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Tensor products of Cuntz semigroups associated with Choquet simplices

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Introduction

The Cuntz semigroup is an invariant for C^*-algebras. It was introduced by Cuntz in [Cun78] as a means of proving the existence of so called dimension functions on unital, simple and stably finite C^*-algebras. He showed that a unital, simple C^*-algebra is stably finite if and only if it admits a 2-quasitracial state (which in the presence of exactness is the same as a tracial state). The Cuntz semigroup proved to be a fine invariant when Toms (in [Tom08]) gave a counterexample to a version of the Elliott conjecture which asserted that the class of all unital, separable, simple, nonelementary, nuclear C^*-algebras is classified by the Elliott invariant, an invariant for C^*-algebras of K-theoretic nature. He gave an example of two C^*-algebras with the aforementioned properties that cannot be distinguished by a number of invariants, including the Elliott invariant, but which are nonisomorphic because their Cuntz semigroups differ. Today, thanks to the work of numerous mathematicians, it is known that smaller classes of C^*-algebras are classified by the Elliott invariant. For example, when substituting 'nuclear' for the stronger condition 'finite nuclear dimension', the above version of the Elliott conjecture holds true.

Subsequently, in [CEI08], Coward, Elliott and Ivanescu introduced the category Cu of abstract Cuntz semigroups. Objects in Cu are ordered monoids that satisfy a number of additional axioms. They showed that a slightly modified version of the Cuntz semigroup of a C^*-algebra A, denoted by $\mathrm{Cu}(A)$, always belongs to Cu. The category Cu is not only a useful framework for the study of Cuntz semigroups of C^*-algebras, but it is also interesting in its own right. For instance in [APT18], Antoine, Perera and Thiel established the existence of tensor products in Cu (relative to a certain notion of Cu-bimorphism). It is then only natural to ask how $\mathrm{Cu}(A \otimes B)$ and $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B)$ are related, for C^*-algebras A and B (say for simplicity that A is nuclear). It turns out that there is a natural morphism $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes B)$, which under certain circumstances is known to be an isomorphism (for example, if A or B is an AF-algebra), but which in general is not an isomorphism.

The Cuntz semigroup of a sufficiently regular C^*-algebra A can be expressed as the disjoint union of two semigroups: the Murray-von Neumann semigroup $V(A)$ and the lower semicontinuous affine functions on the compact convex set $\mathrm{QT}_1(A)$ of all normalized quasitraces on A. In general, for every metrizable compact convex set K, the lower semicontinuous affine functions on K that are either strictly positive or zero, $\mathrm{LAff}(K)^0_{++}$, form a semigroup in the category Cu. One would expect $\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++}$ to be a component of $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B)$, which leads us to the following question: what is the tensor product of $\mathrm{LAff}(K_1)^0_{++}$ and $\mathrm{LAff}(K_2)^0_{++}$, for two metrizable compact convex sets K_1 and K_2? While the answer to this question remains unknown, we provide an answer in the special case that either K_1 or K_2 is a Choquet simplex. The main result of this thesis states that there exists an isomorphism

$$\mathrm{LAff}(K_1)^0_{++} \otimes \mathrm{LAff}(K_2)^0_{++} \cong \mathrm{LAff}(K_1 \otimes K_2)^0_{++}$$

whenever K_1 and K_2 are two metrizable compact convex sets, one of which is a Choquet simplex. Here $K_1 \otimes K_2$ denotes the so called biprojective tensor product of compact convex sets, a concept that is known since the late 1960s.

This thesis is organized as follows: in the first chapter, we recapitulate a few general aspects of the theory of Cuntz semigroups. Beginning with the definition of the category Cu and the construction of the Cuntz semigroup of a C^*-algebra, we discuss how certain properties of a C^*-algebra are encoded in its Cuntz semigroup. We give an overview of the construction of tensor products and inductive limits in the category Cu. At the end of the chapter, we see that a large class of simple Cu-semigroups can be expressed as the disjoint union of two subsemigroups: the so called compact part and the so called soft part.

In the second chapter, we are concerned with compact convex sets. Two important classes of examples of compact convex sets are examined, both of which will show up constantly: probability measures on compact Hausdorff spaces and state spaces of partially ordered abelian groups with order unit. A rough duality between the category of compact convex sets and the category of partially ordered abelian groups with order unit is developed. This duality allows us to define a tensor product for compact convex sets later on. We prove a few approximation results - most notably that any lower semicontinuous affine function on a compact convex set can be approximated by continuous affine functions. This allows us to demonstrate that $\mathrm{LAff}(K)^0_{++}$ is a Cu-semigroup whenever K is a metrizable compact convex set.

Then, in the third chapter, we concentrate on Choquet simplices, i.e. compact convex sets with a certain property that ensures a large supply of continuous affine functions. The subclass of Bauer simplices is discussed. We are particularly interested in how the property of being a Choquet simplex or a Bauer simplex is encoded in the (lower semi-)continuous affine functions.

Before we arrive at the main result in the fourth chapter, we take a look at a certain notion of tensor products for compact convex sets. We verify that the tensor product of two Choquet simplices (Bauer simplices, metrizable compact convex sets) is a Choquet simplex (a Bauer simplex, metrizable). One difficulty in the proof of the main theorem is overcome by applying a structural theorem of Effros, Handelman and Shen concerning inductive limits of finite powers of $\mathbb{Z}$ in the category of partially ordered abelian groups.

In the fifth chapter, we determine which Cu-semigroups are of the form $\mathrm{LAff}(K)^0_{++}$, for some metrizable compact convex set K. Our argumentation relies on a theorem of Robert that gives

a concrete description of the so called realification of a Cu-semigroup.
In the final chapter, we briefly discuss how the main theorem could be applied to obtain a new criterion for which the natural map $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes B)$ is an isomorphism.

As for the notation, we use $\mathbb{N}$ to denote the set of all natural numbers, including zero. If we want the zero to be excluded, we will write $\mathbb{N}^*$. If $x_1, \ldots, x_n, y_1, \ldots, y_m$ is a finite collection of elements in some partially ordered set $(X, \leq)$, the expression $x_1, \ldots, x_n \leq y_1, \ldots, y_m$ shall mean that $x_i \leq y_j$ holds for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$.

I would like to thank my supervisor Dr. Hannes Thiel for introducing me to Cuntz semigroups, for his support, the frequent discussions on the development of this thesis with its occurring obstacles, and overall for successfully guiding me through this project.

1 Cu-semigroups

In this chapter, we recall a few results concerning Cu-semigroups. Our general reference for this chapter, especially for the first two sections, is [Thi16].

1.1 The category Cu

Let x and y be elements in some partially ordered set $(X, \leq)$ (from now on denoted by X). We say that x is way-below y, $x \ll y$, if for every increasing net $(z_i)_{i\in I}$ in X that has a supremum such that $y \leq \sup_i z_i$, there exists some index i_0 such that $x \leq z_{i_0}$. Similarly, we say that x is sequentially way-below y, $x \ll_\omega y$, if for every increasing sequence $(z_n)_{n\in\mathbb{N}}$ in X that has a supremum, we have that $y \leq \sup_n z_n$ implies that there exists some index n_0 such that $x \leq z_{n_0}$. The expression 'increasing net in X' may be substituted for 'upward directed subset of X'. Similarly, 'increasing sequence in X' may be changed to 'upward directed subset of X admitting a countable cofinal subset'. In general, $x \ll y$ implies that $x \ll_\omega y$ which itself implies that $x \leq y$. Also, the inequality $x \leq x' \ll y' \leq y$ implies that $x \ll y$. An analogous statement holds for $\ll_\omega$. If X has a smallest element 0, then $0 \ll x$ holds for all $x \in X$. The relation $\ll$ ($\ll_\omega$) is certainly most interesting if every increasing net (increasing sequence) in X has a supremum, which leads us to the following definition.

1.1 Definition A partially ordered set X is called directed complete (sequentially complete) if every increasing net (increasing sequence) in X has a supremum. We abbreviate the term directed complete partially ordered set to dcpo. Similarly, we refer to a sequentially complete partially ordered set as ω-dcpo.
We say that X is continuous (ω-continuous) if for every $x \in X$, there exists an increasing net $(x_i)_{i\in I}$ (increasing sequence $(x_n)_{n\in\mathbb{N}}$) in X such that $x_i \ll x$ for each $i \in I$ ($x_n \ll_\omega x$ for each $n \in \mathbb{N}$) and such that $x = \sup_i x_i$ ($x = \sup_n x_n$). In the sequential case, this is equivalent to saying that there exists a $\ll_\omega$-increasing sequence $(x_n)_n$ in X with supremum x.
Finally, a domain (ω-domain) is a continuous dcpo (ω-continuous ω-dcpo).

There is a suitable separability condition for which $\ll$ and $\ll_\omega$ agree:

1.2 Definition A subset B of some ω-dcpo X is called basis if for all $x, y \in X$ satisfying

$x \ll_\omega y$, we can find an element $b \in B$ such that $x \leq b \ll_\omega y$. In this case, every element $x \in X$ can be written as the supremum of an increasing (or even $\ll_\omega$-increasing) sequence in B. We say that X is countably based if it has a countable basis.

1.3 Proposition A countably based ω-dcpo is a domain if and only if it is a ω-domain. In this case, the relations $\ll$ and $\ll_\omega$ agree.

1.4 Definition A partially ordered set X is called an inf-semilattice if every nonempty finite subset of X has an infimum. The term sup-semilattice is defined analogously. If X is both an inf-semilattice and a sup-semilattice, we say that X is a lattice. If any subset of X has a supremum and an infimum, we say that X is a complete lattice.

1.5 Example Let X be a locally compact Hausdorff space, and denote the topology on X by $\mathcal{O}(X)$. When equipped with the inclusion relation, $\mathcal{O}(X)$ becomes a partially ordered set. Any subset $\mathcal{A} \subseteq \mathcal{O}(X)$ has a supremum, which is given by $\bigcup \mathcal{A}$, and an infimum, which is given by the interior of $\bigcap \mathcal{A}$, so $\mathcal{O}(X)$ is a complete lattice. In fact, $\mathcal{O}(X)$ is a domain. The way-below relation is given by $U \ll V$ if and only if there exists a compact subset K of X such that $U \subseteq K \subseteq V$ (for $U, V \in \mathcal{O}(X)$). For this reason, the way-below relation is also known as the compact containment relation. If X is second countable, then $\mathcal{O}(X)$ is countably based.

Let M and N be abelian monoids. Recall that a monoid homomorphism from M to N is a map $M \to N$ that preserves the zero element and addition. We let AM denote the category of all abelian monoids with monoid homomorphisms.

A positively ordered monoid is an abelian monoid M, equipped with a compatible partial order such that every element in M is positive (i.e. $0 \leq x$ for all $x \in M$). A PoM-morphism between positively ordered monoids M and N is an order-preserving monoid homomorphism $M \to N$. Let PoM be the category of all positively ordered monoids and PoM-morphisms.

1.6 Definition ([CEI08]) A Cu-semigroup is a positively ordered monoid S that satisfies the following axioms:

($\mathcal{O}$1) Every increasing sequence in S has a supremum.

($\mathcal{O}$2) For every $s \in S$, there exists a $\ll_\omega$-increasing sequence in S with supremum s.

($\mathcal{O}$3) If $s, s', t, t' \in S$ satisfy $s \ll_\omega s'$ and $t \ll_\omega t'$, then $s + t \ll_\omega s' + t'$.

($\mathcal{O}$4) If $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ are increasing sequences in S, then the equation $\sup_n(s_n + t_n) = (\sup_n s_n) + (\sup_n t_n)$ holds.

A generalized Cu-morphism from a Cu-semigroup S to a Cu-semigroup T is a map $S \to T$ that preserves the zero element, addition, order and suprema of increasing sequences. If this map also preserves the sequential way-below relation, then we refer to it as a Cu-morphism. Let Cu be the category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms.

Remark Axiom ($\mathcal{O}1$) states that S is a ω-dcpo, while axiom ($\mathcal{O}2$) just means that S is ω-continuous. Thus, axioms ($\mathcal{O}1$) and ($\mathcal{O}2$) say that S is a ω-domain.

1.7 Definition An ideal in some Cu-semigroup S is a submonoid J that is downward hereditary, i.e. if $x \in S$ and $y \in J$ satisfy $x \leq y$, then x lies in J, and that is closed under suprema of increasing sequences. Let us denote the set of all ideals in S by $\mathrm{Lat}(S)$. We say that S is simple if the only ideals in S are the trivial ideals $\{0\}$ and S.

The set $\mathrm{Lat}(S)$ is naturally ordered by inclusion. Since S itself is an ideal and since arbitrary intersections of ideals are again ideals, it follows that for any subset $D \subseteq S$, there exists a smallest ideal J in S containing D. In this case, J equals the intersection of all ideals in S containing D. It follows that $\mathrm{Lat}(S)$ is a complete lattice. If a is an element in S, set $\infty \cdot a \coloneqq \sup_n n \cdot a$. It is straightforward to show that the ideal generated by a is given by $\{s \in S \mid s \leq \infty \cdot a\}$. If S is simple and a is nonzero, this ideal must be equal to S. In this case, $\infty \cdot a$ is the unique largest element of S, and we denote it by ∞.

1.8 Definition A functional on a Cu-semigroup S is a generalized Cu-morphism $S \to [0, \infty]$. We denote the set of all functionals on S by $F(S)$.

1.2 The Cuntz semigroup of a C^*-algebra

Probably the most important reason for the study of Cu-semigroups is that one can assign to each C^*-algebra A an object $\mathrm{Cu}(A)$ in Cu. This Cu-semigroup is known as the Cuntz semigroup of the C^*-algebra A. This assignment defines a functor from the category of C^*-algebras to the category Cu, so the Cuntz semigroup can be regarded as an invariant for C^*-algebras. In this section, we will recall some details of the construction of $\mathrm{Cu}(A)$ and how certain properties of A are encoded in its Cuntz semigroup.

1.9 Definition Let a and b be positive elements in some C^*-algebra A. We write $a \precsim b$ if there exists a sequence $(z_n)_n$ in A such that $a = \lim_n z_n b z_n^*$. In this case we say that a is Cuntz subequivalent to b. If both $a \precsim b$ and $b \precsim a$ hold, we say that a is Cuntz equivalent to b, and denote it by $a \sim b$.

It is easy to see that the relation $\precsim$ is reflexive (use an approximate identity for A) and transitive. Thus, the relation $\sim$ defines an equivalence relation on A_+.

1.10 Definition The Cuntz semigroup of a C^*-algebra A is defined as follows: As a set, it is given by $\mathrm{Cu}(A) \coloneqq (A \otimes \mathcal{K})_+/\sim$, where $\mathcal{K}$ denotes the compact operators on a separable, infinite-dimensional Hilbert space $\mathcal{H}$. We define a partial order on $\mathrm{Cu}(A)$ by setting $[a] \leq [b]$ if and only if $a \precsim b$. Finally, we can define an addition on $\mathrm{Cu}(A)$ as follows: Let $\psi\colon \mathcal{K} \otimes M_2 \to \mathcal{K}$ be any *-isomorphism. Then $\mathrm{id}_A \otimes \psi\colon A \otimes \mathcal{K} \otimes M_2 \to A \otimes \mathcal{K}$ is also a *-isomorphism. We set

$$[a] + [b] \coloneqq [(\mathrm{id}_A \otimes \psi)(a \oplus b)],$$

where $a \oplus b \coloneqq \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in A \otimes \mathcal{K} \otimes M_2$.

Remark That the binary relation in Definition 1.10 is a well defined partial order follows from the fact that Cuntz subequivalence is transitive and reflexive. However, it is more difficult to show that the addition is well defined. It is well known that any *-isomorphism $\varphi\colon \mathcal{K} \to \mathcal{K}$ has the form $\varphi(\_) = u \cdot \mathrm{id}(\_) \cdot u^*$ for some unitary $u \in \mathcal{B}(\mathcal{H})$. If $\psi_1, \psi_2\colon \mathcal{K} \otimes M_2 \to \mathcal{K}$ are two *-isomorphisms, it follows that there exists a unitary $u \in \mathcal{B}(\mathcal{H})$ such that $\psi_1(\_) = u \cdot \psi_2(\_) \cdot u^*$. Then the element $u' \coloneqq 1_{\tilde{A}} \otimes u$ (where $\tilde{A}$ denotes the minimal unitalization of A) is a unitary in the multiplier algebra of $A \otimes \mathcal{K}$ and satisfies

$$(\mathrm{id}_A \otimes \psi_1)(a \oplus b) = u' \cdot (\mathrm{id}_A \otimes \psi_2)(a \oplus b) \cdot (u')^*,$$

for all $a, b \in (A \otimes \mathcal{K})_+$. In particular, $(\mathrm{id}_A \otimes \psi_1)(a \oplus b)$ and $(\mathrm{id}_A \otimes \psi_2)(a \oplus b)$ are unitarily equivalent with unitaries from the multiplier algebra of $A \otimes \mathcal{K}$. It follows from [Thi16, 2.20] that $[(\mathrm{id}_A \otimes \psi_1)(a \oplus b)] = [(\mathrm{id}_A \otimes \psi_1)(a \oplus b)]$, so this value is independent of the choice of the *-isomorphism $\mathcal{K} \otimes M_2 \to \mathcal{K}$. It is not very hard to show that the value $[(\mathrm{id}_A \otimes \psi)(a \oplus b)]$ does not depend on the choice of the representatives a and b either and consequently, the addition on $\mathrm{Cu}(A)$ is well defined. Again, one easily sees that the order and addition as defined in 1.10 give $\mathrm{Cu}(A)$ the structure of a positively ordered monoid.

1.11 Theorem ([CEI08, Theorem 1]) If A is a C^*-algebra, then $\mathrm{Cu}(A)$ is a Cu-semigroup. For $[a], [b] \in \mathrm{Cu}(A)$, we have $[a] \ll_\omega [b]$ if and only if there exists $\varepsilon > 0$ such that $a \precsim (b - \varepsilon)_+$ (the element $(b - \varepsilon)_+$ is defined as $f(b)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) \coloneqq \max\{x - \varepsilon, 0\}$).
Let $\varphi\colon A \to B$ be a *-homomorphism into another C^*-algebra B. The induced *-homomorphism $\varphi \otimes \mathrm{id}\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ preserves Cuntz subequivalence, and $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ is a Cu-morphism. The assignment above defines a functor $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$, where C^* denotes the category of all C^*-algebras with *-homomorphisms.
If A is separable, then $\mathrm{Cu}(A)$ is countably based.

