_i] \text{ for some } i \in \mathbb{N} \\ y(t), & \text{if } t \in (t'_i, t_i] \text{ for some } i \in \mathbb{N} \\ x(1), & \text{if } t = 1. \end{cases}$$

Next, choose an order isomorphism $\alpha\colon \bigcup_{i\in\mathbb{N}}(s'_i, s_i] \cup (t'_i, t_i] \cup \{1\} \to (0,1]$ and set $z(t) := \tilde{z}(\alpha^{-1}(t))$. Then z is a c-path in S with $z(1) = x(1)$. Set $I_k := \alpha((s'_k, s_k])$ and $J_k := \alpha((t'_k, t_k])$ for $k \in \mathbb{N}$. Since α is an order isomorphism, the intervals I_k and J_k for $k \in \mathbb{N}$ can be written as $I_k := (a'_k, a_k]$ and $J_k = (b'_k, b_k]$ for some $a'_k, a_k, b'_k, b_k \in (0,1)$. By construction, these elements have the desired properties.
. □

Lemma 5.29: *Let S be a* Cu*-semigroup and $a, b \in S$ with $a \ll b$. Then there is a c-path $f \in P_c(S)$ such that $f(\frac{1}{2}) = a$ and $f(1) = b$.*

Proof: By Lemma 1.10, we can choose c-paths g with $g(1) = a$ and h with $h(1) = b$. Since we have $a \ll h(1) = \sup_{\lambda<1} h(\lambda)$, we find a $\lambda_0 \in (0,1)$ such that $a \leq h(\lambda_0)$. Choose an order isomprphism $\alpha\colon (\lambda_0, 1] \to (\frac{1}{2}, 1]$ and define $f\colon (0,1] \to S$ as follows:

$$f(\lambda) := \begin{cases} g(2\lambda), & t \leq \frac{1}{2} \\ h(\alpha^{-1}(\lambda)), & t > \frac{1}{2}. \end{cases}$$

Then f has the desired properties. □

Theorem 5.30: *Let S and T be* Cu*-semigroups and $f\colon S \to T$ be a* Cu*-morphism. Then the following statements hold:*

(a) f is a monomorphism if and only if $P_c(f)$ is injective.
(b) f is an isomorphism if and only if $P_c(f)$ is bijective.
(c) f is an epimorphism if $P_c(f)$ is surjective.

Proof: (a) This follows directly by applying Lemma 5.5 and using that the functors P_c and $\mathrm{Cu}(\mathbb{G}, -)$ are naturally equivalent.
(b) The forward implication is trivial since isomorphisms are preserved by any functor. We divide the proof of the backward implication into four steps:
Step 1: f is surjective:

Let $t \in T$. By Lemma 1.10, we can choose a c-path $b \in P_c(T)$ with $b(1) = t$. Since $P_c(f)$ is surjective, we find a c-path $a \in P_c(S)$ such that $f \circ a = P_c(f)(a) = b$. Set $s := a(1)$. We deduce

$$f(s) = f(\sup_{\lambda<1} a(\lambda)) = \sup_{\lambda<1} f(a(\lambda)) = \sup_{\lambda<1} b(\lambda) = t.$$

Step 2: f is injective:
Let $s, s' \in S$ with $f(s) = f(s')$. By Lemma 1.10, we can choose c-paths $x, y \in P_c(S)$ with $x(1) = s$ and $y(1) = s'$. We obtain $(f \circ x)(1) = f(s) = f(s') = (f \circ y)(1)$. Therefore, we can apply Lemma 5.28 to the c-paths $f \circ x$ and $f \circ y$ in $P_c(T)$ and get a c-path $z \in P_c(T)$ with $z(1) = f(s)$ and a sequence $0 = a'_0 < a_0 \leq b'_0 < b_0 \leq a'_1 < \dots$ in $(0, 1)$ that converges to 1 such that $z\big|_{(a'_k, a_k]} = f \circ x$ and $z\big|_{(b'_k, b_k]} = f \circ y$ for every $k \in \mathbb{N}$. Since $P_c(f)$ is surjective, we find $z' \in P_c(S)$ with $f \circ z' = P_c(f)(z') = z$.
Now, we want to show that $z'(1) = x(1)$.
If there was an increasing sequence $(r_k)_{k\in\mathbb{N}}$ with $\sup_k r_k = 1$ such that $z'(r_k) = x(r_k)$ for all $k \in \mathbb{N}$, this would be true.
Therefore, we assume that no such sequence exists. Then there is a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $t \in (a'_k, a_k]$ we have $x(t) \neq z'(t)$. Choose an order isomorphism $\alpha\colon \bigcup_k (a'_k, a_k] \cup \{1\} \to (0, 1]$ and consider the paths $x \circ \alpha^{-1}$ and $z' \circ \alpha^{-1}$. We have $x \circ \alpha^{-1} \neq z' \circ \alpha^{-1}$ since x and z' differ on $(a'_{k_0}, a_{k_0}]$. But we have $f \circ x \circ \alpha^{-1} = f \circ z' \circ \alpha^{-1}$ since $f \circ z'$ and $f \circ x$ have the same endpoint and agree an $(a'_k, a_k]$ for all $k \in \mathbb{N}$. This contradicts $P_c(f)$ to be injective, so that we obtain $z'(1) = x(1) = s$.
Analogously by using y instead of x and $(b'_k, b_k]$ instead of $(a'_k, a_k]$, we deduce $\sup_\lambda z'(\lambda) = \sup_\lambda b(\lambda) = s'$ and therefore finally have $s = s'$.
Step 3: f^{-1} preserves the way-below realtion:
Let $t, t' \in T$ with $t' \ll t$. By Lemma 5.29, we can choose a c-path $y \in P_c(T)$ with $y(\frac{1}{2}) = t'$ and $y(1) = t$. Since $P_c(f)$ is surjective, we find a c-path $x \in P_c(S)$ with $f \circ x = y$. As f is bijective, this is equivalent to $x = f^{-1} \circ y$. We obtain $x(\frac{1}{2}) = f^{-1}(t')$ and $x(1) = f^{-1}(t)$. Since x is a c-path, we deduce $f^{-1}(t') \ll f^{-1}(t)$.
Step 4: f^{-1} is order-preserving:
Let $t, t' \in T$ with $t' \leq t$. Choose a c-path $x \in P_c(T)$ with $x(1) = t'$. Then $f^{-1} \circ x$ is a path with endpoint $f^{-1}(t')$ and we have for every $n \geq 1$ that $x(1 - \frac{1}{n}) \ll t$. As f^{-1} preserves the way-below relation, we obtain $f^{-1}(x(1 - \frac{1}{n})) \ll f^{-1}(t)$ for all $n \geq 1$. We conclude $f^{-1}(t') = \sup_n f^{-1}(x(1 - \frac{1}{n})) \leq f^{-1}(t)$.
We have shown that f is an order-isomorphism and therefore a Cu-isomorphism.
(c) In step 1 of the proof of (b), we have already shown that f is surjective whenever $P_c(f)$ is surjective and surjective maps between sets are always epimorphisms. □

Corollary 5.31: *The generator* $\mathbb{G}$ *for* Cu *is a strong generator.*

Remark 5.32: This example is inspired by the fact that the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is a non-surjective epimorphism in the category of monoids and shows that the converse of statement (c) in Theorem 5.30 is not true in general. Consider the sets

$$N := ((0, \infty) \times \mathbb{N}) \cup \{(0, 0)\} \text{ and } M := ((0, \infty) \times \mathbb{Z}) \cup \{(0, 0)\}$$

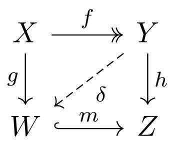
with the order given by $(x', m') \leq (x, m)$ if $x' < x$ or if $x' = x$ and $m' = m$ for $(x', m'), (x, m) \in (0, \infty) \times \mathbb{Z}$. This defines a partial order on N and M and with the componentwise addition, N and M becomes a positively ordered monoid. Let $\iota\colon N \to M$ be the inclusion map. We can apply the γ completion to M and N by considering them as W-semigroups with $\leq$ as transitive relation. Set $S := \gamma(N, \leq)$ and $T := \gamma(M, \leq)$ and let $\gamma(\iota)\colon S \to T$ be the induced map. It is easy to see that the map $\gamma(\iota)$ is not surjective. Though, it is an epimorphism. To see this, let R be another Cu-semigroup and $f, g\colon T \to R$ be Cu-morphisms such that $f \circ \gamma(\iota) = g \circ \gamma(\iota)$. In order to show $f = g$, it is enough to show $f\big|_M = g\big|_M$ because M is dense in $\gamma(M)$ (see Remark 2.6). By assumption, we have $f(x, n) = g(x, n)$ for all $(x, n) \in (0, \infty) \times \mathbb{N}$. Let $x \in (0, \infty)$ and $m \in \mathbb{Z} \setminus \mathbb{N}$. Then we get

$$\begin{aligned} f(x, m) &= f(\tfrac{x}{3}, m) + f(\tfrac{2x}{3}, 0) \\ &= f(\tfrac{x}{3}, m) + g(\tfrac{2x}{3}, 0) \\ &= f(\tfrac{x}{3}, m) + g(\tfrac{x}{3}, -m) + g(\tfrac{x}{3}, m) \\ &= f(\tfrac{x}{3}, m) + f(\tfrac{x}{3}, -m) + g(\tfrac{x}{3}, m) \\ &= f(\tfrac{2x}{3}, 0) + g(\tfrac{x}{3}, m) \\ &= g(x, m). \end{aligned}$$

So we get $f = g$ and have shown that $\gamma(\iota)$ is a non surjective epimorphism. This also implies that $P_c(\gamma(\iota))$ cannot be surjective since, as already seen in the proof of 5.30 (b), this would imply that f is surjective.
It is also easy to see that $\gamma(\iota)$ is injective and therefore in particular a monomorphism. Therefore, this example also shows that the category Cu is not balanced since there is a morphism that is a monomorphism and an epimorphism but not an isomorphism.

Since the last implication of Theorem 5.30 fails, we will consider some stronger versions of epimorphisms.

Definition 5.33: Let $\mathscr{C}$ be a category and $f\colon X \twoheadrightarrow Y$ an epimorphism.
(a) We call f a *strong epimorphism* if for any monomorphism $m\colon W \hookrightarrow Z$ and for any morphisms $g\colon X \to W$ and $h\colon Y \to Z$ such that $h \circ f = m \circ g$ there is a morphism $\delta\colon Y \to W$ such that $\delta \circ f = g$ and $m \circ \delta = h$.