1.12 Example The Cuntz semigroup of $\mathcal{K}$ can be computed as $\mathrm{Cu}(\mathcal{K}) \cong \overline{\mathbb{N}}$, where $\overline{\mathbb{N}} \coloneqq \mathbb{N} \cup \{\infty\}$ is equipped with the obvious order and addition. The isomorphism is given by $[a] \mapsto \mathrm{rank}(a)$, for $a \in \mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$. In general, any C^*-algebra A satisfies $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{K})$. In particular, we obtain $\mathrm{Cu}(\mathbb{C}) \cong \mathrm{Cu}(\mathcal{K}) \cong \overline{\mathbb{N}}$.

1.13 Definition For a positively ordered monoid S, consider the following axioms:

($\mathcal{O}5'$) For all $x, x', y \in S$ satisfying $x' \ll_\omega x \leq y$, there exists $s \in S$ such that $x'+s \leq y \leq x+s$.

($\mathcal{O}5$) For all $x, x', y, y', z \in S$ satisfying $x' \ll_\omega x$, $y' \ll_\omega y$ and $x + y \leq z$, there exists $s \in S$ such that $x' + s \leq z \leq x + s$ while also $y' \leq s$.

($\mathcal{O}6$) For all $x, x', y, z \in S$ satisfying $x' \ll_\omega x \leq y + z$, there exist $s, t \in S$ such that $s \leq x, y$, and $t \leq x, z$, and $x' \leq s + t$.

We say that S has almost algebraic order if S satisfies ($\mathcal{O}5$). Also, S is said to have almost Riesz decomposition if it satisfies ($\mathcal{O}6$). It is clear that ($\mathcal{O}5$) is stronger than ($\mathcal{O}5'$).

The axiom ($\mathcal{O}5$) has first been considered in [APT18], where it was also shown that the Cuntz semigroup of every C^*-algebra satisfies ($\mathcal{O}5$). The weaker version ($\mathcal{O}5'$) was introduced by Rørdam and Winter in [RW10]. Axiom ($\mathcal{O}6$) is due to Robert in [Rob13]. In the same paper, it was shown that the Cuntz semigroup of any C^*-algebra satisfies ($\mathcal{O}6$).

1.14 Theorem Let A be a C^*-algebra. If $I \lhd A$ is a (closed, two-sided) ideal, then the set

$$J_I := \{[a] \in \mathrm{Cu}(A) \mid a \in (I \otimes \mathcal{K})_+\}$$

is an ideal in $\mathrm{Cu}(A)$. The map $\mathrm{Cu}(I) \to \mathrm{Cu}(A)$, induced by the inclusion $I \to A$, is an embedding of Cu-semigroups, and the image of the former map is given by J_I. Thus, we may identify $\mathrm{Cu}(I)$ with J_I. Conversely, if J is an ideal in $\mathrm{Cu}(A)$, then the set

$$I_J := \{a \in A \mid [aa^*] \in J\}$$

is an ideal in A.

The assignments above establish an order isomorphism $\mathrm{Lat}(A) \cong \mathrm{Lat}(\mathrm{Cu}(A))$, $I \mapsto \mathrm{Cu}(I)$. In particular, a C^*-algebra A is simple if and only if its Cuntz semigroup is simple.

Since we can recover the spectrum of a C^*-algebra from the lattice of all ideals, the Cuntz semigroup is a complete invariant for abelian C^*-algebras.

1.15 Corollary Two abelian C^*-algebras A and B are isomorphic if and only if their Cuntz semigroups are isomorphic.

1.16 Definition A 1-quasitrace on a C^*-algebra A is a function $\tau\colon A_+ \to [0, \infty]$ with the following properties:

- $\tau(0) = 0$ and $\tau(t \cdot a) = t \cdot \tau(a)$ for all $a \in A_+$ and $t \in (0, \infty)$,
- $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in A_+$ satisfying $ab = ba$,
- $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.

Let $n \in \mathbb{N}^*$. A n-quasitrace on A is a 1-quasitrace on A that extends to a 1-quasitrace on $A \otimes M_n$. Every 2-quasitrace is a n-quasitrace, for all $n \in \mathbb{N}^*$ (see [BH82, II.4.1]). Additionally, the extension to $A \otimes M_n$ is unique. It follows that every 2-quasitrace extends uniquely to a 1-quasitrace on $A \otimes \mathcal{K}$. We refer to a 2-quasitrace just as a quasitrace. Let us denote the set of all lower semicontinuous quasitraces on A by $\mathrm{QT}(A)$. A trace on A is a 1-quasitrace on A that is additive on all elements (not just commutating ones). We use $\mathrm{T}(A)$ to denote the set of all lower semicontinuous traces on A. Finally, if A is unital, we use $\mathrm{QT}_1(A)$ and $\mathrm{T}_1(A)$ to denote the sets of all normalized (quasi-)traces on A. We remark without proof that every normalized (quasi-)trace on A is automatically real-valued, order-preserving and continuous. Thus, $\mathrm{T}_1(A)$ may be identified with the set of all tracial states on A.

For the upcoming theorem, we use the following notation: for $n \in \mathbb{N}^*$, let $f_n \colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f_n(x) \coloneqq \begin{cases} 0 & \text{if } x \leq \frac{1}{n} \\ nx - 1 & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 1 & \text{if } \frac{2}{n} \leq x \end{cases} .$$

1.17 Theorem Let A be a C^*-algebra. If $\tau \in \mathrm{QT}(A)$ is a lower semicontinuous quasitrace on A, then the map $d_\tau \colon \mathrm{Cu}(A) \to [0, \infty]$, defined by $d_\tau([a]) \coloneqq \lim_n \tau(f_n(a))$, is a well defined functional on $\mathrm{Cu}(A)$. Conversely, if $\lambda \in F(\mathrm{Cu}(A))$ is a functional on $\mathrm{Cu}(A)$, then the function $\tau_\lambda \colon (A \otimes \mathcal{K})_+ \to [0, \infty]$, which is defined by the formula

$$\tau_\lambda(a) \coloneqq \int_0^\infty \lambda([(a-t)_+])dt,$$

is a lower semicontinuous quasitrace on A.
These assignments define a natural bijection $\mathrm{QT}(A) \cong F(\mathrm{Cu}(A))$.

1.3 Tensor products

Throughout this thesis, we will discuss various different tensor products. All of them, except the tensor product of compact convex sets, are defined relative to a notion of bimorphism. The following definition is informal, but it should make clear what we mean by a tensor product associated with a notion of bimorphism, and should therefore avoid repetition.

1.18 Definition Let $\mathcal{C}$ be a category whose objects are sets endowed with additional structures (such as addition, order, etc.) and whose morphisms are structure-preserving maps. For objects A, B, C in $\mathcal{C}$, a $\mathcal{C}$-bimorphism from $A \times B$ to C is a map $A \times B \to C$ (where $A \times B$ denotes the cartesian product of the underlying sets) that preserves the structures of A and B in a certain sense that must be specified beforehand. Usually, this map should preserve some structure of A and B in each variable while preserving other structure of A and B jointly. We denote the set of all $\mathcal{C}$-bimorphisms $A \times B \to C$ by $\mathrm{Bi}\mathcal{C}(A \times B, C)$. If D is another object in $\mathcal{C}$, the composition of any $\mathcal{C}$-bimorphism $A \times B \to C$ with any $\mathcal{C}$-morphism $C \to D$ should always result in a $\mathcal{C}$-bimorphism $A \times B \to D$.

Let A and B be objects in $\mathcal{C}$. A tensor product of A and B is a pair (C, ω), consisting of some object C in $\mathcal{C}$ and a $\mathcal{C}$-bimorphism $\omega\colon A \times B \to C$, such that the following universal property is satisfied: For any object D in $\mathcal{C}$ and for every $\mathcal{C}$-bimorphism $\varphi\colon A \times B \to D$, there exists a unique $\mathcal{C}$-morphism $\tilde{\varphi}\colon C \to D$ such that $\varphi = \tilde{\varphi} \circ \omega$.

If the tensor product of A and B exists, it is unique up to isomorphism. We write $A \otimes^{\mathcal{C}} B$ instead of C. The universal property may be restated by saying that there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes^{\mathcal{C}} B, \_) \cong \mathrm{Bi}\mathcal{C}(A \times B, \_).$$

For $a \in A$ and $b \in B$, we write $a \odot b$ (for 'algebraic tensor products', namely the tensor products in AM, PoM, PreW, OAG and GOU) or $a \otimes b$ (for the tensor product of compact convex sets and the Cu-tensor product) instead of $\omega(a, b)$. An element of this form is called elementary tensor. If the tensor product exists for all objects A and B in $\mathcal{C}$, we say that the category $\mathcal{C}$ has tensor products, or that the tensor product of $\mathcal{C}$-objects exists. Note that although the notion of the tensor product heavily depends on the definition of $\mathcal{C}$-bimorphisms, this dependence is neither represented in the terminology nor in the notation of the tensor product. This should not lead to confusion though, since every category in this thesis will have at most one notion of bimorphisms associated with it.

It was shown in [APT18] that the category Cu has tensor products (with respect to a certain notion of Cu-bimorphism). The general strategy involves regarding Cu as a full, reflective subcategory of a certain category PreW. It is easier to construct a tensor product in the category PreW since the objects in that category are algebraic in nature. The tensor product in Cu is then obtained by taking a certain completion of the tensor product in PreW.

1.19 Definition An auxiliary relation on a positively ordered monoid S is a relation $\prec$ that satisfies the following conditions:

- For all $s \in S$, we have that $0 \prec s$.
- For all $s_1, s_2 \in S$, the inequality $s_1 \prec s_2$ implies that $s_1 \leq s_2$.
- If $s_1, s_2, s_3, s_4 \in S$ satisfy $s_1 \leq s_2 \prec s_3 \leq s_4$, then $s_1 \prec s_4$.

1.20 Definition Suppose that S is a positively ordered monoid, equipped with a certain auxiliary relation $\prec$. For any element s in S, we define the set $s^{\prec} := \{t \in S \mid t \prec s\}$. Consider the following axioms:

(W1) For every $s \in S$, there exists a $\prec$-increasing sequence $(s_n)_n \subseteq s^{\prec}$ that is cofinal in $s^{\prec}$.

(W2) Each $s \in S$ is the supremum of $s^{\prec}$.

(W3) If $s, s', t, t' \in S$ satisfy $s \prec s'$ and $t \prec t'$, then $s + s' \prec t + t'$.

(W4) If $s, t, u \in S$ satisfy $s \prec t + u$, then there exist $t', u' \in S$ such that $t' \prec t$ and $u' \prec u$ while also $s \prec t' + u'$.

A pair $(S, \prec)$, consisting of a positively ordered monoid S and a fixed auxiliary relation $\prec$ on S, is called PreW-semigroup if the axioms (W1), (W3) and (W4) are satisfied. A W-semigroup is a PreW-semigroup $(S, \prec)$ that also satisfies axiom (W2). If $(S, \prec)$ and $(T, \prec)$ are two PreW-semigroups, a generalized W-morphism $(S, \prec) \to (T, \prec)$ is a PoM-morphism $\varphi\colon S \to T$ that satisfies the following continuity condition: If $s \in S$ and $t \in T$ satisfy $t \prec \varphi(s)$, then there exists an element $s' \in S$ with $s' \prec s$ and $t \leq \varphi(s')$. If this map also preserves the auxiliary relations, then we refer to it as a W-morphism. Let PreW be the category whose objects are PreW-semigroups and whose morphisms are W-morphisms. Then let W be the full subcategory of PreW whose objects are W-semigroups.

1.21 Example If S is a Cu-semigroup, then $(S, \ll_\omega)$ is a W-semigroup. For any other Cu-semigroup T, the Cu-morphisms $S \to T$ are exactly the W-morphisms $(S, \ll_\omega) \to (T, \ll_\omega)$. Thus, we may regard Cu as a full subcategory of W, which itself is a full subcategory of PreW. We will write S instead of $(S, \ll_\omega)$.

1.22 Theorem ([APT18, 3.1.6]) Let $(S, \prec)$ be a PreW-semigroup. There exists a Cu-semigroup $\gamma(S, \prec)$ and a W-morphism $\alpha\colon (S, \prec) \to \gamma(S, \prec)$ with the following universal property: For any Cu-semigroup T and any W-morphism $\varphi\colon (S, \prec) \to T$, there exists a unique Cu-morphism $\tilde{\varphi}\colon \gamma(S, \prec) \to T$ such that $\varphi = \tilde{\varphi} \circ \alpha$.

As usual, $\gamma(S, \prec)$ is unique up to Cu-isomorphism. We refer to $\gamma(S, \prec)$ as the Cu-completion of $(S, \prec)$. The completion functor $\gamma\colon \mathrm{PreW} \to \mathrm{Cu}$ is left adjoint to the inclusion functor $\mathrm{Cu} \to \mathrm{PreW}$ (this is just a restatement of the universal property). In particular, Cu is a full, reflective subcategory of PreW.

1.23 Definition Let S, T and U be sets, and let $\varphi\colon S \times T \to U$ be a map.

1) Suppose that S, T and U are abelian monoids. We say that φ is a AM-bimorphism if it is a AM-morphism in each variable.
2) Suppose that S, T and U are positively ordered monoids. We say that φ is a PoM-bimorphism if it is a PoM-morphism in each variable.
3) Suppose that S, T and U are PreW-semigroups. We say that φ is a W-bimorphism if it is a generalized W-morphism in each variable and if for all $s, s' \in S$ and $t, t' \in T$ satisfying $s \prec s'$ and $t \prec t'$, we have that $\varphi(s, t) \prec \varphi(s', t')$.
4) Suppose that S, T and U are Cu-semigroups. We say that φ is a Cu-bimorphism if it is a generalized Cu-morphism in each variable and if for all $s, s' \in S$ and $t, t' \in T$ satisfying $s \ll_\omega s'$ and $t \ll_\omega t'$, we have that $\varphi(s, t) \ll \varphi(s', t')$.

1.24 Theorem ([APT18]) 1) The tensor product of abelian monoids exists.

2) The tensor product of positively ordered monoids exists, more precisely: Let S and T be positively ordered monoids, and let $(S\otimes^{\mathrm{AM}}T,\, \omega)$ be their tensor product as abelian monoids. Then there exists a partial order $\leq$ on $S \otimes^{\mathrm{AM}} T$ such that $S \otimes^{\mathrm{PoM}} T \coloneqq (S \otimes^{\mathrm{AM}} T, \leq)$ is a positively ordered monoid, such that ω is a PoM-bimorphism and such that $(S \otimes^{\mathrm{PoM}} T,\, \omega)$ is the tensor product of S and T as positively ordered monoids.

3) The tensor product of PreW-semigroups exists, more precisely: Let $(S, \prec)$ and $(T, \prec)$ be PreW-semigroups, and let $(S \otimes^{\mathrm{PoM}} T,\, \omega)$ be the PoM-tensor product. There exists an auxiliary relation $\prec$ on $S \otimes^{\mathrm{PoM}} T$ such that $(S, \prec) \otimes^{\mathrm{PreW}} (T, \prec) \coloneqq (S \otimes^{\mathrm{PoM}} T, \prec)$ is a PreW-semigroup, such that ω is a W-bimorphism and such that $(S, \prec) \otimes^{\mathrm{PreW}} (T, \prec)$ is the tensor product of $(S, \prec)$ and $(T, \prec)$ as PreW-semigroups.

4) The tensor product of Cu-semigroups exists, more precisely: If S and T are Cu-semigroups, let $((S, \ll) \otimes^{\mathrm{PreW}} (T, \ll), \omega)$ be the tensor product of S and T, considered as PreW-semigroups. Set $S \otimes T \coloneqq \gamma((S, \ll) \otimes^{\mathrm{PreW}} (T, \ll))$, and let $\alpha\colon (S, \ll) \otimes^{\mathrm{PreW}} (T, \ll) \to S \otimes T$ be the PreW-morphism belonging to the completion. Then $(S \otimes T, \alpha \circ \omega)$ is the tensor product of S and T as Cu-semigroups.

Remark Suppose that S, T and U are Cu-semigroups. A map $\varphi\colon S \times T \to U$ is called a generalized Cu-bimorphism if it is a generalized Cu-morphism in each variable. The Cu-tensor product of S and T also has the universal property that generalized Cu-bimorphisms $S \times T \to U$ correspond to generalized Cu-morphisms $S \otimes T \to U$.

1.25 Theorem ([APT18, 6.3.5]) There is a Cu-isomorphism

$$\gamma((S, \prec) \otimes^{\mathrm{PreW}} (T, \prec)) \cong \gamma(S, \prec) \otimes \gamma(T, \prec)$$

that is natural in PreW-semigroups $(S, \prec)$ and $(T, \prec)$.

1.4 Inductive limits

The category Cu has inductive limits. Similarly to the case of the tensor product, this follows from the fact that the category PreW has inductive limits. The inductive limit in PreW is based on the construction of the inductive limit in the category Set of all sets. Let us recall some details.

Suppose that $((M_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ is an inductive system in Set. For $a \in M_i$ and $b \in M_j$, we write $a \sim b$ if there exists an index $i, j \leq k \in I$ such that $\varphi_{i,k}(a) = \varphi_{j,k}(b)$. This defines an equivalence relation on the disjoint union $\bigsqcup_{i\in I} M_i$. Set $M \coloneqq (\bigsqcup_{i\in I} M_i)/{\sim}$, and define $\varphi_{i,\infty}\colon M_i \to M$ by $\varphi_{i,\infty}(x) \coloneqq [x]$. It is straightforward to check that $(M, (\varphi_{i,\infty})_{i\in I})$ is the inductive limit of the inductive system $((M_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$. We refer to this inductive limit as the algebraic inductive limit.

There are many examples of categories that have inductive limits which are based on the algebraic inductive limit. For instance, the category PoM has inductive limits. The inductive limit is just the algebraic inductive limit, equipped with the induced order and addition. In the same vein, the category PreW has inductive limits which is again given by the algebraic inductive limit, equipped with the induced order, addition and auxiliary relation. Since Cu is a full, reflective subcategory of PreW, it follows that Cu has inductive limits. More precisely, if $((S_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I, i\leq j})$ is an inductive system in Cu, the inductive limit is the completion of the inductive limit in PreW:

$$\mathrm{Cu}\text{-}\varinjlim S_i \cong \gamma(\mathrm{PreW}\text{-}\varinjlim S_i).$$

1.26 Theorem ([APT18, 3.2.9]) The functor $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$ preserves inductive limits.

The above theorem is a useful tool for computing Cuntz semigroups of C^*-algebras arising as inductive limits of simpler C^*-algebras. For example, using Theorem 1.26, one can compute the Cuntz semigroup of the CAR-algebra M_{2^∞} as

$$\mathrm{Cu}(M_{2^\infty}) \cong \mathbb{N}[\tfrac{1}{2}] \sqcup (0,\infty],$$

where $\mathbb{N}[\frac{1}{2}]$ denotes the set of all positive dyadic numbers, and where addition and order are defined as follows:

$$a+b := \begin{cases} a+b\in\mathbb{N}[\frac{1}{2}] & \text{if } a,b\in\mathbb{N}[\frac{1}{2}] \\ a+b\in(0,\infty] & \text{otherwise} \end{cases} \quad \text{and } a\leq b :\Leftrightarrow \begin{cases} a<b & \text{if } a\in\mathbb{N}[\frac{1}{2}], b\in(0,\infty] \\ a\leq b & \text{otherwise} \end{cases},$$

for $a,b\in\mathbb{N}[\frac{1}{2}]\sqcup(0,\infty]$. A detailed computation can be found in [Thi16, 4.33].

1.5 Composition and decomposition

In many cases, a Cu-semigroup consists of two components: a 'compact' part and a 'soft' part. For example, the Cuntz semigroup of M_{2^∞} is the disjoint union of the compact part $\mathbb{N}[\frac{1}{2}]$ and the soft part $(0,\infty]$. More general, a large class of simple, stably finite Cu-semigroups decomposes into a compact part and a soft part. On the other hand, given a triple (C, D, γ), consisting of certain ordered semigroups C and D and a so called composition map $\gamma\colon C\to D$, one can construct a Cu-semigroup $C\sqcup_\gamma D$ in which the compact and soft parts are given by C and D respectively.

The composition of semigroups and decomposition of Cu-semigroups was studied in [Eng14]. In this section, we will recall some of Engbers' results. All Cu-semigroups in [Eng14] are assumed to satisfy $(\mathcal{O}5')$ and $(\mathcal{O}6)$, so we will add these axioms to our assumptions.

Let S be a Cu-semigroup. An element $s \in S$ is called compact if $s \ll_\omega s$. We denote the set of all compact elements in S by $C(S)$. Clearly, $C(S)$ is a submonoid of S. Let us use $D(S)$ to denote the set of all elements that are either zero or noncompact. For any monoid M, we set $M^\times := M \setminus \{0\}$.

1.27 Definition ([Eng14, 6.1.1]) We say that a Cu-semigroup S is decomposable if there exists a monoid homomorphism $\gamma_S \colon C(S) \to S$ such that

$$\gamma_S(x) = \max\{y \in S \mid y \leq x \text{ and } y \neq x\} \neq 0$$

holds for each $x \in C(S)^\times$. If such a map exists, it is necessarily unique. The element $\gamma_S(x)$ is called the predecessor of x, and the map γ_S is called the predecessor map of S.

Let $\mathcal{C}$ be the full subcategory of PoM consisting of all algebraically ordered PoMs (a PoM M is algebraically ordered if for all $x, z \in M$, we have $x \leq z$ if and only if there exists $y \in M$ such that $x + y = z$). Let $\mathcal{D}$ denote the full subcategory of Cu which consists of all Cu-semigroups satisfying ($\mathcal{O}5'$) and ($\mathcal{O}6$) and that have no compact elements beside the zero element.
Suppose that C is an object in $\mathcal{C}$. An ideal in C is a downward hereditary submonoid. If the only ideals in C are $\{0\}$ and C, we say that C is simple.

1.28 Definition ([Eng14, 8.1.2]) Suppose that C and D are simple semigroups in $\mathcal{C}$ and $\mathcal{D}$. We say that a map $\gamma \colon C \to D$ is a composition map if the following conditions are satisfied:

- For all $x \in C$, we have $\gamma(x) = 0$ if and only if $x = 0$.
- γ is additive.
- For all $x, y \in C^\times$, the inequality $\gamma(x) \ll_\omega \gamma(y)$ implies that $x \leq y$ and $x \neq y$.
- For all $x \in C$ and $y \in D^\times$, we have $\gamma(x) \ll_\omega y$ if and only if there exists $z \in D^\times$ such that $\gamma(x) + z = y$.
- For all $x \in C^\times$ and $y, z \in D^\times$ satisfying $\gamma(x) \ll_\omega x + y$, there exist $y', z' \in D^\times$ such that $y' \ll_\omega y, \gamma(x)$, and $z' \ll_\omega z, \gamma(x)$, and $\gamma(x) \leq y' + z'$.

In [Eng14, 8.1.3 and 8.2.1], Engbers demonstrates that there is a one-to-one correspondence between the following classes:

1) Simple, decomposable Cu-semigroups satisfying ($\mathcal{O}5'$) and ($\mathcal{O}6$).
2) Triples (C, D, γ), consisting of a simple semigroup C in $\mathcal{C}$, a simple semigroup D in $\mathcal{D}$ and a composition map $\gamma \colon C \to D$.

To each Cu-semigroup S as in 1), one assigns the triple $(C(S), D(S), \gamma_S)$. Conversely, a triple (C, D, γ) as in 2) gives rise to a Cu-semigroup $S := C \sqcup_\gamma D$, which is defined as follows: as a set, S is defined as $S \coloneqq C^\times \sqcup D^\times \sqcup \{0_S\}$. Addition and order are defined by

$$x +_S y \coloneqq \begin{cases} y & \text{if } x = 0_S \\ x & \text{if } y = 0_S \\ x +_C y & \text{if } x, y \in C^\times \\ \gamma(x) +_D y & \text{if } x \in C^\times, y \in D^\times \\ x +_D \gamma(y) & \text{if } x \in D^\times, y \in C^\times \\ x +_D y & \text{if } x, y \in D^\times \end{cases}, \text{ and}$$

$$x \leq_S y :\Leftrightarrow \begin{cases} y \in S & \text{if } x = 0_S \\ x = 0_S & \text{if } y = 0_S \\ x \leq_C y & \text{if } x, y \in C^\times \\ \gamma(x) \ll_{\omega,D} y & \text{if } x \in C^\times, y \in D^\times \\ x \leq_D \gamma(y) & \text{if } x \in D^\times, y \in C^\times \\ x \leq_D y & \text{if } x, y \in D^\times \end{cases},$$

for $x, y \in S$. Engbers then shows that this one-to-one correspondence can, when suitably interpreted, be understood as an equivalence of categories. As shown in [Eng14, 6.2.4 and 6.3.2], the class of all simple and decomposable Cu-semigroups satisfying ($\mathcal{O}5'$) and ($\mathcal{O}6$) includes the following subclasses:

- Cuntz semigroups of simple, separable, nonelementary, stably finite C^*-algebras, and
- simple, countably based, nonelementary, weakly cancellative Cu-semigroups satisfying ($\mathcal{O}5'$) and ($\mathcal{O}6$).

A Cu-semigroup S is called weakly cancellative if for all $x, y, z \in S$, the condition $x+z \ll_\omega y+z$ implies that $x \ll_\omega y$. The definition of an elementary Cu-semigroup can be found in [Eng14, 3.1.6].

An element s in some Cu-semigroup S is called soft if for all $s' \ll_\omega s$, there exists some $k \in \mathbb{N}$ such that $(k+1)s' \leq ks$. A nice feature of soft elements is that their order is determined by functionals, assuming that S almost unperforated (meaning that for all $x, y \in S$, the condition $(k+1)x \leq ky$ for some $k \in \mathbb{N}$ implies that $x \leq y$).

1.29 Theorem ([APT18, 5.3.12]) Let S be an almost unperforated Cu-semigroup. Suppose that x and y lie in S, that x is soft and that $\lambda(x) \leq \lambda(y)$ holds for every functional $\lambda \in F(S)$. Then $x \leq y$.

1.30 Definition Let S be a Cu-semigroup. An element $s \in S$ is called infinite if there exists a nonzero element $x \in S$ such that $s = s + x$. We say that s is finite if it is not infinite, and we call S stably finite if for all $s \in S$, the condition $s \ll_\omega s'$ for some $s' \in S$ implies that s is finite.

1.31 Proposition ([APT18, 5.3.16]) If S is a simple and stably finite Cu-semigroup that satisfies ($\mathcal{O}5'$), then every element in S is either compact or soft. The zero element is the only element that is both compact and soft.

Remark Let S be a simple Cu-semigroup satisfying ($\mathcal{O}5'$). Under the mild assumption that S is stably finite, it follows from Proposition 1.31 that $D(S)$ equals the set of all soft elements. It is shown in [APT18, 5.2.10] that a nonzero, simple Cu-semigroup S is stably finite if and only if ∞ is not compact. It follows that S is stably finite if and only if ∞ lies in $D(S)$.
A unital, simple C^*-algebra A is stably finite if and only if $\mathrm{Cu}(A)$ is stably finite (see [Thi16, 6.22]).

The upcoming proposition is taken from [Thi16, 6.40]. Let me explain some notation. We use $\mathcal{Z}$ to denote the Jiang-Su algebra. If A is a (unital) C^*-algebra, we use $V(A)$ to denote the Murray-von Neumann semigroup of A. Recall that $\mathrm{QT}_1(A)$ denotes the set of all normalized quasitraces on A. It turns out that $\mathrm{QT}_1(A)$ is a so called Choquet simplex, so we can consider the set $\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++}$ of all lower semicontinuous affine functions on $\mathrm{QT}_1(A)$ that are either strictly positive or zero. Then $\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++}$ is a particularly nice Cu-semigroup, and the following holds:

1.32 Proposition For a unital, simple, separable, stably finite, $\mathcal{Z}$-stable C^*-algebra A, we have that

$$\mathrm{Cu}(A) \cong V(A) \sqcup_\gamma \mathrm{LAff}(\mathrm{QT}_1(A))^0_{++},$$

where the composition map is given by $\gamma\colon V(A) \to \mathrm{LAff}(\mathrm{QT}_1(A))^0_{++}$, $[a] \mapsto (\tau \mapsto \tau(a))$.

As you can see, the Cu-semigroup $\mathrm{LAff}(K)^0_{++}$ for some metrizable compact convex set K appears naturally in the study of Cuntz semigroups of sufficiently regular C^*-algebras.

With the help of Proposition 1.32, one can compute the Cuntz semigroup of the Jiang-Su algebra as $\mathrm{Cu}(\mathcal{Z}) \cong \mathbb{N} \sqcup_\gamma [0,\infty]$, with the obvious composition map $\gamma\colon \mathbb{N} \to [0,\infty]$, $x \mapsto x$. Other authors prefer to write $\mathrm{Cu}(\mathcal{Z}) \cong \mathbb{N} \sqcup (0,\infty]$ for the same Cu-semigroup. On a similar note, we have seen that $\mathrm{Cu}(M_{2^\infty}) \cong \mathbb{N}[\frac{1}{2}] \sqcup (0,\infty]$. In Engbers' notation, this should be written as $\mathbb{N}[\frac{1}{2}] \sqcup_\gamma [0,\infty]$, with the obvious composition map γ.

From now on, we will use $\ll$ instead of $\ll_\omega$ to denote the way-below relation in a general Cu-semigroup S.

2 Compact convex sets

Compact convex sets play a central role in this thesis; so in this chapter, we cover their basic theory. By a (compact) convex set, we always mean a (compact) convex subset of a locally convex, Hausdorff, real topological vector space. Up until section 2.3, we closely follow the approach taken in [Goo86].

Let X be a set, and let $f, g\colon X \to [-\infty, \infty]$ be functions. We write $f \leq g$ if $f(x) \leq g(x)$ holds for all $x \in X$. Similarly, the inequality $f < g$ shall mean that $f(x) < g(x)$ holds for every $x \in X$. For any $c \in [-\infty, \infty]$, we also use c to denote the constant function that takes the value c.

Now let X be a topological space. Recall that a function $f\colon X \to [-\infty, \infty]$ is called lower semicontinuous if $f^{-1}((t, \infty])$ is open for every $t \in \mathbb{R}$. Equivalently, for all $x \in X$ and for every net $(x_i)_i$ in X that converges to x, the inequality $f(x) \leq \liminf_i f(x_i)$ holds. We say that f is upper semicontinuous if $-f$ is lower semicontinuous.

Finally, let K be a convex set, and suppose that $f\colon K \to [-\infty, \infty]$ is a function such that the image of f is contained in $[-\infty, \infty)$ or in $(-\infty, \infty]$. We say that f is convex if the inequality $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ holds for all $x, y \in K$ and $t \in [0, 1]$ (where $0 \cdot \infty$ and $0 \cdot (-\infty)$ are defined to be 0). Also, f is called concave if $-f$ is convex. We say that f is affine if it is both convex and concave, i.e. the equality

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$. Throughout this thesis, we will consider the following sets:

$$\begin{aligned}
&\mathrm{Aff}(K) \coloneqq \{f\colon K \to \mathbb{R} \text{ continuous and affine}\},\\
&\mathrm{Aff}(K)_+ \coloneqq \{f \in \mathrm{Aff}(K) \mid f \geq 0\},\\
&\mathrm{Aff}(K)^0_{++} \coloneqq \{f \in \mathrm{Aff}(K) \mid f > 0 \text{ or } f = 0\},\\
&\mathrm{LAff}(K) \coloneqq \{f\colon K \to (-\infty, \infty] \text{ lower semicontinuous and affine}\},\\
&\mathrm{LAff}(K)_+ \coloneqq \{f \in \mathrm{LAff}(K) \mid f \geq 0\},\\
&\mathrm{LAff}(K)^0_{++} \coloneqq \{f \in \mathrm{LAff}(K) \mid f > 0 \text{ or } f = 0\}.
\end{aligned}$$

We equip $\mathrm{LAff}(K)$ with pointwise order and addition. Observe that $\mathrm{LAff}(K)$ is a dcpo, where the supremum of an increasing net is just given by the pointwise supremum. Taking suprema of increasing nets is compatible with addition.

In the first half of this chapter, we will build a sufficiently large supply of intersesting examples of compact convex sets.
Let X be a compact Hausdorff space. A probability measure on X is a Radon measure μ on X satisfying $\mu(X) = 1$ (we refer to [Arv96] for details). We denote the set of all probability measures on X by $M_1^+(X)$. According to the Riesz-Markov theorem, the assignment

$$\mu \mapsto (f \mapsto \int_X f d\mu)$$

defines a bijection

$$M_1^+(X) \cong \{\varphi\colon \mathcal{C}(X;\mathbb{R}) \to \mathbb{R} \text{ positive, linear, } \varphi(1) = 1\} \subseteq \mathcal{C}(X;\mathbb{R})^*,$$

where $\mathcal{C}(X;\mathbb{R})^*$ denotes the topological dual space of the continuous, real valued functions on X. From now on, we will identify $M_1^+(X)$ with the set of all normalized positive, linear functionals on $\mathcal{C}(X;\mathbb{R})$. For $x \in X$, let δ_x denote the dirac measure at x. It is defined by $\delta_x(B) = 1$ if $x \in B$, and $\delta_x(B) = 0$ if $x \notin B$, for any Borel set B. Note that the dirac measure at x is the unique probability measure on X satisfying $\delta_x(\{x\}) = 1$.

2.1 Proposition Let X be a compact Hausdorff space. When equipped with the weak-* topology, $\mathcal{C}(X;\mathbb{R})^*$ is a locally convex, Hausdorff, real topological vector space and contains $M_1^+(X)$ as a compact convex subset.

Proof. (cf. [Goo86, 5.22]) It is clear that $M_1^+(X)$ is convex and closed in the weak-* topology. Let $\varphi \in M_1^+(X)$. Any f in the unit ball of $\mathcal{C}(X;\mathbb{R})$ satisfies $-1 \le f \le 1$, which implies that

$$-1 = \varphi(-1) \le \varphi(f) \le \varphi(1) = 1,$$

hence the norm of φ is bounded by 1. It follows that $M_1^+(X)$ is contained in the unit ball of $\mathcal{C}(X;\mathbb{R})^*$ which is weak-*-compact by the Banach-Alaoglu theorem. It follows that $M_1^+(X)$ is in fact a compact convex set. □

2.2 Proposition For any compact Hausdorff space X, the assignment $x \mapsto \delta_x$ defines a homeomorphism $X \cong \partial_e M_1^+(X)$.

Proof. (cf. [Goo86, 5.24]) Given $x \in X$, let us show that δ_x is extreme. Consider a convex combination $\delta_x = t\mu_1 + (1-t)\mu_2$, where $\mu_1, \mu_2 \in M_1^+(X)$ and $t \in (0,1)$. We have that

$$1 = \delta_x(\{x\}) = t\mu_1(\{x\}) + (1-t)\mu_2(\{x\}),$$

which is only possible if $\mu_1(\{x\}) = \mu_2(\{x\}) = 1$ or equivalently $\mu_1 = \mu_2 = \delta_x$. Thus, δ_x is extreme and the map

$$\delta\colon X \to \partial_e M_1^+(X),\ x \mapsto \delta_x$$

is well defined.

Let $(x_i)_{i\in I} \subseteq X$ be a net that converges to some point $x \in X$. For every $f \in \mathcal{C}(X;\mathbb{R})$, we have

$$\delta_{x_i}(f) = f(x_i) \to f(x) = \delta_x(f),$$

so the net $(\delta_{x_i})_{i\in I}$ converges to δ_x. This means that δ is continuous. Clearly δ is injective. Since $\partial_e M_1^+(X)$ is Hausdorff and X is compact, the only thing left to show is the surjectivity.