(b) We call f an *extremal epimorphism* if $f = m \circ g$ for a morphism $g\colon X \to Z$ and a monomorphism $m\colon Z \hookrightarrow Y$ implies that m is an isomorphism.

Remark 5.34: In the category Cu, extremal epimorphisms and strong epimorphisms are the same since Cu is finitely complete (see [Bor94, Proposition 4.3.7]).

Lemma 5.35: *Let S and T be* Cu*-semigroups and $f\colon S \to T$ be a* Cu*-morphism. If $P_c(f)$ is surjective, then f is an extremal (or equivalently a strong) epimorphism.*

Proof: Let $f = m \circ g$ a decomposition with a monomorphism $m\colon R \hookrightarrow T$ in Cu for some Cu-semigroup R and a Cu-morphism $g\colon X \to R$. Applying the functor P_c gives $P_c(f) = P_c(m) \circ P_c(g)$. Since $P_c(f)$ is surjective, so is $P_c(m)$. By Theorem 5.30, $P_c(m)$ is also injective. Again by 5.30, we deduce that m is an isomorphism. □

Recall that ε_S for a Cu-semigroup S is the counit of the adjunction between the free functor and the path-functor (see Remark 5.26).

Corollary 5.36: *For any* Cu*-semigroup S, the* Cu*-morphism $\varepsilon_S\colon \text{Cu-}\coprod_{P_c(S)} \mathbb{G} \to S$ is a strong epimorphism.*

Proof: We show that $P(\varepsilon_S)$ is surjective. Therefore, let $s \in P_c(S)$. By the definition of ε_S and following Remark 3.31, we have $\varepsilon_S = \varphi_S \circ \tau(f)$, where $\varphi_S\colon \tau(S) \to S$ is the endpoint-map and $f = \coprod_{x\in P_c(S)} \alpha(x)\colon \mathcal{P}\text{-}\coprod_{x\in P_c(S)} \mathbb{G} \to S$. The maps $\alpha(x)\colon \mathbb{G} \to S$ for $x \in P_c(S)$ are given by $\alpha(x)(\chi_{(\lambda,1]}) = x(1-\lambda)$ for $\lambda \in (0,1]$.
For $\lambda \in (0,1]$, define $g_\lambda\colon (0,1) \to \mathcal{P}\text{-}\coprod_{P_c(S)} \mathbb{G}$ by $g(t) = (\chi_{(1-\lambda+1-t,1]}\,\delta_{x,s})_{x\in P_c(S)}$. Then we get $[g_\lambda] \in \text{Cu-}\coprod_{P_c(S)} \mathbb{G}$. Using that $\alpha(s)$ preserves suprema at the third step, we obtain

$$\varepsilon_S([g_\lambda]) = (\varphi_S \circ \tau(f))([g_\lambda]) = \sup_{t\in(0,1)} \alpha(s)(\chi_{(1-\lambda+1-t,1]}) = \alpha(s)(\chi_{(1-\lambda,1]}) = s(\lambda).$$

Finally, it is easy to see that $g\colon (0,1] \to \text{Cu-}\coprod_{P_c(S)} \mathbb{G}, \lambda \mapsto [g_\lambda]$ defines an alement in $P_c(\text{Cu-}\coprod_{P_c(S)} \mathbb{G})$. We deduce that $P_c(\varepsilon_S)(g) = s$ which shows that $P_c(\varepsilon_S)$ is surjective. □

5.5 Subobjects in Cu

In order to find a notion of sub-Cu-semigroups, we consider the general notion of subobjects in arbitrary categories and investigate what this means in the category Cu. The following definitions can be found in [Bor94, Section 4.1].

Definition 5.37: Let $\mathscr{C}$ be a category, let A, B and C be objects of $\mathscr{C}$ and let further $\iota\colon A \hookrightarrow C$ and $\kappa\colon B \hookrightarrow C$ be monomorphisms. We set $\iota \sim \kappa$ if and only if there is an isomorphism $\varphi\colon A \to B$ such that $\kappa \circ \varphi = \iota$. We call the equivalence class $[\iota]$ of a monomorphism $\iota\colon A \hookrightarrow C$ a *subobject* of C and denote the class of all subobject by $\mathrm{Sub}(C)$.

Although this seems to be very unnatural, in general, the subobjects of a fixed object need not form a set.

Definition 5.38: A category $\mathscr{C}$ is called *well-powered* if $\mathrm{Sub}(C)$ forms a set for every object C of $\mathscr{C}$.

Example 5.39: For a group G, the subobjects of G are the subgroups of G. More precisely, the map

$$\begin{aligned} \mathrm{Sub}(G) &\to \{H \mid H \subseteq \text{ G is a subgroup}\} \\ [\varphi] &\mapsto \mathrm{Im}(\varphi) \end{aligned}$$

is a bijection. For C*-algebras, one analogously obtains that the subobjects of a C*-algebra A are precisely the sub-C*-algebras of A.