Let $\mu \in \partial_e M_1^+(X)$. Assume that there exists a Borel set C such that $\mu(C) \in (0,1)$. Then the formulae

$$\mu_1(B) \coloneqq \mu(C)^{-1}\mu(B \cap C) \text{ and } \mu_2(B) \coloneqq \mu(X \setminus C)^{-1}\mu(B \cap (X \setminus C)),$$

for any Borel set B, define elements μ_1, μ_2 in $M_1^+(X)$. Moreover, $\mu = \mu(C)\mu_1 + \mu(X \setminus C)\mu_2$. It follows that $\mu_1 = \mu_2 = \mu$, since μ is assumed to be an extreme point. But then we get $1 = \mu_1(C) = \mu_2(C) = 0$, which is a contradiction. Thus, μ only attains the values 0 and 1.

Let Γ denote the set of all compact subsets of X that have measure 1. Given $B_1, B_2 \in \Gamma$, their union also lies in Γ, hence

$$1 = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) - \mu(B_1 \cap B_2) = 2 - \mu(B_1 \cap B_2),$$

or equivalently $\mu(B_1 \cap B_2) = 1$. This shows that Γ is closed under finite intersections. As a consequence, Γ has the finite intersection property. Since X is compact, the set $C \coloneqq \bigcap \Gamma$ is nonempty. Notice that any compact set $B \subseteq X \setminus C$ cannot lie in Γ, hence $\mu(B) = 0$. The inner regularity of μ implies $\mu(X \setminus C) = 0$ or equivalently $\mu(C) = 1$.

Assume that C contains two distinct points x and y. By Urysohn's lemma, there exists a continuous function $f\colon X \to [0,1]$ satisfying $f(x) = 0$ and $f(y) = 1$. Then C can be written as the union of the proper compact subsets $f^{-1}([0,\frac{1}{2}])$ and $f^{-1}([\frac{1}{2},1])$. Neither of them is contained in Γ, so both have measure 0. But then C must have measure 0 as well which is a contradiction. Thus, C contains a unique element x. We have that $\mu(\{x\}) = \mu(C) = 1$, which shows that $\mu = \delta_x$, finishing the proof. □

2.1 Partially ordered abelian groups with order unit

Another important class of examples for compact convex sets consists of state spaces of partially ordered abelian groups with order unit. On the other hand, one can assign to each compact convex set a certain partially ordered abelian group with order unit. This rough duality between compact convex sets and partially ordered abelian groups with order unit is the key ingredient that allows us to define a tensor product for compact convex sets later on.

2.3 Definition A partially ordered abelian group is an abelian group G, equipped with a compatible partial order. A OAG-morphism between partially ordered abelian groups G and H is an order-preserving group homomorphism $G \to H$. Note that a group homomorphism $G \to H$ is order-preserving if and only if it is positive, i.e. maps positive elements to positive elements. Let OAG be the category whose objects are ordered abelian groups and whose morphisms are OAG-morphisms. We use G_+ to denote the set of all positive elements in G. An element $u \in G_+$ is called order unit if every element in G is dominated by some multiple of u, that is for every $g \in G$, there exists $n \in \mathbb{N}$ such that $g \leq nu$.

A partially ordered abelian group with order unit is a pair (G, u), consisting of a partially ordered abelian group G and an order unit u in G. A GOU-morphism between partially ordered abelian groups with order unit (G, u) and (H, v) is a OAG-morphism $G \to H$ that maps u to v. The category GOU shall consist of all partially ordered abelian groups with order unit and all GOU-morphisms. We denote the set of all GOU-morphisms $(G, u) \to (\mathbb{R}, 1)$ by $S(G, u)$. The elements of $S(G, u)$ are called states.

We may shorten 'partially ordered abelian group' to 'ordered abelian group'. Also, we may abbreviate 'partially ordered abelian group with order unit' to 'group with order unit'.

Remark Observe that an ordered abelian group G is directed if and only if $G = G_+ - G_+$. Every group with order unit is directed.

It is not trivial that a general group with order unit (G, u) has a state. However, as long as (G, u) is nonzero, this turns out to be true. This is a consequence of the following Proposition, the proof of which requires Zorn's Lemma.

2.4 Proposition Let H be a subgroup of a partially ordered abelian group G such that each element in G is dominated by an element in H. Then every positive homomorphism $H \to \mathbb{R}$ extends to a positive homomorphism $G \to \mathbb{R}$.

Proof. [Goo86, 4.2]. □

2.5 Corollary Let (G, u) be a partially ordered abelian group with order unit and suppose that H is a subgroup containing u. Then every state on (H, u) extends to a state on (G, u).

2.6 Corollary Let (G, u) be a partially ordered abelian group with order unit. There exists a state on (G, u) if and only if G is nonzero.

Proof. [Goo86, 4.4]. □

If (G, u) is a group with order unit, we can recover all OAG-morphisms $G \to \mathbb{R}$ from the states on (G, u), as follows.

2.7 Lemma For any partially ordered abelian group with order unit (G, u), we have that

$$\mathrm{Hom}_{\mathrm{OAG}}(G, \mathbb{R}) = ((0, \infty) \cdot S(G, u)) \cup \{0\}.$$

Proof. [Goo86, 4.5]. □

2.8 Proposition Let (G, u) be a partially ordered abelian group with order unit. Then $S(G, u)$ is a compact convex subset of the locally convex, Hausdorff, real topological vector space $\mathbb{R}^G$ of all real-valued functions on G.

Given two partially ordered abelian groups with order unit (G, u) and (H, v) and a morphism $\varphi\colon (G, u) \to (H, v)$, the map

$$\varphi^*\colon S(H, v) \to S(G, u),\ \varphi^*(f) \coloneqq f \circ \varphi$$

is continuous and affine. Thus, the assignment $(G, u) \mapsto S(G, u)$ defines a contravariant functor from GOU to the category of compact convex sets (with continuous affine maps as morphisms).

Proof. (cf. [Goo86, 6.2]) It is evident that $S(G, u)$ is a closed, convex subset of $\mathbb{R}^G$ (closedness can easily be checked via nets). For each $x \in G$, choose a natural number n_x such that $-n_x u \leq x \leq n_x u$. It follows that $-n_x \leq s(x) \leq n_x$ holds for each state $s \in S(G, u)$, whence

$$S(G, u) \subseteq \prod_{x \in G} [-n_x, n_x].$$

The product of intervals is compact by Tychonoff's theorem. It follows that $S(G, u)$ is a compact convex set.

The second part of the statement is just a routine verification and will be omitted. □

2.9 Lemma Let X be a compact Hausdorff space. Then $(\mathcal{C}(X; \mathbb{R}), 1)$ is a partially ordered abelian group with order unit. Moreover, $S(\mathcal{C}(X; \mathbb{R}), 1)$ and $M_1^+(X)$ are naturally isomorphic as compact convex sets.

Proof. (cf. [Goo86, 6.8]) It is clear that $\mathcal{C}(X; \mathbb{R})$ is a partially ordered abelian group. Since X is compact, every function in $\mathcal{C}(X; \mathbb{R})$ is bounded, implying that 1 is a order unit.

The inclusion $\mathcal{C}(X; \mathbb{R})^* \hookrightarrow \mathbb{R}^{\mathcal{C}(X; \mathbb{R})}$ is an embedding of topological vector spaces, since both carry the topology of pointwise convergence. Thus, we may regard both $S(\mathcal{C}(X; \mathbb{R}), 1)$ and $M_1^+(X)$ as compact convex subsets of $\mathbb{R}^{\mathcal{C}(X; \mathbb{R})}$. The inclusion $M_1^+(X) \subseteq S(\mathcal{C}(X; \mathbb{R}), 1)$ is trivial. For the converse inclusion, let $\varphi \in S(\mathcal{C}(X; \mathbb{R}), 1)$. We have to show that φ is homogeneous. For $t \in \mathbb{R}$ and $f \in \mathcal{C}(X; \mathbb{R})$, let us prove that the formula $\varphi(tf) = t\varphi(f)$ holds. We may assume without loss of generality that f is positive. It is clear that the formula holds if t is rational. In the general case, let $\alpha, \beta \in \mathbb{Q}$ such that $\alpha \leq t \leq \beta$. Then $\alpha f \leq tf \leq \beta f$, which implies that

$$\alpha\varphi(f) = \varphi(\alpha f) \leq \varphi(tf) \leq \varphi(\beta f) = \beta\varphi(f).$$

Since α and β may be arbitrarily close to t, we obtain $\varphi(tf) = t\varphi(f)$, as desired. □

2.10 Proposition Let K be a compact convex set. Then $(\mathrm{Aff}(K), 1)$ is a partially ordered abelian group with order unit.

Given two compact convex sets K_1 and K_2 and a continuous affine map $f\colon K_1 \to K_2$, the map

$$f^*\colon (\mathrm{Aff}(K_2), 1) \to (\mathrm{Aff}(K_1), 1),\ \ f^*(g) := g \circ f$$

is a GOU-morphism. Thus, the assignment $K \mapsto (\mathrm{Aff}(K), 1)$ defines a contravariant functor from the category of compact convex sets to GOU.

Proof. Using the compactness of K, we deduce that every continuous map $K \to \mathbb{R}$ is bounded from above. It follows that 1 is an order unit for $\mathrm{Aff}(K)$. The rest of the proof consists of trivial verifications and will be omitted. □

We have seen so far that there are contravariant functors

$$\begin{array}{ccc} \text{Compact convex sets} & \rightleftarrows & \text{GOU} \\ K & \longmapsto & (\mathrm{Aff}(K), 1) \\ S(G,u) & \longleftarrow & (G,u) \end{array}$$

For any partially ordered abelian group with order unit (G, u), the map

$$\iota\colon (G,u) \to (\mathrm{Aff}(S(G,u)), 1),\ \ g \mapsto \mathrm{ev}_g$$

is a GOU-morphism. In fact, ι is a natural transformation from the identity functor to the functor $(\mathrm{Aff}(S(\_)), 1)$. Thus, we will refer to ι as the natural map. In general, ι will not be an isomorphism. This is not surprising as (G, u) may lack characteristics that $(\mathrm{Aff}(S(G, u)), 1)$ always has, such as being divisible and unperforated. For example $(\mathrm{Aff}(S(\mathbb{Z}, 1)), 1) \cong (\mathbb{R}, 1)$, which is not isomorphic to $(\mathbb{Z}, 1)$. It is known that ι is an embedding if and only if G is archimedean (see [Goo86, 7.7]).

Similarly, if K is a compact convex set, the map

$$\kappa\colon K \to S(\mathrm{Aff}(K), 1),\ \ x \mapsto \mathrm{ev}_x$$

is continuous and affine. Again, κ is a natural transformation from the identity functor to the functor $S(\mathrm{Aff}(\_))$, so we refer to κ as the natural map. Contrary to ι, this map is always an isomorphism, as shown in the following proposition.

2.11 Proposition For any compact convex set K, the natural map $\kappa\colon K \to S(\mathrm{Aff}(K), 1)$ is an isomorphism.

Proof. (cf. [Goo86, 7.1]) Suppose that K is located in a locally convex, Hausdorff, real topological vector space E. For the injectivity, let $x, y \in K$ such that $x \neq y$. By the Hahn-Banach theorem, there exists a continuous linear map $f\colon E \to \mathbb{R}$ such that $f(x) \neq f(y)$. The restriction of f to K is continuous and affine, and we have that

$$\kappa(x)(f|_K) = f(x) \neq f(y) = \kappa(y)(f|_K),$$

hence $\kappa(x) \neq \kappa(y)$.
To show that κ is surjective, let $s \in S(\mathrm{Aff}(K), 1)$. For now, let us assume that s is an extreme point. Clearly, the restriction map

$$\rho\colon M_1^+(K) = S(\mathcal{C}(K; \mathbb{R}), 1) \to S(\mathrm{Aff}(K), 1)$$

is continuous and affine. Moreover, it is surjective according to Corollary 2.5. Notice that $\rho^{-1}(\{s\})$ is a nonempty compact face of $M_1^+(K)$. By the Krein-Milman theorem, there exists at least one extreme point $\delta \in \partial_e \rho^{-1}(\{s\})$. It follows that $\delta \in \partial_e M_1^+(K)$, hence by Proposition 2.2, there exists an element $x \in K$ such that $\delta = \delta_x$. For all $f \in \mathrm{Aff}(K)$, we get

$$s(f) = \rho(\delta)(f) = \delta_x(f) = f(x) = \kappa(x)(f),$$

which implies that $s = \kappa(x)$. So far, we have shown that $\partial_e S(\mathrm{Aff}(K), 1)$ is contained in the image of κ. Note that the latter is a compact convex set, so applying the Krein-Milman theorem once again yields

$$S(\mathrm{Aff}(K), 1) = \overline{\mathrm{conv}(\partial_e S(\mathrm{Aff}(K), 1))} \subseteq \kappa(K),$$

which just means that κ is in fact surjective. As a continuous bijection from a compact space to a Hausdorff space, κ is a homeomorphism. □

It follows from Proposition 2.11 that every compact convex set is isomorphic to a state space of some partially ordered abelian group with order unit.

2.2 Approximations of (semi-)continuous functions

One of our highest priorities right now is to show that $\mathrm{LAff}(K)^0_{++}$ is a Cu-semigroup. The difficulty lies in showing that every element in $\mathrm{LAff}(K)^0_{++}$ may be approximated by elements that are way-below. The key observation is that every lower semicontinuous, affine function may be approximated by an increasing net of continuous, affine functions (see Proposition

2.15). Throughout this section, we will prove various other approximation results which are needed in Chapter 3.

Let X be a compact Hausdorff space. We use $\mathrm{Lsc}(X)$ to denote the set of all lower semicontinuous functions $X \to (-\infty, \infty]$. The sets $\mathrm{Lsc}(X)_+$ and $\mathrm{Lsc}(X)^0_{++}$ are defined analogously to the case of (lower semi-)continuous, affine functions.

2.12 Lemma 1) Let X be a nonempty, compact Hausdorff space. Then any $f \in \mathrm{Lsc}(X)$ attains its minimum.

2) Let K be a nonempty, compact convex set. Then any $f \in \mathrm{LAff}(K)$ attains its minimum on the extreme boundary of K.

Proof. 1) For any $n \in \mathbb{N}$, the set $f^{-1}((-n, \infty])$ is open in X by the lower semicontinuity of f. Since $X = \bigcup_{n\in\mathbb{N}} f^{-1}((-n, \infty])$ is compact, there exists a number $n \in \mathbb{N}$ such that $X = f^{-1}((-n, \infty])$. Therefore f is bounded below, so $\alpha := \inf_{x\in X} f(x)$ is a real number. Choose a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ such that $\alpha = \lim_\lambda f(x_\lambda)$. Since X is compact, this net has a convergent subnet. In order to keep notation simple, we may assume without loss of generality that the net $(x_\lambda)_{\lambda\in\Lambda}$ is already converging to some $z \in X$. Using the lower semicontinuity of f at the third step, we deduce

$$\alpha \leq f(z) = f(\lim_\lambda x_\lambda) \leq \lim_\lambda f(x_\lambda) = \alpha,$$

hence f attains its minimum at the point z.

2) Let α denote the infimum of f and consider $F := \{x \in K \mid f(x) = \alpha\} = f^{-1}((-\infty, \alpha])$. Since f is lower semicontinuous and affine, F is a compact convex set which is nonempty by the previous part. It is easily seen that F is a face of K. By the Krein-Milman theorem, F has at least one extreme point z. Then z is also an extreme point of K satisfying $f(z) = \alpha$. $\square$

That every lower semicontinuous, affine function can be approximated by continuous, affine functions heavily relies on the following application of the Hahn-Banach theorem.

2.13 Lemma Let E be a locally convex, Hausdorff, real topological vector space, and let $A, B \subseteq E \oplus \mathbb{R}$ be closed, disjoint convex subsets such that A is compact. Assume that there exist $z \in E$ and $\mu_1, \mu_2 \in \mathbb{R}$ such that $(z, \mu_1) \in A$, $(z, \mu_2) \in B$ and $\mu_1 < \mu_2$. Then there exists a continuous, affine function $e\colon E \to \mathbb{R}$ such that

$$A \subseteq \{(y, t) \in E \oplus \mathbb{R} \mid t < e(y)\} \text{ and } B \subseteq \{(y, t) \in E \oplus \mathbb{R} \mid t > e(y)\}.$$

Proof. By the Hahn-Banach separation theorem, there exists a continuous linear functional $\varphi \in (E \oplus \mathbb{R})^*$ such that

$$\sup_{(y,t)\in A} \varphi(y, t) < \inf_{(y,t)\in B} \varphi(y, t).$$

Let $\psi := \varphi(\_, 0)$, $\gamma := \varphi(0,1)$. For every point $(y,t) \in E \oplus \mathbb{R}$, we have $\varphi(y,t) = \psi(y) + \gamma \cdot t$ by the linearity of φ. The formula above may be rewritten as

$$\sup_{(y,t)\in A} \psi(y) + \gamma \cdot t < \inf_{(y,t)\in B} \psi(y) + \gamma \cdot t.$$

The assumptions $(z,\mu_1) \in A$ and $(z,\mu_2) \in B$ entail $\psi(z) + \gamma \cdot \mu_1 < \psi(z) + \gamma \cdot \mu_2$, and the assumption $\mu_1 < \mu_2$ then implies that $\gamma > 0$. Choose any real number η such that

$$\sup_{(y,t)\in A} \psi(y) + \gamma \cdot t < \eta < \inf_{(y,t)\in B} \psi(y) + \gamma \cdot t.$$

Now define $e\colon E \to K$ via the formula $e(x) := \gamma^{-1} \cdot (\eta - \psi(x))$. As ψ is continuous and linear, e is continuous and affine. Given $(x,\lambda) \in A$, the inequality $\psi(x) + \gamma \cdot \lambda < \eta$ holds. This is equivalent to the inequality $\lambda < e(x)$, proving that $A \subseteq \{(y,t) \in E \oplus \mathbb{R} \mid t < e(y)\}$. The other claimed inclusion can be shown similarly. □

Let K be a compact convex set in some surrounding space E. The epigraph of a function $f\colon K \to [-\infty, \infty]$ is defined as the set $M := \{(x,\lambda) \in K \times \mathbb{R} \mid f(x) \leq \lambda\}$. Notice that M is a convex subset of $E \oplus \mathbb{R}$ if and only if f is convex. Also, M is a closed subset of $E \oplus \mathbb{R}$ if and only if f is lower semicontinuous.