Remark 5.40: We can also define a partial order on $\mathrm{Sub}(C)$. For $\iota\colon A \hookrightarrow C$ and $\kappa\colon B \hookrightarrow C$ define $[\iota] \leq [\kappa]$ if there is a morphism $\psi\colon A \to B$ such that $\kappa \circ \psi = \iota$. This indeed defines a partial order: If we also have $[\kappa] \leq [\iota]$, this means that there is a morphism $\psi'\colon B \to A$ such that $\iota \circ \psi' = \kappa$. Then we obtain $\iota \circ \psi' \circ \psi = \iota$ and $\kappa \circ \psi \circ \psi' = \kappa$. Since ι and κ are monomorphisms, we deduce $\psi' \circ \psi = \mathrm{id}_A$ and $\psi \circ \psi' = \mathrm{id}_B$, so that ψ is an isomorphism and $[\iota] = [\kappa]$.
We can ask if the subobjects of a fixed object form a complete lattice. For groups and C*-algebras, this is true: the infimum of a family of subgroups or sub-C*-algebras is just their intersection. With regard to the bijection above, intersection corresponds to the equivalence class of the inclusion map of the intersection into the object.

We will study now the situation in the category Cu. Using abstract category theory, we can deduce that for a Cu-semigroup S, the subobjects of S form a set and moreover a complete lattice. Note that within a partially ordered set S, we have for a family $(s_i)_{i\in I}$ in S that

$$\begin{aligned} \sup_{i\in I} s_i &= \inf\{s \mid s_i \leq s \text{ for all } i \in I\} \\ \text{and } \inf_{i\in I} s_i &= \sup\{s \mid s \leq s_i \text{ for all } i \in I\}. \end{aligned}$$

Therefore, S is a complete lattice if and only if arbitrary suprema or arbitrary infima exist. Note that the supremum over the empty set is the smallest element in S and the infimum over the empty set is the largest element in S.

Lemma 5.41: *The category* Cu *is well-powered.*

Proof: This follows by [Bor94, Proposition 4.5.15] since Cu is complete and has a strong generator. □

Lemma 5.42: *Let S be a* Cu*-semigroup. Then* $\mathrm{Sub}(S)$ *forms a complete lattice.*

Proof: The category Cu is well-powered and complete. Applying [Bor94, Corollary 4.2.5] proves the claim. □

Now we will state expicitely how we can identify $\mathrm{Sub}(S)$ for a Cu-semigroup S with a set and afterwards we will show how the infimum and the supremum of families of subobjects look like.

Lemma 5.43: *Let S_0, S_1 and T be* Cu*-semigroups and $f\colon S_0 \hookrightarrow T$ and $g\colon S_1 \hookrightarrow T$ be monomorphisms such that* $\mathrm{Im}(P_c(f)) \subseteq \mathrm{Im}(P_c(g))$*. Then there is a* Cu*-morphism $\varphi\colon S_0 \to S_1$ such that $f = g \circ \varphi$. In particular, we get $[f] \leq [g]$ in* $\mathrm{Sub}(T)$*.*

Proof: Let $h\colon P_c(S_1) \to \mathrm{Im}(P_c(g))$ be the morphism $P(g)$ with restricted range. Then h is a bijection. Since we have $\mathrm{Im}(P_c(f)) \subseteq \mathrm{Im}(P_c(g))$, we can consider the map $h^{-1} \circ P_c(f)\colon P_c(S_0) \to P_c(S_1)$. Let

$$\alpha_{S_1,P_c(S_0)}\colon \mathrm{Cu}(\text{Cu-}\coprod_{P_c(S_0)} \mathbb{G}, S_1) \to \mathbf{Set}(P_c(S_0), P_c(S_1))$$

be the natural bijection from Theorem 5.25 and set $\gamma := \alpha^{-1}_{S_1,P_c(S_0)}(h^{-1} \circ P_c(f))$. We claim that $f \circ \varepsilon_{S_0} = g \circ \gamma$. To see this, consider the following diagram:

$$\begin{array}{ccc}
\mathrm{Cu}(\text{Cu-}\coprod_{P_c(S_0)} \mathbb{G}, S_0) & \xrightarrow{\alpha_{S_0,P_c(S_0)}} & \mathbf{Set}(P_c(S_0), P_c(S_0)) \\
\Big\downarrow f_* & & \Big\downarrow P_c(f)_* \\
\mathrm{Cu}(\text{Cu-}\coprod_{P_c(S_0)} \mathbb{G}, T) & \xrightarrow{\alpha_{T,P_c(S_0)}} & \mathbf{Set}(P_c(S_0), P_c(T)) \\
\Big\uparrow g_* & & \Big\uparrow P_c(g)_* \\
\mathrm{Cu}(\text{Cu-}\coprod_{P_c(S_0)} \mathbb{G}, S_1) & \xrightarrow{\alpha_{S_1,P_c(S_0)}} & \mathbf{Set}(P_c(S_0), P_c(S_1))
\end{array}$$

By Theorem 5.25, this diagram commutes. Starting with ε_{S_0} on the upper left corner of the diagram, we on the one hand get $\alpha_{T,P_c(S_0)}(f \circ \varepsilon_{S_0}) = P_c(f)$. On the other hand, starting with γ on the left corner below, we get

$$\alpha_{T,P_c(S_0)}(g \circ \gamma) = P_c(g) \circ h^{-1} \circ P_c(f) = P_c(f).$$