2.14 Lemma ([Alf71, I.1.2]) Let K be a compact convex set, and let $f\colon K \to (-\infty, \infty]$ be a lower semicontinuous, convex function. For every $x \in K$, we have that

$$f(x) = \sup\{e(x) \mid e \in \mathrm{Aff}(K),\ e < f\}.$$

Proof. Let $x \in K$ and $\alpha < f(x)$ be arbitrary. We have to show that there exists a continuous, affine function $e \in \mathrm{Aff}(K)$ such that $e < f$ and $\alpha < e(x)$.

Let us denote the epigraph of f by M. We would like to apply Lemma 2.13 to separate the point (x,α) from M by means of a continuous affine function. However, M does not contain any point of the form (x,λ) if $f(x) = \infty$, which prevents us from doing so. In order to circumvent this inconvenience, we will instead replace M by a bigger set M'. Choose any real number β satisfying $\alpha < \beta < f(x)$ and set $M' := \overline{\mathrm{conv}(\{(x,\beta)\} \cup M)}$. Obviously M' is a nonempty closed convex subset of $E \oplus \mathbb{R}$. To show that (x,α) is not contained in M', we define the following sets:

$$A := \{(z,\lambda) \in K \times \mathbb{R} \mid \beta \leq \lambda\},\ B := \{(z,\lambda) \in K \times \mathbb{R} \mid f(z) \leq \lambda \leq \beta\}.$$

Observe that $\mathrm{conv}(\{(x,\beta)\} \cup M) \subseteq A \cup \mathrm{conv}(\{(x,\beta)\} \cup B)$. Notice that B is a compact convex set because f is lower semicontinuous and convex. It follows that $\mathrm{conv}(\{(x,\beta)\} \cup B)$ is closed and consequently that

$$M' \subseteq A \cup \mathrm{conv}(\{(x,\beta)\} \cup B).$$

Clearly, (x, α) is not contained in the right hand side, so it is not contained in M' either.

We can now apply Lemma 2.13 to obtain a continuous, affine function $e \in \mathrm{Aff}(K)$ satisfying $\alpha < e(x)$ and $M' \subseteq \{(z, \lambda) \in K \times \mathbb{R} \mid \lambda > e(z)\}$. To show that $e < f$, let $y \in K$ be arbitrary. If $f(y) = \infty$, then $e(y) < f(y)$ is trivially satisfied. In the case that $f(y) < \infty$, the element $(y, f(y))$ lies in M', implying that $e(y) < f(y)$. This finishes the proof. □

2.15 Proposition ([Alf71, I.1.4]) Let K be a compact convex set. For any function f in $\mathrm{LAff}(K)$, the set $\Lambda \coloneqq \{e \in \mathrm{Aff}(K) \mid e < f\}$ is upward directed and has supremum f.

Proof. The statement is clear if $f = \infty$, so let us assume that $f \neq \infty$. We know from Lemma 2.12 that f is bounded below, so Λ is nonempty. The epigraph M of f is a nonempty closed convex subset of $E \oplus \mathbb{R}$. Let e_1, $e_2 \in \Lambda$ and choose a real number η such that $\eta \leq e_1, e_2$. For $i = 1, 2$ consider the sets $M_i \coloneqq \{(z, \lambda) \in K \times \mathbb{R} \mid \eta \leq \lambda \leq e_i(x)\}$. It follows from the fact that $e_1, e_2 < f$ that M and $M_1 \cup M_2$ are disjoint. In fact, it it true that $M \cap \mathrm{conv}(M_1 \cup M_2) = \emptyset$, since f is affine. Clearly, each M_i is a nonempty compact convex set, so $\mathrm{conv}(M_1 \cup M_2)$ is also a nonempty compact convex set. We can now apply Lemma 2.13 to obtain a continuous affine function $e \in \mathrm{Aff}(K)$ such that

$$\mathrm{conv}(M_1 \cup M_2) \subseteq \{(z, \lambda) \in K \times \mathbb{R} \mid \lambda < e(z)\} \text{ and } M \subseteq \{(z, \lambda) \in K \times \mathbb{R} \mid \lambda > e(z)\}.$$

Similarly as in the proof of Lemma 2.14, this will imply that $e_1, e_2 < e < f$, proving that Λ is in fact an upward directed set. It follows from Lemma 2.14 that is has supremum f. □

2.16 Lemma Let K be a compact convex set.

1) Let $f\colon K \to [-\infty, \infty)$ be upper semicontinuous and affine, $g \in \mathrm{LAff}(K)$. Then $f \leq g$ if and only if $f|_{\partial_e K} \leq g|_{\partial_e K}$. Moreover, $f < g$ if and only if $f|_{\partial_e K} < g|_{\partial_e K}$ if and only if there exists an $\varepsilon > 0$ such that $f + \varepsilon \leq g$.
2) Let f, $g \in \mathrm{LAff}(K)$. Then $f \leq g$ if and only if $f|_{\partial_e K} \leq g|_{\partial_e K}$.

Proof. We only show the nontrivial implications.

1) The function $g - f$ is lower semicontinuous and affine, so it attains its minimum on $\partial_e K$ by Lemma 2.12. Let ε be the minimum, then $f + \varepsilon \leq g$. The assumption $f|_{\partial_e K} \leq g|_{\partial_e K}$ implies $\varepsilon \geq 0$, while the assumption $f|_{\partial_e K} < g|_{\partial_e K}$ entails $\varepsilon > 0$.

2) By Proposition 2.15, there exists an upward directed set $\Lambda \subseteq \mathrm{Aff}(K)$ with supremum f. For each $e \in \Lambda$, we have that $e|_{\partial_e K} \leq f|_{\partial_e K} \leq g|_{\partial_e K}$, hence $e \leq g$ by the first part. By passing to the supremum, we obtain $f \leq g$. □

2.17 Lemma Let K be a compact convex set and suppose that $f, g\colon K \to \mathbb{R}$ are lower semicontinuous, convex and bounded above (for example, f and g are continuous). There exists a biggest convex function e less than f and g. Moreover, e is lower semicontinuous.

Proof. (cf. [Goo86, 11.10]) Suppose that K is located in some locally convex, Hausdorff, real topological vector space E. Choose real numbers α and β such that $\alpha \leq f, g \leq \beta$. The sets

$$M_f \coloneqq \{(x,\lambda) \in K \times \mathbb{R} \mid f(x) \leq \lambda \leq \beta\} \text{ and } M_g \coloneqq \{(x,\lambda) \in K \times \mathbb{R} \mid g(x) \leq \lambda \leq \beta\}$$

are compact, convex subsets of $E \oplus \mathbb{R}$ since both f and g are assumed to be lower semicontinuous and convex. The set $M_e \coloneqq \operatorname{conv}(M_f \cup M_g)$ is convex, but also compact, because M_f and M_g are both convex and compact. Note that M_e is contained in the set $\{(x,\lambda) \in K \times \mathbb{R} \mid \alpha \leq \lambda\}$, since the latter is a convex set containing M_f and M_g. It follows that $e \colon K \to \mathbb{R}$, $e(x) \coloneqq \inf_{(x,\lambda) \in M_e} \lambda$ is a well defined function. By the compactness of M_e, we may change the infimum to a minimum. We prove that e is convex: Let $x, y \in K$, $t \in [0,1]$. Both the points $(x, e(x))$ and $(y, e(y))$ lie in M_e. As M_e is convex, the point $(tx + (1-t)y, te(x) + (1-t)e(y))$ must also lie in M_e, implying that $e(tx + (1-t)y) \leq te(x) + (1-t)e(y)$.
If x lies in K, then the point $(x, f(x))$ lies in $M_f \subseteq M_e$, implying that $e(x) \leq f(x)$. As x is arbitrary, we conclude that $e \leq f$. Similarly, we obtain that $e \leq g$. Now suppose that h is another convex function such that $h \leq f, g$. The set $M_h \coloneqq \{(x,\lambda) \in K \times \mathbb{R} \mid h(x) \leq \lambda \leq \beta\}$ is again convex by the convexity of h. The inequality $h \leq f, g$ implies that $M_f, M_g \subseteq M_h$. We obtain that $M_e \subseteq M_h$, which again implies that $h \leq e$. Thus, e is the biggest convex function less than f and g.
Before we move on to the last part of this proof, we claim that

$$M_e = \{(x,\lambda) \in K \times \mathbb{R} \mid e(x) \leq \lambda \leq \beta\}.$$

If (y,μ) lies in M_e, then $e(y) \leq \mu$ holds by definition. Just like in the beginning of the proof, M_e is contained in the set $\{(x,\lambda) \in K \times \mathbb{R} \mid \lambda \leq \beta\}$, so in particular $\mu \leq \beta$. Conversely, suppose that $(y,\mu) \in K \times \mathbb{R}$ satisfies $e(y) \leq \mu \leq \beta$. The set $A \coloneqq \{\lambda \mid (y,\lambda) \in M_e\}$ is a convex subset of $\mathbb{R}$ (and in particular connected). Clearly, the points $e(y)$ and β lie in A, implying that $[e(y), \beta] \subseteq A$. Thus, μ lies in A, or in other words $(y,\mu) \in M_e$.
Finally, let us show that e is lower semicontinuous. Given $t \in \mathbb{R}$, we will prove that $e^{-1}((-\infty, t])$ is closed. This is clear if $\beta < t$, since $e^{-1}((-\infty, t]) = K$ in this case. Now let us assume that $t \leq \beta$. The function $\varphi \colon K \to E \oplus \mathbb{R}$, $\varphi(x) \coloneqq (x, t)$ is continuous. It follows that

$$\begin{aligned} e^{-1}((-\infty, t]) &= \{x \in K \mid e(x) \leq t\} = \{x \in K \mid e(x) \leq t \leq \beta\} \\ &= \{x \in K \mid (x,t) \in M_e\} = \varphi^{-1}(M_e) \end{aligned}$$

is a closed subset of K, as desired. □

2.18 Lemma Let K be compact convex set, and let $f\colon K \to [-\infty, \infty)$ be an upper semicontinuous, convex function. For every $x \in K$, we have that

$$f(x) = \inf\{e(x) \mid e\colon K \to \mathbb{R} \text{ continuous, convex}, f < e\}.$$

Proof. (cf. [Goo86, 11.9]) Let $x \in K$. As f is upper continuous, it has an upper bound which we denote by γ. Let α be an arbitrary real number such that $f(x) < \alpha$. We have to show that there exists a continuous, convex function $e\colon K \to \mathbb{R}$ such that $f < e$ and $e(x) < \alpha$. If $\alpha > \gamma$, we can choose e to be the constant function with value $2^{-1}(\gamma + \alpha)$. So let us assume that $\alpha \leq \gamma$. Choose a real number β such that $f(x) < \beta < \alpha$. Since f is upper continuous, the set $f^{-1}([\beta, \infty))$ is compact and does not contain x. Choose an open neighborhood U of x in the surrounding locally convex space E such that $\overline{U}$ and $f^{-1}([\beta, \infty))$ are disjoint (this is a small argument involving compactness of $f^{-1}([\beta, \infty))$). We can write U as $U = x + V$, where V is an open neighborhood of the origin in E. Since E is a locally convex space, we may assume without loss of generality that V is balanced (meaning that $\lambda V \subseteq V$ for every $\lambda \in [0, 1]$), convex and open. Let p denote the Minkowski functional associated with V. Then p is the unique seminorm on E satisfying $V = \{y \in E \mid p(y) < 1\}$. It is also true that $\overline{V} = \{y \in E \mid p(y) \leq 1\}$. Define $e\colon K \to \mathbb{R}$ by the formula

$$e(y) \coloneqq \beta + (\gamma - \beta)p(y - x).$$

Then e is continuous by the continuity of p. Also, since $\gamma - \beta > 0$ and since p is convex, e is convex as well. Moreover, $e(x) = \beta < \alpha$. It is left to show that $f < e$. Given $y \in K$, let us show that $f(y) < e(y)$. Note that $\beta \leq e$, so we may assume without loss of generality that $\beta \leq f(y)$. In this case, y is contained in $f^{-1}([\beta, \infty))$, which implies that it is not contained in $\overline{U} = x + \overline{V}$, therefore $p(y - x) > 1$. It follows that

$$e(y) = \beta + (\gamma - \beta)p(y - x) > \beta + (\gamma - \beta) \cdot 1 = \gamma \geq f(y),$$

as desired. □

2.19 Lemma Let X be a compact Hausdorff space, $f\colon X \to [-\infty, \infty)$ upper semicontinuous and $\Lambda \subseteq \mathrm{Lsc}(X)$ nonempty such that $f < \sup \Lambda$. Then there exist finitely many elements $e_1, \ldots, e_n \in \Lambda$ such that $f < \max\{e_1, \ldots, e_n\}$. If Λ is upward directed, there exists an element $e \in \Lambda$ such that $f < e$.

Proof. For any $x \in X$, we have $f(x) < \sup_{e \in \Lambda} e(x)$. Choose $e_x \in \Lambda$ such that $f(x) < e_x(x)$. Since $e_x - f$ is lower semicontinuous, the set $M(x) \coloneqq \{y \in X \mid 0 < e_x(y) - f(y)\}$ is an open neighborhood of x. By compactness of $X = \bigcup_{x \in X} M(x)$, there exist $x_1, \ldots, x_n \in X$ such that $K = M(x_1) \cup \ldots \cup M(x_n)$, implying that $f < \max\{e_{x_1}, \ldots, e_{x_n}\}$. If Λ is assumed to be upward directed, we can choose an element $e \in \Lambda$ that dominates each e_{x_i} which entails $f < e$. □

2.20 Lemma Let K be a compact convex set, $f\colon K \to [-\infty, \infty)$ upper semicontinuous and convex and $h\colon K \to (-\infty, \infty]$ lower semicontinuous and concave such that $f < h$. Then there exists a lower semicontinuous and convex function $e\colon K \to \mathbb{R}$ such that $f < e < h$.

Proof. We know from Lemma 2.18 that the set

$$\Lambda \coloneqq \{e \mid e\colon K \to \mathbb{R} \text{ continuous, convex}, f < e\}$$

has f as its infimum, hence $\inf \Lambda < h$. By Lemma 2.19 (or rather its dual result), there exist finitely many elements $e_1, \ldots, e_n \in \Lambda$ such that $\min\{e_1, \ldots, e_n\} < h$. Choose a positive real number $\varepsilon > 0$ such that $f + \varepsilon < e_1, \ldots, e_n$. By Proposition 2.17, there exists a biggest convex function $e\colon K \to \mathbb{R}$ less than $e_1, \ldots, e_n$, and e is moreover lower semicontinuous. It follows that $f < f + \varepsilon \leq e \leq \min\{e_1, \ldots, e_n\} < h$. □

2.21 Lemma Let K be a compact convex set, $f\colon K \to [-\infty, \infty)$ upper semicontinuous and convex and $h\colon K \to (-\infty, \infty]$ lower semicontinuous and concave such that $f < h$. There exists a function $f'\colon K \to \mathbb{R}$ that is the pointwise maximum of finitely many functions from $\mathrm{Aff}(K)$ and such that $f < f' < h$.

Proof. By Lemma 2.20, there exists a lower semicontinuous, convex function $e\colon K \to \mathbb{R}$ such that $f < e < h$. We have seen in Lemma 2.14 that the set

$$\Lambda \coloneqq \{e' \in \mathrm{Aff}(K) \mid e' < e\}$$

has supremum e. Using Lemma 2.19, there exist finitely many elements $e_1, \ldots, e_n \in \Lambda$ such that $f < \max\{e_1, \ldots, e_n\}$. The function $f' \coloneqq \max\{e_1, \ldots, e_n\}$ has the desired properties. □

2.22 Proposition Let K be a compact convex set, $f\colon K \to [-\infty, \infty)$ upper semicontinuous and convex and $h\colon K \to (-\infty, \infty]$ lower semicontinuous and concave such that $f < h$. There exists a function $f'\colon K \to \mathbb{R}$ that is the pointwise maximum of finitely many functions from $\mathrm{Aff}(K)$ and a function $h'\colon K \to \mathbb{R}$ that is the pointwise minimum of finitely many functions from $\mathrm{Aff}(K)$ such that $f < f' < h' < h$.

Proof. (cf. [Goo86, 11.11]) The existence of such an f' is given by Lemma 2.21. Then $-h$ is upper semicontinuous and convex, while $-f'$ is continuous and concave with $-h < -f'$. Applying Lemma 2.21 a second time, we get a function e that is the pointwise maximum of finitely many functions from $\mathrm{Aff}(K)$ and such that $-h < e < -f'$. Then $h' \coloneqq -e$ is a function that is the pointwise minimum of finitely many functions from $\mathrm{Aff}(K)$. Moreover, $f < f' < h' < h$. □

2.3 LAff$(K)^0_{++}$ is a Cu-semigroup

Now that we know that every lower semicontinuous, affine function is the supremum of an increasing net of continuous, affine functions, we can show that LAff$(K)^0_{++}$ is a Cu-semigroup. However, in order to pass from nets to sequences, we have to assume that K is metrizable.

2.23 Lemma Let X be a compact Hausdorff space. For any system $\mathcal{B} \subseteq \mathcal{O}(X)$ of open subsets of X, the following are equivalent:

1) $\mathcal{B}$ is a basis for the topology on X.
2) $\mathcal{B}$ separates the points of X, that is: for all $x, y \in X$, $x \neq y$, there exist $U, V \in \mathcal{B}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Also, for all $x \in X$ and $U_1, \ldots, U_n \in \mathcal{B}$ such that $x \in \cap_{i=1}^n U_i$, there exists $U \in \mathcal{B}$ such that $x \in U \subseteq \cap_{i=1}^n U_i$.