Since $\alpha_{T,P_c(S_0)}$ is a bijection, we deduce $f \circ \varepsilon_{S_0} = g \circ \gamma$. By Corollary 5.36, the map ε_{S_0} is a strong epimorphism. Since g is a monomorphism, there is a Cu-morphism $\varphi\colon S_0 \to S_1$ such that $f = g \circ \varphi$. □

Remark 5.44: Let $f\colon S_0 \hookrightarrow T$ and $g\colon S_1 \hookrightarrow T$ be monomorphisms in Cu. Let $(S_0 \times_{f,g} S_1, p_0, p_1)$ the pullback along f and g. In particular, we have the following commutative diagram:

$$\begin{array}{ccc}
S_0 \times_{f,g} S_1 & \xrightarrow{p_0} & S_0 \\
\Big\downarrow p_1 & & \Big\downarrow f \\
S_1 & \xrightarrow{g} & T.
\end{array}$$

Since f is a monomorphism, so is p_1:
Let h and h' be morphisms such that $p_1 \circ h = p_1 \circ h'$. We want to show $h = h'$. We have $g \circ p_1 \circ h = g \circ p_1 \circ h'$ and using the commutativity of the diagram this means $f \circ p_0 \circ h = f \circ p_0 \circ h'$. As f is a monomorphism, we deduce $p_0 \circ h = p_0 \circ h'$. Since we also have $p_1 \circ h = p_1 \circ h'$, by the universal property of the pullback, we conclude

$h = h'$. This shows also that p_0 is a monomorphism if g is one.
Let us denote the infimum of the subobjects $[f]$ and $[g]$ by $[f] \wedge [g]$. We claim $[f] \wedge [g] = [f \circ p_0]$. It is trivial that $[f \circ p_0] = [g \circ p_1]$ is a lower bound for $[f]$ and $[g]$. Let $[h\colon R \hookrightarrow T]$ be another lower bound for $[f]$ and $[g]$. Then there are morphisms $\varphi\colon R \to S_0$ and $\varphi'\colon R \to S_1$ such that $h = f \circ \varphi$ and $h = g \circ \varphi'$. By the universal property of the pullback, we get a morphism $\psi\colon R \to S_0 \times_{f,g} S_1$ such that $p_0 \circ \psi = \varphi$ and $p_1 \circ \psi = \varphi'$. Inserting this gives $h = f \circ p_0 \circ \psi$ and $h = g \circ p_1 \circ \psi$ which implies $[h] \leq [f \circ p_0]$ and $[h] \leq [g \circ p_1] = [f \circ p_0]$.
We can generalize this to infima of arbitrary families of subobjects (see also [Bor94, Proposition 4.2.4]): Let $([f_i\colon S_i \to T])_{i \in I}$ be a family of subobjects of T. Consider the diagram consisting of all the monomorphisms f_i and let $(N, (\varphi_i)_{i\in I})$ be the limit of this diagram. Then the morphisms $f_i \circ \varphi_i\colon N \to T$ agree for all $i \in I$. Set $f := f_i \circ \varphi_i$ for some $i \in I$. Then f is a monomorphism since for morphisms $g, h\colon X \to N$ with $s \circ g = s \circ h$, we get $f_i \circ \varphi_i \circ g = f_i \circ \varphi_i \circ h$ for all $i \in I$. Since the f_i are monomorphisms, we deduce $\varphi_i \circ g = \varphi_i \circ h$ for all $i \in I$ which implies $g = h$ by the universal property of the limit. Using again the universal property of the limit, we analogously to the special case above get $\bigwedge_{i\in I}[f_i] = [f]$. If the set I is empty, then the infimum over I is just the class of id_T.

Theorem 5.45: *The mapping*

$$\begin{aligned} \Phi\colon \mathrm{Sub}(S) &\to \{X \mid X \subseteq P_c(S)\} \\ [f] &\mapsto \mathrm{Im}(P(f)) \end{aligned}$$

defines an order-embedding. We further have $\Phi(\bigwedge_{i\in I}[f_i]) = \bigcap_{i\in I} \mathrm{Im}(P_c(f_i))$ *for a family* $([f_i])_{i\in I}$ *in* $\mathrm{Sub}(S)$.

Proof: By the definition of the partial order in $\mathrm{Sub}(S)$, obviously $[f] \leq [g]$ implies $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$ and therefore also $\mathrm{Im}(P_c(f)) \subseteq \mathrm{Im}(P_c(g))$. We immediately obtain the converse implication by applying Lemma 5.43 and get that Φ is an order-embedding. In order to show that Φ preserves infima, let $([f_i])_{i\in I}$ be a family of subobjects in S and let $(N, (\varphi_i)_{i\in I})$ be the limit of the diagram consisting of the f_i. By the previous remark, we get $\bigwedge[f_i] = [f]$ with $f = f_i \circ \varphi_i$ for some $i \in I$. It is well-known that functors that have a left adjoint are compatible with limits. In our case this means that $(P_c(N), (P_c(\varphi_i))_{i\in I})$ is the limit of the diagram consisting of the maps $P_c(f_i)$ and that $[P_c(f)]$ is the infimum of the subobjects $[P_c(f_i)]$ in **Set**. On the other hand, as the maps $P_c(f_i)$ are injective, this limit is given by $\bigcap_{i\in I} \mathrm{Im}(P_c(f_i))$. Then $[P_c(f)] = [\iota]$, where ι is just the inclusion $\bigcap_{i\in I} \mathrm{Im}(P_c(f_i)) \to P_c(T)$. We obtain

$$\Phi(\bigwedge_{i\in I}[f_i]) = \mathrm{Im}(P_c(f)) = \mathrm{Im}(\iota) = \bigcap_{i\in I} \mathrm{Im}(P_c(f_i)).$$