Proof. 1) $\Rightarrow$ 2): Let $x, y \in X$ such that $x \neq y$. Choose open, disjoint neighborhoods A of x and B of y. Since $\mathcal{B}$ is assumed to be a basis for the topology on X, we can find $U, V \in \mathcal{B}$ such that $x \in U \subseteq A$ and $y \in V \subseteq B$. Clearly U and V are disjoint, so this shows that $\mathcal{B}$ separates the points of X. The second condition in 2) is trivially satisfied.
2) $\Rightarrow$ 1): Let $x \in X$, and let A be an open neighborhood for x. For each $y \in X \setminus A$, we may choose disjoint sets $U_y, V_y \in \mathcal{B}$ such that $x \in U_y$ and $y \in V_y$. Since $X \setminus A$ is compact and contained in $\cup_{y \in X \setminus U} V_y$, we find $y_1, \ldots, y_n \in X \setminus A$ such that $X \setminus A \subseteq \cup_{y=1}^n V_{y_i}$. It follows that

$$x \in \cap_{i=1}^n U_{y_i} \subseteq \cap_{i=1}^n X \setminus V_{y_i} \subseteq A.$$

Finally, we may choose $U \in \mathcal{B}$ such that $x \in U \subseteq \cap_{i=1}^n U_{y_i} \subseteq A$, finishing the proof. □

2.24 Proposition Let K be a compact convex set. Then LAff(K) is a domain. For functions $f, h \in$ LAff(K), we have $f \ll h$ if and only if there exist $g \in$ Aff(K) such that $f \leq g < h$. Additionally, the following conditions are equivalent:

1) K is metrizable.
2) Aff(K) is separable.
3) LAff(K) is countably based.

Proof. Let $f, h \in$ LAff(K). Suppose that $f \ll h$. The set $\Lambda := \{ e \mid e \in \text{Aff}(K), e < h \}$ is upward directed with supremum h according to Proposition 2.15, hence there exists an element $g \in \Lambda$ such that $f \leq g$. Conversely, suppose there exist $g \in$ Aff(K) such that $f \leq g < h$. To show that $f \ll h$, let $\Lambda \subseteq$ LAff(K) be an upward directed set with $h \leq \sup \Lambda$. By Lemma 2.19, there exists an element $e \in \Lambda$ such that $f \leq g < e$.
1) $\Rightarrow$ 2): If we assume that K is metrizable, then $\mathcal{C}(K; \mathbb{R})$ is separable. Since any subspace of a separable metrizable space is separable itself, we obtain that Aff(K) is separable.

2) $\Rightarrow$ 3): Suppose that $\mathrm{Aff}(K)$ is separable, and let $B \subset \mathrm{Aff}(K)$ be a countable dense subset. We claim that B is a base for $\mathrm{LAff}(K)$. Let $f, h \in \mathrm{LAff}(K)$ such that $f \ll h$. By the description of the way-below relation, there exists a function $g \in \mathrm{Aff}(K)$ such that $f \leq g < h$. Choose $\varepsilon > 0$ such that $f \leq g < g + 2\varepsilon \leq h$, then choose an element $b \in B$ such that $\|b - (g+\varepsilon)\| < \varepsilon$. It follows that $-\varepsilon < b - (g+\varepsilon) < \varepsilon$, hence $f \leq g < b < g + 2\varepsilon \leq h$, as desired.
3) $\Rightarrow$ 1): Let $B \subseteq \mathrm{LAff}(K)$ be a countable basis. We claim that the system

$$\mathcal{A} \coloneqq \{f^{-1}((0,\infty]) \mid f \in B\} \subseteq \mathcal{O}(K)$$

separates the points of K. To prove this claim, let $x, y \in K$ such that $x \neq y$. Choose $h \in \mathrm{Aff}(K)$ such that $h(x) \neq h(y)$. We can arrange that $h(x) = 2$ while also $h(y) = -2$. Choose $f, g \in B$ satisfying $h - 1 \leq f < h$ and $-h - 1 \leq g < -h$. We find that

$$f(x) \geq h(x) - 1 = 1, \text{ i.e. } x \in f^{-1}((0,\infty]),$$
$$g(y) \geq -h(y) - 1 = 1, \text{ i.e. } y \in g^{-1}((0,\infty]),$$

and that $f^{-1}((0,\infty]) \cap g^{-1}((0,\infty]) = \emptyset$: if z lies in $f^{-1}((0,\infty])$, then $h(z) > f(z) > 0$, which implies that $g(z) < -h(z) < 0$, hence $z \notin g^{-1}((0,\infty])$. This proves the claim that $\mathcal{A}$ separates the points of K. Now let $\mathcal{B}$ be the set of all finite intersections of elements in $\mathcal{A}$. Then $\mathcal{B}$ is countable, separates the points of K and is trivially closed under finite intersections. It follows from Lemma 2.23 that $\mathcal{B}$ is a basis for the topology on K. Therefore K is second countable and consequently metrizable. □

2.25 Proposition Let K be a metrizable compact convex set. Then $\mathrm{LAff}(K)^0_{++}$ is a Cu-semigroup. It is countably based, simple and satisfies ($\mathcal{O}5$). Given $f, h \in \mathrm{LAff}(K)^0_{++}$ with $h \neq 0$, we have $f \ll h$ in $\mathrm{LAff}(K)^0_{++}$ if and only if $f \ll h$ in $\mathrm{LAff}(K)$.

Proof. Most of the statements follow easily from Proposition 2.24, and it is clear that $\mathrm{LAff}(K)^0_{++}$ is simple. In order to prove that $\mathrm{LAff}(K)^0_{++}$ satisfies ($\mathcal{O}5$), let $f, f', g, g', h \in \mathrm{LAff}(K)^0_{++}$ such that

$$f' \ll f, \text{ and } g' \ll g, \text{ and } f + g \leq h.$$

Choose a continuous affine function f'' such that $f' \leq f'' \ll f$. One can then show that the function $e \coloneqq h - f''$ lies in $\mathrm{LAff}(K)^0_{++}$ and satisfies $f' + e \leq h \leq f + e$ and $g' \leq e$. □

3 Choquet simplices

3.1 Definition and examples

Let G be an abelian group. A cone in G is just another term for a submonoid in G. Any cone C in G defines a reflexive, transitive order on G by setting

$$x \leq_C y :\Longleftrightarrow y - x \in C,$$

for $x, y \in C$. Notice that C equals the set of all positive elements in $(G, \leq_C)$, and that the order is compatible with addition. This order is a partial order if and only if C is strict, that is if $C \cap (-C) = \{0\}$, or equivalently, if C is a conical monoid, that is if the equality $x + y = 0$ implies that $x = y = 0$, for $x, y \in C$.

3.1 Definition A convex cone in some locally convex, Hausdorff, real topological vector space E is a convex subset C that is also a cone in the abelian group $(E, +)$. Notice that any convex cone is closed under multiplication with positive scalars. We call C strict if the underlying cone is strict. A lattice cone is a strict convex cone C such that $(C, \leq_C)$ is a lattice, or equivalently, such that $(C - C, \leq_C)$ is a lattice.
A map $C \to D$ between two convex cones is called linear if it is affine and preserves addition. Such a map will automatically be homogeneous for positive scalars.

3.2 Proposition Let K be a compact convex set. There exists a strict convex cone $C(K)$ and a continuous affine map $\kappa\colon K \to C(K)$ such that the following universal property holds: For every convex cone D and every affine map $\varphi\colon K \to D$, there exists a unique linear map $\tilde{\varphi}\colon C(K) \to D$ such that $\varphi = \tilde{\varphi} \circ \kappa$. Moreover, φ is continuous (lower semicontinuous, upper semicontinuous) if and only if $\tilde{\varphi}$ is continuous (lower semicontinuous, upper semicontinuous). Furthermore, $C(K)$ is unique up to linear homeomorphism and is referred to as the cone with base K.

Proof. Let $\kappa\colon K \to S(\mathrm{Aff}(K), 1)$ be the natural isomorphism. Then $C(K)$ can be realized as

$$C(K) := \mathrm{Hom}_{\mathrm{OAG}}(\mathrm{Aff}(K), \mathbb{R}) = \{\lambda \cdot \kappa(x) \mid \lambda > 0,\ x \in K\} \cup \{0\},$$

where the equality follows from Lemma 2.7. Notice that $C(K)$ is a strict convex cone, and that $\kappa\colon K \to C(K)$ is continuous and affine.
Assume that D is a convex cone and that $\varphi\colon K \to D$ is an affine map. We define $\tilde{\varphi}\colon C(K) \to D$ by $\tilde{\varphi}(0) := 0$ and $\tilde{\varphi}(\lambda \cdot \kappa(x)) := \lambda \cdot \varphi(x)$, for $\lambda > 0$ and $x \in K$. The representation of an element in $C(K) \setminus \{0\}$ as $\lambda \cdot \kappa(x)$ is unambiguous, so $\tilde{\varphi}$ is well defined. It is a straightforward verification that $\tilde{\varphi}$ has the desired properties and will be omitted. The statements concerning (semi-)continuity follow from the fact that the map $(0,\infty) \cdot \kappa(K) \to (0,\infty) \times K$, given by $\lambda \cdot \kappa(x) \mapsto (\lambda, x)$, is continuous. □

Remark We say that K is regularly embedded in the surrounding space E if K lies in a hyperplane that misses the origin. If this is the case, the cone with base K can be realized as $C(K) = ((0,\infty) \cdot K) \cup \{0\}$. In fact, the proof of Proposition 3.2 relies on the fact that K is isomorphic to $S(\mathrm{Aff}(K), 1)$, and that the latter is regularly embedded in $\mathbb{R}^{\mathrm{Aff}(K)}$.

3.3 Definition A Choquet simplex is a compact convex set K such that the cone with base K is a lattice cone. It follows from Proposition 3.2 that this definition does not depend on the concrete realization of the cone with base K.

Before we give examples of Choquet simplices, let us briefly discuss which compact convex sets are not Choquet simplices. It is shown in [Goo86, 10.8] that the extreme boundary of any Choquet simplex K is affinely independent. It follows that squares, cubes, disks, balls and in fact most compact convex sets are not Choquet simplices. With this in mind, one can show that up to affine homeomorphism, the only Choquet simplices that can be embedded into finite dimensional spaces are the empty set and the standard n-dimensional simplices, for $n \in \mathbb{N}$. The standard n-simplex is defined as the convex hull of the $(n+1)$ standard basis vectors in $\mathbb{R}^{n+1}$. For more interesting examples of Choquet simplices, one has to consider infinite dimensional compact convex sets.

3.4 Definition We say that a partially ordered set $(X, \leq)$ has the Riesz interpolation property (or just interpolation) if for all $x_1, x_2, z_1, z_2 \in X$ satisfying $x_1, x_2 \leq z_1, z_2$, there exists $y \in X$ such that $x_1, x_2 \leq y \leq z_1, z_2$.
An abelian semigroup M is said to have the Riesz refinement property if for all $x_1, x_2, z_1, z_2 \in M$ with $x_1 + x_2 = z_1 + z_2$, there exist $y_{i,j} \in M$, for $i, j \in \{1,2\}$, such that $x_i = y_{i,1} + y_{i,2}$ and $z_j = y_{1,j} + y_{2,j}$, for $i, j \in \{1,2\}$.
A partially ordered abelian semigroup $(M, \leq)$ has the Riesz decomposition property if for all $x, z_1, z_2 \in M$ satisfying $x \leq z_1, z_2$, there exist $y_1, y_2 \in M$ such that $y_1 \leq z_1$, $y_2 \leq z_2$ and $x = y_1 + y_2$.

It is well known that an ordered abelian group $(G, \leq)$ has the Riesz interpolation property if and only if G_+ has the Riesz refinement property if and only if $(G_+, \leq)$ has the Riesz decomposition property. A proof can be found in [Goo86, 2.1]. An ordered abelian group with the Riesz interpolation property is also called interpolation group.

3.5 Proposition Let (G, u) be a partially ordered group with order unit. If G has interpolation, then $S(G, u)$ is a Choquet simplex.

Proof. As $S(G, u)$ is regularly embedded in $\mathbb{R}^G$, the cone with base $S(G, u)$ can be realized as

$$C = ((0, \infty) \cdot S(G, u)) \cup \{0\} = \mathrm{Hom}_{\mathrm{OAG}}(G, \mathbb{R}).$$

The last equality is given by Lemma 2.7. Let $f, g \in C$ and define $m\colon G_+ \to \mathbb{R}_+$ as the pointwise maximum of f and g.

The following vocubulary refers solely to this proof: a decomposition of an element $x \in G_+$ is a finite collection of elements $x_1, \ldots, x_n \in G_+$ such that $x_1 + \ldots + x_n = x$.

Let $x \in G_+$. For every decomposition $x = x_1 + \ldots + x_n$, we have that

$$m(x_1) + \ldots + m(x_n) \leq (f + g)(x_1) + \ldots + (f + g)(x_n) = (f + g)(x),$$

hence the set

$$M(x) \coloneqq \{m(x_1) + \ldots + m(x_n) \mid x_1, \ldots, x_n \text{ is a decomposition of } x\}$$

is bounded above by $(f + g)(x)$. Therefore the function $h\colon G_+ \to \mathbb{R}_+,\ h(x) \coloneqq \sup M(x)$ is well defined. We claim that the equation $h(x + y) = h(x) + h(y)$ holds for all $x, y \in G_+$. For $x, y \in G_+$, let

$$x = x_1 + \ldots + x_n \text{ and } y = y_1 + \ldots + y_m$$

be arbitrary decompositions. Then $x_1 + \ldots + x_n + y_1 + \ldots + y_m$ is a decomposition of $x + y$, implying that

$$m(x_1) + \ldots + m(x_n) + m(y_1) + \ldots + m(y_m) \leq h(x + y).$$

This means that $t_x + t_y \leq h(x+y)$ whenever $t_x \in M(x)$ and $t_y \in M(y)$, from which we deduce

$$h(x) + h(y) = \sup_{t_x \in M(x)} \sup_{t_y \in M(y)} t_x + t_y \leq h(x + y).$$

The converse inequality is where the interpolation comes into play. Let $x + y = z_1 + \ldots + z_n$ be a decomposition. Since G has interpolation, or equivalently, since G_+ has the Riesz refinement property, there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in G_+$ such that

$$\begin{aligned} &x = x_1 + \ldots + x_n, \\ &y = y_1 + \ldots + y_n \text{ and} \\ &z_i = x_i + y_i \text{ for each } i \in \{1, \ldots, n\}. \end{aligned}$$

For each $i \in \{1, \ldots, n\}$, we have that $m(z_i) \leq m(x_i) + m(y_i)$. This implies that

$$m(z_1) + \ldots + m(z_n) \leq m(x_1) + \ldots + m(x_n) + m(y_1) + \ldots + m(y_n) \leq h(x) + h(y),$$

which yields $h(x + y) \leq h(x) + h(y)$ after passing to the supremum.
We have shown that $h(x+y) = h(x)+h(y)$ if $x, y \in G_+$. Note that $h(0) = 0$. As G is directed, h extends uniquely to an element $h \in C$. We claim that h is the supremum of f and g in C. For every $x \in G^+$, we have $f(x)$, $g(x) \leq m(x) \leq h(x)$, which implies that f, $g \leq_C h$. Thus, h is an upper bound for f and g. Let $e \in C$ be another upper bound for f and g. For every $x \in G^+$, we get $m(x) = \max\{f(x), g(x)\} \leq e(x)$. Just as we did before, one can show that $e(x)$ is an upper bound for $M(x)$ which means that $h(x) = \sup M(x) \leq e(x)$. It follows that $h \leq_C e$, so h is in fact the supremum of f and g in C. We have shown that $(C, \leq_C)$ is a sup-semilattice. That $(C, \leq_C)$ is an inf-semilattice can be shown similarly. It follows that $S(G, u)$ is in fact a Choquet simplex. □

3.6 Corollary If X is a compact Hausdorff space, then $M_1^+(X)$ is a Choquet simplex.

Proof. We have seen in Lemma 2.9 that $M_1^+(X) \cong S(\mathcal{C}(X; \mathbb{R}), 1)$. Clearly $\mathcal{C}(X; \mathbb{R})$ has interpolation, so the assertion follows from the previous proposition. □

3.2 Continuous and affine functions on Choquet simplices

Choquet simplices are interesting for us because their continuous and affine functions satisfy the Riesz interpolation property. The proof requires the following technical Lemma, which is adapted from [Goo86, 11.1 and 11.2].
But first, let us settle some notation. Let X be a compact Hausdorff space. On $\mathcal{C}(X; \mathbb{R})$, we define a partial order $<^=$ by setting $f <^= g$ if and only if $f = g$ or $f < g$. We refer to $<^=$ as the strict ordering. For a linear subspace $A \subseteq \mathcal{C}(X; \mathbb{R})$ containing 1, observe that

$$\mathrm{Hom}_{\mathrm{OAG}}((A, <^=), \mathbb{R}) = \mathrm{Hom}_{\mathrm{OAG}}((A, \leq), \mathbb{R}) \subseteq A^*,$$

where A^* denotes the topological dual space of A. That any $\varphi \in \mathrm{Hom}_{\mathrm{OAG}}((A, \leq), \mathbb{R})$ is linear can be shown similarly as in the proof of Lemma 2.9. It follows just like in the proof of Proposition 2.1 that φ is bounded (the assumption that $1 \in A$ is used here). We use $\leq_+$ to denote the order on A^* induced by the cone $\mathrm{Hom}_{\mathrm{OAG}}((A, <^=), \mathbb{R})$.

3.7 Lemma Let X be a compact Hausdorff space, and let $A \subseteq \mathcal{C}(X; \mathbb{R})$ be a linear subspace containing 1. Equip A with the strict ordering. Suppose that $(A^*, \leq_+)$ is a lattice, and let $p \in A^*$. The function $r\colon A_+ \to \mathbb{R}$, defined by

$$r(f) \coloneqq \sup\{p(e) \mid e \in A,\ 0 <^= e <^= f\},$$

is additive.