□

Definition 5.46: Let S be a Cu-semigroup and $A \subseteq P_c(S)$. We set

$$\overline{A} := \bigcap \{\mathrm{Im}(P_c(m)) \mid [m] \in \mathrm{Sub}(S), A \subseteq \mathrm{Im}(P_c(m))\}$$

and refer to $\overline{A}$ as the closure of A.

Definition 5.47: Let S be a Cu-semigroup and $A \subseteq P_c(S)$. We say that A is closed if $\overline{A} = A$ and we say that a subset $D \subseteq A$ is dense in A if $\overline{D} = A$.

Remark 5.48: It is obvious that $A \subseteq B \subseteq P_c(S)$ implies $\overline{A} \subseteq \overline{B}$, that $A \subseteq \overline{A}$ and that $\overline{\overline{A}} = \overline{A}$. By definition, $\overline{A} = \mathrm{Im}(P_c(m))$ where $[m]$ is the infimum of all subobjects of S corresponding to a subset of $P_c(S)$ that contains A. The last definition can be done analogously in other categories. For example in the category of groups, for a subset $X \subseteq G$ of a group G, the „closure" would be the smallest subgroup of G containing X, which is the intersection over all subgroups containing X. For groups, we just consider the usual forgetful functor instead of the functor P_c.

5.6 Strong epimorphisms

We have already considered strong epimorphisms before, but in this section, we want to give a precise characterization of them with the help of the closure from Definition 5.46.

Proposition 5.49: *Let $f\colon S \to T$ be a* Cu*-morphism. Then there is a* Cu*-semigroup R, a monomorphism $m\colon R \hookrightarrow T$ and a strong epimorphism $p\colon S \twoheadrightarrow R$ such that $f = m \circ p$.*

Proof: Since the category Cu is well-powered and complete, it is finitely well-complete (see [Bor94, Definition 4.4.1, Proposition 4.4.2]). In finitely well-complete categories, such a factorization exists (see [Bor94, Proposition 4.4.3]). □

Theorem 5.50: *Let $\varphi\colon S \to T$ be a* Cu*-morphism. Then φ is a strong epimorphism if and only if the image of $P_c(\varphi)$ is dense in the sense that $\overline{\mathrm{Im}(P_c(\varphi))} = P_c(T)$.*

Proof: For the forward implication, assume that φ is a strong epimorphism. Since the composition of strong epimorphisms is again a strong epimorphism, $\varphi \circ \varepsilon_S$ is a strong epimorphism. Let $m\colon T_0 \hookrightarrow T$ be a monomorphism with the property that $\mathrm{Im}(P_c(m)) = \overline{\mathrm{Im}(P_c(\varphi))}$. We show now that $P_c(m)$ is surjective. We use the argument from the proof of Lemma 5.43. We have $\mathrm{Im}(P_c(\varphi)) \subseteq \mathrm{Im}(P_c(m))$. As in the proof of 5.43, there is a morphism γ such that $\varphi \circ \varepsilon_S = m \circ \gamma$. Since $\varphi \circ \varepsilon_S$ is a strong epimorphism, we find a morphism $\delta\colon T \to T_0$ such that $m \circ \delta = \mathrm{id}_T$. This implies that $P_c(m)$ is surjective and hence $P_c(T) = \mathrm{Im}(P_c(m)) = \overline{\mathrm{Im}(P_c(\varphi))}$.
In order to show the backward implication, let m be a monomorphism and p be a strong epimorphism such that $\varphi = m \circ p$. Then $\mathrm{Im}(P_c(\varphi)) \subseteq \mathrm{Im}(P_c(m))$ and since we trivially have $\overline{\mathrm{Im}(P_c(m))} = \mathrm{Im}(P_c(m))$, we obtain

$$P_c(T) = \overline{\mathrm{Im}(P_c(\varphi))} \subseteq \overline{\mathrm{Im}(P_c(m))} = \mathrm{Im}(P_c(m)) \subseteq P_c(T)$$

and see that $P_c(m)$ is surjective. As $P_c(m)$ is also injective, m is an isomorphism. Since isomorphisms are in particular strong epimorphisms and the composition of strong epimorphisms is again a strong epimorphism (see [Bor94, Proposition 4.3.6]), we obtain that φ is a strong epimorphism. □

5.7 Further questions

An important reason why we studied the epimorphisms in Cu was that we wanted to decide whether this category is regular or not. In order to give the definition, we also need the term *regular epimorphism*, which means an epimorphism that appears as the coequalizer morphism of some pair of morphisms in the category.

Definition 5.51: A category $\mathscr{C}$ is *regular* if the following properties hold:

(1) The category $\mathscr{C}$ is finitely complete.