Proof. Let p_+ denote the supremum of p and 0 in $(A^*, \leq_+)$, and let $f \in A_+$. Given $e \in A$ such that $0 <^= e <^= f$, we get $p(e) \leq p_+(e) \leq p_+(f)$. Thus, the set $\{p(e) \mid e \in A,\ 0 <^= e <^= f\}$ is bounded above by $p_+(f)$. This shows that r is well defined and that $r(f) \leq p_+(f)$.
The converse inequality is clear if $f = 0$, so let us assume that $f > 0$. Let $\varepsilon > 0$, and choose $n \in \mathbb{N}^*$ such that $r(f) - p(f) \leq n\varepsilon$. Consider the following subspaces of $A \oplus A$:

$$C := \{(e, -e) \mid e \in A\} \text{ and } D := \mathbb{R} \cdot (nf, f) + C.$$

Define $g\colon D \to \mathbb{R}$ by $g(\gamma \cdot (nf, f) + (e, -e)) := \gamma n(r(f) + \varepsilon) + p(e)$. Note that (nf, f) does not lie in C, hence the representation of the input is unique and g is well defined. Evidently, g preserves the addition. We claim that g is positive. Let $\gamma \cdot (nf, f) + (e, -e)$ be positive, that is

$$0 <^= \gamma nf + e \text{ and } 0 <^= \gamma f - e.$$

Adding these inequalities yields $0 <^= \gamma(n+1)f$, or equivalently $0 \leq \gamma$. The assumption $\gamma = 0$ results in $e = 0$, so $g(\gamma \cdot (nf, f) + (e, -e)) = 0$. Let us therefore assume that $\gamma > 0$. We have

$$\begin{aligned} &0 <^= \gamma f - e <^= \gamma f + \gamma nf = \gamma(n+1)f, \text{ or equivalently} \\ &0 <^= \gamma^{-1}(n+1)^{-1}(\gamma f - e) <^= f. \end{aligned}$$

By the definition of r, this results in

$$\begin{aligned} &\gamma^{-1}(n+1)^{-1}p(\gamma f - e) = p(\gamma^{-1}(n+1)^{-1}(\gamma f - e)) \leq r(f), \text{ or equivalently} \\ &\gamma(n+1)r(f) - p(\gamma f - e) \geq 0. \end{aligned}$$

We conclude that

$$\begin{aligned} g(\gamma \cdot (nf, f) + (e, -e)) &= \gamma n(r(f) + \varepsilon) + p(e) \\ &\geq \gamma nr(f) + \gamma(r(f) - p(f)) + p(e) \\ &= \gamma(n+1)r(f) - p(\gamma f - e) \geq 0, \end{aligned}$$

proving that g is in fact positive. Our assumption $f > 0$ entails that (nf, f) is an order unit in $A \oplus A$. By Proposition 2.4, g extends to a positive homomorphism $\tilde{g}\colon A \oplus A \to \mathbb{R}$. The function $q\colon A \to \mathbb{R}$, defined by $q(e) := \tilde{g}(e, 0)$, lies in $(A^*)_+$. For every $e \in A_+$, we have that

$$q(e) = \tilde{g}(e, 0) \geq \tilde{g}(e, 0) - \tilde{g}(0, e) = \tilde{g}(e, -e) = p(e),$$

hence $p \leq_+ q$. Then $p_+ \leq_+ q$ by the positivity of q. As f lies in A_+, we obtain

$$\begin{aligned} p_+(f) \leq q(f) = n^{-1}\tilde{g}(nf, 0) &\leq n^{-1}(\tilde{g}(nf, 0) + \tilde{g}(0, f)) \\ &= n^{-1}\tilde{g}(nf, f) = n^{-1} \cdot n(r(f) + \varepsilon) = r(f) + \varepsilon. \end{aligned}$$

As ε was arbitrary, we get $p_+(f) \leq r(f)$.
Overall, we have shown that $r = p_+|_{A_+}$. In particular, r is additive. □

3.8 Theorem Let X be a compact Hausdorff space, and let $A \subseteq \mathcal{C}(X;\mathbb{R})$ be a linear subspace containing 1. If $(A^*, \leq_+)$ is a lattice, then $(A, <^=)$ is an interpolation group.

Proof. (cf. [Goo86, 11.3]) Endow A with the strict ordering. We will show that A satisfies the Riesz decomposition property which is equivalent to the Riesz interpolation property. Let $f_1, f_2 \in A_+$. Consider the following sets:

$$\begin{aligned} F_i &:= \{f \in A \mid 0 <^= f <^= f_i\}, \text{ for } i = 1,2 \text{ and} \\ F &:= \{f \in A \mid 0 <^= f <^= f_1 + f_2\}. \end{aligned}$$

We have to show that $F \subseteq F_1 + F_2$. This is trivial if either f_1 or f_2 equals zero. Therefore, we may assume that $0 < f_1, f_2$.
Suppose that $F_1 + F_2$ is not a dense subset of F. Choose an element $h \in F$ that does not lie in the closure of $F_1 + F_2$. As $F_1 + F_2$ is convex, its closure is convex as well. According to the Hahn-Banach theorem, there exists a continuous linear functional $p \in A^*$ satisfying

$$\sup\{p(f) \mid f \in \overline{F_1 + F_2}\} < p(h).$$

Define $r\colon A_+ \to \mathbb{R}$ by the formula

$$r(f) := \sup\{p(e) \mid e \in A,\ 0 <^= e <^= f\}.$$

By Lemma 3.7, r is an additive function. However, the computation

$$\begin{aligned} r(f_1) + r(f_2) = \sup(p(F_1)) + \sup(p(F_2)) &= \sup(p(F_1 + F_2)) \\ < p(h) \leq \sup(p(F)) &= r(f_1 + f_2) \end{aligned}$$

results in a contradiction, proving that $F_1 + F_2$ is a dense subset of F.
To show that $F \subseteq F_1 + F_2$, let $f \in F$. We may assume that $0 \neq f \neq f_1 + f_2$, such that $0 < f < f_1 + f_2$. Choose $\varepsilon > 0$ such that $\varepsilon < f_1, f_2$ and $\varepsilon < f < f_1 + f_2 - \varepsilon$, then choose $0 < \delta < 1/2$ such that $2\delta f < \varepsilon$. We have that

$$2\delta\varepsilon < \varepsilon < (1 + 2\delta)f < f_1 + f_2 - \varepsilon + 2\delta f < f_1 + f_2,$$

hence

$$0 < (1 + 2\delta)f - 2\delta\varepsilon < f_1 + f_2,$$

in other words $(1+2\delta)f - 2\delta\varepsilon \in F$. As $F_1 + F_2$ is a dense subset of F, there exist $g_1 \in F_1$ and $g_2 \in F_2$ such that $\|(1 + 2\delta)f - 2\delta\varepsilon - g_1 - g_2\| < 2\delta\varepsilon$. Let

$$\begin{aligned} h &:= ((1 + 2\delta)f - 2\delta\varepsilon - g_1 - g_2)/2, \\ h_i &:= (1 + 2\delta)^{-1}(g_i + \delta\varepsilon + h), \text{ for } i = 1, 2. \end{aligned}$$

As $\|h\| < \delta\varepsilon$, we have that $-\delta\varepsilon < h < \delta\varepsilon$. For $i = 1, 2$, it follows that

$$0 < g_i < g_i + \delta\varepsilon + h < f_i + 2\delta\varepsilon < (1 + 2\delta) f_i.$$

Dividing by $1 + 2\delta$ yields $0 < h_i < f_i$, that is $h_i \in F_i$. Then the computation

$$(1 + 2\delta)(h_1 + h_2) = g_1 + g_2 + 2\delta\varepsilon + 2h = (1 + 2\delta) f$$

shows that $f = h_1 + h_2 \in F_1 + F_2$. □

3.9 Theorem For a compact convex set K, the following conditions are equivalent:

(i) K is a Choquet simplex.

(ii) $(\mathrm{Aff}(K), <^=)$ is an interpolation group.

(iii) $(\mathrm{Aff}(K), \leq)$ is an interpolation group.

Proof. (cf. [Goo86, 11.4]) (i) $\Rightarrow$ (ii): As seen in Proposition 3.2, K is a base for the positive cone in $(\mathrm{Aff}(K)^*, \leq_+)$. It follows from [Goo86, 7.3] that $\mathrm{Aff}(K)^*$ is directed, or in other words that it is the linear space generated by its positive cone. So if K is assumed to be a Choquet simplex, then $(\mathrm{Aff}(K)^*, \leq_+)$ is a lattice. Theorem 3.8 states that $(\mathrm{Aff}(K), <^=)$ is an interpolation group.
(ii) $\Rightarrow$ (iii): Suppose that $(\mathrm{Aff}(K), <^=)$ is an interpolation group, and let $f_1, f_2, h_1, h_2 \in \mathrm{Aff}(K)$ such that $f_1, f_2 \leq h_1, h_2$. We will inductively construct a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{Aff}(K)$ satisfying

$$g_{n-1} - 1/2^n <^= g_n <^= g_{n-1} + 1/2^n \text{ for all } n > 1, \text{ as well as}$$
$$f_1 - 1/2^n, f_2 - 1/2^n <^= g_n <^= h_1 + 1/2^n, h_2 + 1/2^n \text{ for all } n.$$

As $f_1 - 1, f_2 - 1 <^= h_1 + 1, h_2 + 1$, we may choose g_0 directly. Suppose that $g_0, \ldots, g_{n-1}$ have already been constructed. Note that

$$f_1 - 1/2^n, f_2 - 1/2^n, g_{n-1} - 1/2^n <^= h_1 + 1/2^n, h_2 + 1/2^n, g_{n-1} + 1/2^n,$$

so we may again make use of interpolation to obtain g_n with the desired properties. This finishes the induction. By the first property, $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{Aff}(K)$ is a Cauchy sequence. Since $\mathrm{Aff}(K)$ is a complete metric space, the sequence converges to some $g \in \mathrm{Aff}(K)$. The second property guarantees that $f_1, f_2 \leq g \leq h_1, h_2$.
(iii) $\Rightarrow$ (i): If $(\mathrm{Aff}(K), \leq)$ is an interpolation group, then $K \cong S(\mathrm{Aff}(K), 1)$ is a Choquet simplex according to Proposition 3.5. □

Theorem 3.9 guarantees the existence of a large stock of continuous affine functions on a Choquet simplex K. However, a more efficient tool for constructing continuous affine functions on K is given by the following theorem, which was first proved by Edwards in [Edw65].

3.10 Theorem For a compact convex set K, the following conditions are equivalent:

(i) K is a Choquet simplex.

(ii) For any upper semicontinuous convex function $f\colon K \to [-\infty, \infty)$ and any lower semicontinuous concave function $h\colon K \to (-\infty, \infty]$ satisfying $f < h$, there exists a continuous affine function $g \in \mathrm{Aff}(K)$ such that $f < g < h$.

(iii) For any upper semicontinuous convex function $f\colon K \to [-\infty, \infty)$ and any lower semicontinuous concave function $h\colon K \to (-\infty, \infty]$ satisfying $f \leq h$, there exists a continuous affine function $g \in \mathrm{Aff}(K)$ such that $f \leq g \leq h$.

Proof. (cf. [Goo86, 11.12 and 11.13]) (i) $\Rightarrow$ (ii): By Proposition 2.22, there exist a function f' that is the pointwise maximum of finitely many continuous affine functions and a function h' that is the pointwise minimum of finitely many continuous affine functions such that $f < f' < h' < h$. Theorem 3.9 implies that $(\mathrm{Aff}(K), <^=)$ has interpolation. Thus, we can find $g \in \mathrm{Aff}(K)$ such that $f' <^= g <^= h'$, implying that $f < g < h$.

(ii) $\Rightarrow$ (iii): We will inductively construct a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{Aff}(K)$ such that

$$g_{n-1} - 1/2^n < g_n < g_{n-1} + 1/2^n \text{ for all } n > 1, \text{ while also}$$
$$f - 1/2^n < g_n < h + 1/2^n \text{ for all } n.$$

As $f - 1 < h + 1$, we can use (ii) to obtain $g_0 \in \mathrm{Aff}(K)$ such that $f - 1 < g_0 < h + 1$. Suppose that we have constructed $g_0, \ldots, g_{n-1}$ with the desired properties. Consider the functions

$$f' := \max\{f - 1/2^n, g_{n-1} - 1/2^n\}$$
$$h' := \min\{h + 1/2^n, g_{n-1} + 1/2^n\}.$$

Notice that f' is upper semicountinuous and convex while h' is lower semicontinuous and concave such that $f' < h'$. Use (ii) to obtain $g_n \in \mathrm{Aff}(K)$ such that $f' < g_n < h$. Then g_n is as desired, finishing the induction. The first property will guarantee that $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{Aff}(K)$ is a Cauchy sequence. Since $\mathrm{Aff}(K)$ is a complete metric space, the sequence converges to some $g \in \mathrm{Aff}(K)$. The second property guarantees that $f \leq g \leq h$.

(iii) $\Rightarrow$ (i): Let $f_1, f_2, h_1, h_2 \in \mathrm{Aff}(K)$ such that $f_1, f_2 \leq h_1, h_2$. Then the pointwise maximum of f_1 and f_2 is continuous and convex while the pointwise minimum of h_1 and h_2 is continuous and concave. We can therefore apply (iii) to obtain $g \in \mathrm{Aff}(K)$ such that $f_1, f_2 \leq g \leq h_1, h_2$. This shows that $(\mathrm{Aff}(K), \leq)$ has interpolation. By Theorem 3.9, K is a Choquet simplex. □

Theorem 3.10 is incredibly useful because lower semicontinuous concave functions (upper semicontinuous convex functions) appear naturally, for example as the pointwise minimum of lower semicontinuous affine functions (pointwise minimum of upper semicontinuous affine functions). Our first application consists of showing that $\mathrm{LAff}(K)$ is a inf-semilattice whenever K is a Choquet simplex.

3.11 Lemma Let K be a Choquet simplex, and suppose that $h\colon K \to (-\infty, \infty]$ is a lower semicontinuous concave function. There exists a biggest lower semicontinuous affine function $f \in \mathrm{LAff}(K)$ such that $f \leq h$.

Proof. Consider the set $\Lambda := \{e \in \mathrm{Aff}(K) \mid e \leq h\}$. The lower semicontinuity of h implies that h is bounded below, so Λ is nonempty. We claim that Λ is upward directed. Given $e_1, e_2 \in \Lambda$, their pointwise maximum is continuous and convex, satisfying $\max\{e_1, e_2\} \leq h$. By Theorem 3.10, there exists $e \in \mathrm{Aff}(K)$ such that $\max\{e_1, e_2\} \leq e \leq h$. It follows that $e \in \Lambda$ and $e_1, e_2 \leq e$.

If $f \in \mathrm{LAff}(K)$ denotes the supremum of Λ, then $f \leq h$. Let $f' \in \mathrm{LAff}(K)$ be another function such that $f' \leq h$. Choose an increasing net $(e_i)_{i \in I} \subseteq \mathrm{Aff}(K)$ with supremum f'. For each i, the element e_i lies in Λ, hence $e_i \leq f$. Passing to the supremum shows that $f' \leq f$, as desired. □

3.12 Proposition For a compact convex set K, the following conditions are equivalent:

(i) K is a Choquet simplex.

(ii) $\mathrm{LAff}(K)$ has interpolation.

(iii) $\mathrm{LAff}(K)$ is an inf-semilattice.

Proof. The implication (iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i): Let $f_1, f_2, h_1, h_2 \in \mathrm{Aff}(K)$ such that $f_1, f_2 < h_1, h_2$. Choose an $\varepsilon > 0$ such that $f_1 + \varepsilon, f_2 + \varepsilon < h_1, h_2$. Since $\mathrm{LAff}(K)$ has interpolation, we can choose $h \in \mathrm{LAff}(K)$ such that $f_1 + \varepsilon, f_2 + \varepsilon \leq h \leq h_1, h_2$. But then $f_1, f_2 \ll h$, hence we can choose an $e \in \mathrm{Aff}(K)$ such that $f_1, f_2 < e < h \leq h_1, h_2$. We have shown that $(\mathrm{Aff}(K), <)$ has interpolation. It is fairly obvious that this is equivalent to the condition that $(\mathrm{Aff}(K), <^=)$ has interpolation. By Theorem 3.9, K is a Choquet simplex.

(i) $\Rightarrow$ (iii): Given $h_1, h_2 \in \mathrm{LAff}(K)$, their pointwise minimum is lower semicontinuous and concave. By Lemma 3.11, there exists a biggest lower semicontinuous affine function f in $\mathrm{LAff}(K)$ such that $f \leq \min\{h_1, h_2\}$. Clearly, f is the infimum of h_1 and h_2 in $\mathrm{LAff}(K)$. □

3.13 Lemma Let K be a Choquet simplex, and let $F \subseteq \partial_e K$ be a compact subset of the extreme boundary. Any continuous function $g\colon F \to \mathbb{R}$ can be extended to a continuous affine function $\tilde{g} \in \mathrm{Aff}(K)$.

Proof. (cf. [Goo86, 11.14]) Define functions $f\colon K \to [-\infty, \infty)$, $h\colon K \to (-\infty, \infty]$ by setting

$$f(x) := \begin{cases} g(x) & \text{if } x \in F \\ -\infty & \text{if } x \notin F \end{cases} \text{ and } h(x) := \begin{cases} g(x) & \text{if } x \in F \\ \infty & \text{if } x \notin F \end{cases}.$$

We claim that h is lower semicontinuous and concave. For any $t \in \mathbb{R}$, the set

$$\begin{aligned} h^{-1}((t,\infty]) &= \{x \in K \mid h(x) > t\} \\ &= \{x \in K \setminus F \mid \infty > t\} \cup \{x \in F \mid g(x) > t\} \\ &= (K \setminus F) \cup g^{-1}((t,\infty]) \end{aligned}$$

is open because F is compact and g is (lower semi-)continuous. Therefore h is lower semicontinuous. Let $x, y \in K$ and $t \in (0,1)$. If $tx+(1-t)y$ does not lie in F, then $h(tx+(1-t)y) = \infty$, so the equation

$$h(tx + (1-t)y) \geq th(x) + (1-t)h(y)$$

is trivially satisfied. On the other hand, if $tx + (1-t)y$ lies in $F \subseteq \partial_e K$, then $x = y$ and the same equation still holds trivially. This proves that h is lower semicontinuous and concave. Similarly, f is upper semicontinuous and convex and $f \leq h$. We use Theorem 3.10 to obtain $\tilde{g} \in \mathrm{Aff}(K)$ such that $f \leq \tilde{g} \leq h$. As $f|_F = h|_F = g$, the function $\tilde{g}$ has no choice but to be an extension of g. □

3.3 Bauer simplices

Lemma 3.13 is particularly useful if $\partial_e K$ is compact to begin with. In this case, every continuous function on the extreme boundary extends to a continuous affine function on K. As a result, there is an abundance of continuous affine functions on K.