(2) If $f\colon X \to Y$ is a morphism in $\mathscr{C}$ and if we have a pullback diagram

$$\begin{array}{ccc} X \times_{f,f} X & \xrightarrow{p_0} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y, \end{array}$$

then the coequalizer of p_0 and p_1 exists. The pair (p_0, p_1) is also called the *kernel pair* of f.

(3) If $f\colon X \to Z$ is a regular epimorphism and

$$\begin{array}{ccc} X \times_{f,g} Y & \xrightarrow{p_0} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

is a pullback diagram, then p_1 is also a regular epimorphism.

Regular categories generalize abelian categories and have some nice properties; for example every morphism can be factorized into a regular epimorphism followed by a monomorphism and we also have a notion of exact sequences. A sequence

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{p} C$$

is called *exact* if on the one hand it is a coequalizer diagram and on the other hand the pair (f, g) is the kernel pair of p.

The first two properties for being regular are satisfied for Cu. To verify the third property, we need to understand the regular epimorphisms in Cu.

Question 5.52: How can regular epimorphisms in Cu be characterized?

We could also use an equivalent characterization of being regular. It is known that a category is regular if and only if it is finitely complete, if every monomorphism can be factorized into a strong epimorphism followed by a monomorphism and if the pullback along a strong epimorphism is again a strong epimorphism (see [Kel91]). Again, the first two properties are satisfied in Cu. By Theorem 5.50, we have to anwer the following question to decide whether Cu is regular or not.

Question 5.53: Let $f\colon S \to T$ be a Cu-morphism such that $\overline{\mathrm{Im}(P_c(f))} = P_c(T)$ and let

$$\begin{array}{ccc} S \times_{f,g} R & \xrightarrow{p_0} & S \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ R & \xrightarrow{g} & T \end{array}$$

be a pullback diagram. Has p_1 dense image, i.e. do we have $\overline{\mathrm{Im}(P_c(p_1))} = P_c(R)$?

A famous open problem with regard to the classification of C*-algebras is the UCT problem. A separable C*-algebra A is said to satisfy the UCT if for every separable C*-algebra B, there is a certain short exact sequence

$$0 \to \bigoplus_{i=0,1} \mathrm{Ext}(K_i(A), K_i(B)) \to KK_0(A,B) \to \bigoplus_{i=0,1} \mathrm{Hom}(K_i(A), K_i(B)) \to 0.$$

It is not known how large the class of C*-algebras satisfying the UCT is. It would be great if there was an analogue sequence for Cuntz semigroups. In [APT17], the authors established the internal hom functor $[\![-,-]\!]$ as the analogue of the Hom-functor for abelian groups. A next step would be to search for a Cuntz semigroup version of the Ext-functor for abelian groups. We will sketch a possible way how this could be done, although it is not sure if we are on the right track. First, we have to find a notion of projective Cu-semigroups.

Definition 5.54: A Cu-semigroup P is called *projective* if for every Cu-morphism $f\colon P \to T$ and any epimorphism $\pi\colon S \twoheadrightarrow T$ in Cu such that $P_c(\pi)$ is surjective, there is a Cu-morphism $\hat{f}\colon P \to T$ such that $f = \hat{f} \circ \pi$.

$$\begin{array}{ccc} & & S \\ & {\scriptstyle \hat{f}}\nearrow & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow[f]{} & T. \end{array}$$

It would also be natural to consider other epimorphisms in the definition of projectiveness (see e.g. [Bor94, Definition 4.6.1]). We have chosen those which are preserved by the functor $P_c(-)$ because we want the free objects to be projective.

Lemma 5.55: *For every set X, the* Cu-*semigroup* Cu-$\coprod_X \mathbb{G}$ *is projective.*

Proof: Let $f\colon$ Cu-$\coprod_X \mathbb{G} \to T$ be a Cu-morphism and $\pi\colon S \twoheadrightarrow T$ be an epimorphism such that $P_c(\pi)$ is surjective. Consider the natural bijection

$$\alpha_{T,X}\colon \mathrm{Cu}(\mathrm{Cu}\text{-}\coprod_X \mathbb{G}, T) \to \mathbf{Set}(X, P_c(T))$$

from Theorem 5.25. Since $P_c(\pi)$ is surjective, for every $x \in X$ there is an element $s_x \in P_c(S)$ such that $P_c(\pi)(s_x) = \alpha_{T,X}(f)(x)$. Define a map $g\colon X \to P_c(S)$ by $g(x) = s_x$. Since $\alpha_{S,X}$ is surjective, we can find $\hat{f} \in \mathrm{Cu}(\mathrm{Cu}\text{-}\coprod_X \mathbb{G}, S)$ such that $\alpha_{S,X}(\hat{f}) = g$. We claim that $f = \pi \circ \hat{f}$. By 5.25, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Cu}(\mathrm{Cu}\text{-}\coprod_X \mathbb{G}, S) & \xrightarrow{\alpha_{S,X}} & \mathbf{Set}(X, P_c(S)) \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle P(\pi)_*} \\ \mathrm{Cu}(\mathrm{Cu}\text{-}\coprod_X \mathbb{G}, T) & \xrightarrow{\alpha_{T,X}} & \mathbf{Set}(X, P_c(T)). \end{array}$$

Using the commutativity of the diagram at the third step, we obtain

$$\alpha_{T,X}(f) = P_c(\pi) \circ g = P_c(\pi) \circ \alpha_{S,X}(\hat{f}) = \alpha_{T,X}(\pi_*(\hat{f}) = \alpha_{T,X}(\pi \circ \hat{f}).$$

Since $\alpha_{T,X}$ is injective, we deduce $f = \pi \circ \hat{f}$. □

The last lemma says that Cu has enough projective objects in the sense that for every Cu-semigroup S, there is a projective Cu-semigroup P and an epimorphism $p\colon P \twoheadrightarrow S$. We can choose the projective object $\coprod_{P_c(S)} \mathbb{G}$ and the counit map $\varepsilon_S\colon \coprod_{P_c(S)} \mathbb{G} \to S$. We have already seen that $P_c(\varepsilon_S)$ is surjective and in particular that ε_S is a strong epimorphism.
A *projective resolution* of a Cu-semigroup S shall be an exact sequence

$$\cdots \overset{\cdots}{\rightrightarrows} P_4 \overset{\cdots}{\rightrightarrows} P_3 \overset{d_{2,1},d_{2,2}}{\underset{d_{1,1},d_{1,2}}{\rightrightarrows\rightrightarrows}} P_2 \overset{d_2}{\underset{d_1}{\rightrightarrows}} P_1 \xrightarrow{\varepsilon} S$$

where all P_i are projective. Exact means here that on the one hand ε is the coequalizer map of the pair (d_1, d_2) and on the other hand the pair (d_1, d_2) is the kernel pair of ε, that d_1 is the coequalizer map of the pair $(d_{1,1}, d_{1,2})$ and that the pair $(d_{1,1}, d_{1,2})$ is the kernel pair of d_1 and so on. Note that we in particular have $\varepsilon \circ d_1 = \varepsilon \circ d_2$.
For a Cu-semigroup S, we define $F(S) := \coprod_{P_c(S)} \mathbb{G}$.

Question 5.56: Does every Cu-semigroup S admit a projective resolution?

We could proceed as follows: we start with the counit map ε_S and go futher with the kernel pair (d_1, d_2) of ε_S and so on:

$$\cdots \overset{\cdots}{\rightrightarrows} F(S) \times_{\varepsilon_S,\varepsilon_S} F(S) \overset{d_2}{\underset{d_1}{\rightrightarrows}} F(S) \xrightarrow{\varepsilon_S} S.$$

It is not clear that this is an exact sequence. If we had a projective resolution, we could apply the functor $[\![-,T]\!]$ for another Cu-semigroup T and obtain

$$\begin{array}{ccccccccc} \cdots \overset{\cdots}{\longleftarrow\!\!\longleftarrow} & [\![P_4,T]\!] & \overset{\cdots}{\longleftarrow\!\!\longleftarrow} & [\![P_3,T]\!] & \underset{d^*_{1,1},d^*_{1,2}}{\overset{d^*_{2,1},d^*_{2,2}}{\longleftarrow\!\!\longleftarrow\!\!\longleftarrow}} & [\![P_2,T]\!] & \underset{d^*_1}{\overset{d^*_2}{\longleftarrow\!\!\longleftarrow}} & [\![P_1,T]\!] \xleftarrow{\varepsilon^*} & [\![S,T]\!] \\ & & & & & & & \nwarrow^{e} & \downarrow \alpha \\ & & & & & & & & E. \end{array}$$

Let E be the equalizer of the pair (d_1^*, d_2^*). Because we have $d_1^* \circ \varepsilon^* = d_2^* \circ \varepsilon^*$, by the universal property of the equalizer we get a unique Cu-morphism $\alpha\colon [\![S,T]\!] \to E$ that makes the diagram commute. We define

$$\text{Cu-Ext}(S,T) := \text{coeq}(\alpha, 0).$$

This would be the analogue of the Ext-functor for abelian groups. Note that in the category of abelian groups, we have for a morphism $f\colon X \to Y$ that $\ker(f) = \text{eq}(f,0)$ and $\text{coker}(f) = \text{coeq}(f,0)$. Of course, this was just a rough idea how an Ext-functor for Cuntz semigroups could be defined. It is not sure if the definition makes really sense, e.g. that it is independent of the choice of the projective resolution, that it behaves functorial or even if indeed there is a projective resolution for every Cu-semigroup. So, there are many things that remain unclear. The next example shows that it satisfies some minimum requirements.

Example 5.57: Let P be a projective Cu-semigroup. We want to show that Cu-Ext$(P,T) = \{0\}$ for every Cu-semigroup T (assuming it is well-defined). It is easy to see that the sequence

$$P \underset{\text{id}}{\overset{\text{id}}{\rightrightarrows}} P \xrightarrow{\text{id}} P$$

is a projective resolution. Applying the functor $[\![-,T]\!]$ results in

$$[\![P,T]\!] \underset{\text{id}}{\overset{\text{id}}{\leftleftarrows}} [\![P,T]\!] \xleftarrow{\text{id}} [\![P,T]\!].$$

Since this is already an equalizer diagram, the morphism α from above is an isomorphism and therefore we obtain Cu-Ext$(P,T) = \text{coeq}(\alpha, 0) = \{0\}$. In particular, we have for every set X that Cu-Ext$(\text{Cu-}\coprod_X \mathbb{G}, T) = \{0\}$.

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