3.14 Definition A Bauer simplex is a Choquet simplex K such that $\partial_e K$ is compact.

3.15 Proposition Let K be a Bauer simplex. Every continuous function $g\colon \partial_e K \to \mathbb{R}$ extends uniquely to a continuous affine function $\tilde{g}\colon K \to \mathbb{R}$. The restriction map $\mathrm{Aff}(K) \to \mathcal{C}(\partial_e K;\mathbb{R})$ is an isomorphism of partially ordered abelian groups.
Similarly, every lower semicontinuous function $h\colon \partial_e K \to (-\infty,\infty]$ extends uniquely to a lower semicontinuous affine function $\tilde{h}\colon K \to (-\infty,\infty]$, and the restriction map $\mathrm{LAff}(K) \to \mathrm{Lsc}(\partial_e K)$ is an isomorphism of partially ordered abelian monoids.

Proof. That g extends to a continuous affine function $\tilde{g}$ is an immediate consequence of Lemma 3.13 when applied to $F = \partial_e K$, and the uniqueness follows from Lemma 2.16. It is clear that the restriction map is a bijective OAG-morphism. Using Lemma 2.16 again, we deduce that its inverse is order preserving.
We can express h as the supremum of an increasing net $(h_i)_i$ in $\mathcal{C}(\partial_e K;\mathbb{R})$. It is easy to check that $\tilde{h} := \sup_i \tilde{h}_i$ is the unique extension to an element in $\mathrm{LAff}(K)$. The rest follows just like in the first case. □

3.16 Proposition For a compact convex set K, the following conditions are equivalent:
1) K is a Bauer simplex.
2) $\mathrm{Aff}(K)$ is a lattice.
3) $\mathrm{LAff}(K)$ is a lattice.

Proof. 1) $\Rightarrow$ 3): It was shown in Proposition 3.15 that the restriction map $\mathrm{LAff}(K) \to \mathrm{Lsc}(\partial_e K)$ is an order-isomorphism. Since $\mathrm{Lsc}(\partial_e K)$ is a lattice, it follows that $\mathrm{LAff}(K)$ is a lattice.
3) $\Rightarrow$ 2): It follows from Proposition 3.12 that K is a Choquet simplex. Given $f, g \in \mathrm{Aff}(K)$, their pointwise maximum h is continuous and convex. By definition, we have $f, g \leq \sup\{f, g\}$ (where the supremum is taken in $\mathrm{LAff}(K)$), or equivalently $h \leq \sup\{f, g\}$. Use Theorem 3.10 to find $e \in \mathrm{Aff}(K)$ such that $h \leq e \leq \sup\{f, g\}$. Since $h \leq e$ is again equivalent to $f, g \leq e$, we obtain $\sup\{f, g\} \leq e \leq \sup\{f, g\}$. In particular, $\sup\{f, g\} = e$ is continuous. It follows easily from this that the supremum of f and g in $\mathrm{Aff}(K)$ exists and coincides with $\sup\{f, g\}$. Therefore $\mathrm{Aff}(K)$ is a sup-semilattice. Then $\mathrm{Aff}(K)$ is also a lattice, where the infimum of f and g is given by $\inf\{f, g\} = -\sup\{-f, -g\}$.
2) $\Rightarrow$ 1): The assumption implies that $\mathrm{Aff}(K)$ has interpolation, or equivalently that K is a Choquet simplex (see 3.9). It is shown in [Goo86, 11.17] that a point x in a Choquet simplex K lies in $\partial_e K$ if and only if for all $f, g \in \mathrm{Aff}(K)$, there exists $h \in \mathrm{Aff}(K)$ such that $h \leq f, g$ while also $h(x) = \min\{f(x), g(x)\}$. In our case, this implies that

$$\partial_e K = \{x \in K \mid \inf\{f, g\}(x) = \min\{f(x), g(x)\} \text{ for all } f, g \in \mathrm{Aff}(K)\},$$

which is compact. Thus, K is a Bauer simplex. □

The class of all Bauer simplices is well understood. We will see in the upcoming proposition that up to isomorphism, Bauer simplices are exactly the compact convex sets of the form $M_1^+(X)$, where X is a compact Hausdorff space.

3.17 Proposition If X is a compact Hausdorff space, then $M_1^+(X)$ is a Bauer simplex such that $\partial_e M_1^+(X) \cong X$. Conversely, if K is a Bauer simplex, then $\partial_e K$ is a compact Hausdorff space such that $K \cong M_1^+(\partial_e K)$.

Proof. Suppose that X is a compact Hausdorff space. By Corollary 3.6, $M_1^+(X)$ is a Choquet simplex, and we have $\partial_e M_1^+(X) \cong X$ according to Proposition 2.2. In particular $\partial_e M_1^+(X)$ is compact, hence $M_1^+(X)$ is a Bauer simplex.
Now suppose that K is a Bauer simplex. Then $\partial_e K$ is a compact Hausdorff space by definition. Proposition 3.15 implies that $(\mathrm{Aff}(K), 1) \cong (\mathcal{C}(\partial_e K; \mathbb{R}), 1)$ as groups with order unit. We apply Proposition 2.11 at the first step and Lemma 2.9 at the third step to deduce

$$K \cong S(\mathrm{Aff}(K), 1) \cong S(\mathcal{C}(\partial_e K; \mathbb{R}), 1) \cong M_1^+(\partial_e K),$$

just as desired. □

Remark So far, we have yet to encounter a single concrete example of a Choquet simplex K that is not a Bauer simplex. The so called Poulsen simplex is a metrizable Choquet simplex that is far away from being a Bauer simplex: its exteme points form a dense subset. Also, every metrizable Choquet simplex is affinely homeomorphic to a closed face of K. For more information about the Poulsen simplex, we refer to [LOS78].

3.18 Proposition If K is a Bauer simplex, then $\mathrm{LAff}(K)_+$ is a domain. Let us denote the way-below relation in $\mathrm{LAff}(K)_+$ ($\mathrm{LAff}(K)$) by $\ll_+$ ($\ll$). For $e \in \mathrm{LAff}(K)$, we set $e_+ := \sup\{e, 0\}$. Given $f, h \in \mathrm{LAff}(K)_+$, we have $f \ll_+ h$ if and only if there exists $g \in \mathrm{LAff}(K)$ such that $f \leq g_+$ while also $g \ll h$. If K is metrizable, then $\mathrm{LAff}(K)_+$ is countably based.

Proof. Since $\mathrm{LAff}(K)$ is a domain, the set $\Lambda := \{e \in \mathrm{LAff}(K) \mid e \ll h\}$ is upward directed with supremum h. It follows that the set $\Lambda_+ := \{e_+ \mid e \in \Lambda\} \subseteq \mathrm{LAff}(K)_+$ is also upward directed with supremum h. So if $f \ll_+ h$, then there exists an element $g \in \mathrm{LAff}(K)$ such that $g \ll h$ and $f \leq g_+$. Conversely, assume that there exists a function g as above. Let $D \subseteq \mathrm{LAff}(K)_+$ be upward directed such that $h \leq \sup D$. As $g \ll h$, there exists an element $e \in D$ such that $g \leq e$. It follows that $f \leq g_+ \leq e$, proving that $f \ll_+ h$. We have shown that the claimed characterization of the way-below relation holds, and it follows from the first part that $\mathrm{LAff}(K)_+$ is indeed a domain.
If K is metrizable, then $\mathrm{LAff}(K)$ is countably based according to Proposition 2.24. If B is a countable basis for $\mathrm{LAff}(K)$, it is straightforward to check that $B_+ := \{e_+ \mid e \in B\}$ is a countable basis for $\mathrm{LAff}(K)_+$. Thus, $\mathrm{LAff}(K)_+$ is countably based. □

3.19 Proposition Let K be a metrizable Bauer simplex. Then $\mathrm{LAff}(K)_+$ is a countably based Cu-semigroup.

3.20 Lemma A compact Hausdorff space X is metrizable if and only if $M_1^+(X)$ is metrizable.

Proof. The nontrivial implication is proved in [Goo86, 5.23]. □

If X is a metrizable compact Hausdorff space, it follows from Lemma 3.20 and Proposition 3.19 that $\mathrm{LAff}(M_1^+(X))_+$ is a countably based Cu-semigroup. Since the natural map $\rho\colon \mathrm{LAff}(M_1^+(X))_+ \to \mathrm{Lsc}(X)_+$ is a PoM-isomorphism, it follows that $\mathrm{Lsc}(X)_+$ is a countably based Cu-semigroup and that ρ is a Cu-isomorphism. A similar statement holds for $\mathrm{Lsc}(X)^0_{++}$ (of course, both of these statements can be proved more directly, without making the detour to Bauer simplices). The situation is restated in the following corollary.

3.21 Corollary Let X be a metrizable compact Hausdorff space. Then $\mathrm{Lsc}(X)_+$ and $\mathrm{Lsc}(X)_{++}^0$ are countably based Cu-semigroups. Moreover, the natural maps

$$\mathrm{LAff}(M_1^+(X))_+ \to \mathrm{Lsc}(X)_+ \text{ and}$$
$$\mathrm{LAff}(M_1^+(X))_{++}^0 \to \mathrm{Lsc}(X)_{++}^0$$

are Cu-isomorphisms.

4 A tensor product for compact convex sets and the main theorem

Recall that we have established a rough duality between compact convex sets and partially ordered abelian groups with order unit. To each compact convex set K, we assign the group with order unit $(\mathrm{Aff}(K), 1)$, and to each group with order unit (G, u), we assign the compact convex set $S(G, u)$. There is a fairly obvious way to define a tensor product for groups with order unit. This allows us to define a tensor product of two compact convex sets K_1 and K_2, by taking the tensor product of $(\mathrm{Aff}(K_1), 1)$ and $(\mathrm{Aff}(K_2), 1)$, then applying the state functor.

4.1 The tensor product of groups with order unit

We denote the category of abelian groups with group homomorphisms by AG. If G_1, G_2 and H are abelian groups, a map $G_1 \times G_2 \to H$ is called AG-bimorphism if it is a group homomorphism in each variable. It is well known that the tensor product of abelian groups ($\mathbb{Z}$-modules) exists.

4.1 Definition 1) Let G_1, G_2 and H be ordered abelian groups. A map $\varphi\colon G_1 \times G_2 \to H$ is called OAG-bimorphism if it is a AG-bimorphism and if $\varphi(g_1, g_2)$ is positive whenever g_1 and g_2 are positive.

2) Suppose that (G_1, u_1), (G_2, u_2) and (H, v) are partially ordered abelian groups with order unit. A map $\varphi\colon G_1 \times G_2 \to H$ is called GOU-bimorphism if it is a OAG-bimorphism and if $\varphi(u_1, u_2) = v$. We may write $\varphi\colon (G_1, u_1) \times (G_2, u_2) \to (H, v)$ instead of $\varphi\colon G_1 \times G_2 \to H$.

4.2 Proposition 1) The tensor product of partially ordered abelian groups exists, more precisely: Let G_1 and G_2 be partially ordered abelian groups, and let $(G_1 \otimes^{\mathrm{AG}} G_2, \omega)$ be the tensor product of the underlying abelian groups. We can endow $G_1 \otimes^{\mathrm{AG}} G_2$ with a partial order $\leq$ such that $G_1 \otimes^{\mathrm{OAG}} G_2 := (G_1 \otimes^{\mathrm{AG}} G_2, \leq)$ is a partially ordered abelian group, such that ω is a OAG-bimorphism and such that $(G_1 \otimes^{\mathrm{OAG}} G_2, \omega)$ is the tensor product of G_1 and G_2 as partially ordered abelian groups. Furthermore, the positive cone of $G_1 \otimes^{\mathrm{OAG}} G_2$ is generated by $\omega((G_1)_+, (G_2)_+)$.

2) The tensor product of partially ordered abelian groups with order unit exists, more precisely: Let (G_1, u_1) and (G_2, u_2) be partially ordered abelian groups with order unit, and

let $(G_1 \otimes^{\mathrm{OAG}} G_2, \omega)$ be their tensor product as partially ordered abelian groups with order unit. Then $(G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2) := (G_1 \otimes^{\mathrm{OAG}} G_2, \omega(u_1, u_2))$ is a partially ordered abelian group with order unit, and $((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2), \omega)$ is the tensor product of (G_1, u_1) and (G_2, u_2) as partially ordered abelian groups with order unit.

Proof. For 1), let $C \subseteq G_1 \otimes^{\mathrm{AG}} G_2$ be the cone generated by $\omega((G_1)_+, (G_2)_+)$. The difficulty lies in showing that C is strict. However, this is done in [GH86, 2.1]. It follows that the order induced by C is a partial order. The rest of 1) is straightforward and will be omitted. It is clear that 2) holds. □

4.3 Proposition Let (G_1, u_1) and (G_2, u_2) be two groups with order unit. If s_1 and s_2 are states on (G_1, u_1) and (G_2, u_2) respectively, there is a unique state on $(G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2)$ which maps an elementary tensor $x \odot y$ to $s_1(x) \cdot s_2(y)$. The map

$$S(G_1, u_1) \times S(G_2, u_2) \to S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2))$$

is jointly continuous and affine in each variable. The continuous and affine maps

$$\begin{aligned} S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2)) &\to S(G_1, u_1),\ s \mapsto s(\_ \odot u_2) \\ S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2)) &\to S(G_2, u_2),\ s \mapsto s(u_1 \odot \_) \end{aligned}$$

are referred to as the projections onto the first and the second variable respectively. The composition

$$S(G_1, u_1) \times S(G_2, u_2) \to S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2)) \to S(G_1, u_1) \times S(G_2, u_2)$$

equals the identity. In particular, the first map is injective.

Proof. Let s_1 and s_2 be states on (G_1, u_1) and (G_2, u_2) respectively. The map

$$(G_1, u_1) \times (G_2, u_2) \to (\mathbb{R}, 1),\ (x, y) \mapsto s_1(x) \cdot s_2(y)$$

is a GOU-bimorphism. The corresponding GOU-morphism $(G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2) \to (\mathbb{R}, 1)$ is the desired state. Let us denote the map

$$S(G_1, u_1) \times S(G_2, u_2) \to S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2))$$

by φ. Fix some state $s_2 \in S(G_2, u_2)$. Let $s_1, s_1' \in S(G_1, u_1)$, $t \in [0, 1]$. For every $x \in G_1$ and $y \in G_2$, we have that

$$\begin{aligned} \varphi(t \cdot s_1 + (1-t) \cdot s_1', s_2)(x \odot y) &= (t \cdot s_1 + (1-t) \cdot s_1')(x) \cdot s_2(y) \\ &= t \cdot s_1(x) \cdot s_2(y) + (1-t) \cdot s_1'(x) \cdot s_2(y) \\ &= (t \cdot \varphi(s_1, s_2) + (1-t) \cdot \varphi(s_1', s_2))(x \odot y), \end{aligned}$$

hence $\varphi(t \cdot s_1 + (1-t) \cdot s_1', s_2) = t \cdot \varphi(s_1, s_2) + (1-t) \cdot \varphi(s_1', s_2)$ by uniqueness. Since s_1, s_1' and t were arbitrary, the map $\varphi(\_, s_2)$ is affine. Similarly, φ is affine in the second variable.
For $n \in \{1, 2\}$, let $(s_{n,\lambda})_{\lambda\in\Lambda} \subseteq S(G_n, u_n)$ be a net converging to some $s_n \in S(G_1, u_1)$. By the compactness of $S((G_n, u_n) \otimes^{\mathrm{GOU}} (G_2, u_2))$, the net $(\varphi(s_{1,\lambda}, s_{2,\lambda}))_{\lambda\in\Lambda}$ has at least one convergent subnet. Now suppose that $(\varphi(s_{1,\alpha(i)}, s_{2,\alpha(i)}))_{i\in I}$ is an arbitrary convergent subnet, and let $f \in S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2))$ denote the limit. For every $x \in G_1$ and $y \in G_2$, we have that

$$\begin{aligned} f(x \odot y) &= (\lim_i \varphi(s_{1,\alpha(i)}, s_{2,\alpha(i)}))(x \odot y) = \lim_i (\varphi(s_{1,\alpha(i)}, s_{2,\alpha(i)})(x \odot y)) \\ &= \lim_i (s_{1,\alpha(i)}(x) \cdot s_{2,\alpha(i)}(y)) = s_1(x) \cdot s_2(y) = \varphi(s_1, s_2)(x \odot y). \end{aligned}$$

Using uniqueness again, we deduce $f = \varphi(s_1, s_2)$. We have shown that the net $(\varphi(s_{1,\lambda}, s_{2,\lambda}))_{\lambda\in\Lambda}$ has exactly one limit point, namely $\varphi(s_1, s_2)$. Since $S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2))$ is compact, the net must converge to that point. We have shown that φ is jointly continuous.
It is easy to check that the projections

$$\begin{aligned} S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2)) &\to S(G_1, u_1), \;\; s \mapsto s(\_ \odot u_2) \\ S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2)) &\to S(G_2, u_2), \;\; s \mapsto s(u_1 \odot \_) \end{aligned}$$

are well defined, continuous and affine. It remains to show that the composition

$$S(G_1, u_1) \times S(G_2, u_2) \to S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2)) \to S(G_1, u_1) \times S(G_2, u_2)$$

equals the identity. For $s_1 \in S(G_1, u_1)$ and $s_2 \in S(G_2, u_2)$, we use s to denote their image in $S((G_1, u_1) \otimes^{\mathrm{GOU}} (G_2, u_2))$. Then s is the unique state that maps an elementary tensor $x \odot y$ to the value $s_1(x) \cdot s_2(y)$. For all $x \in G_1$, we have that

$$s(x \odot u_2) = s_1(x) \cdot s_2(u_2) = s_1(x) \cdot 1 = s_1(x),$$

hence $s(\_ \odot u_2) = s_1$. Similarly $s(u_1 \odot \_) = s_2$, so the second map sends s back to (s_1, s_2), finishing the proof. □

4.2 Dimension groups

Before we move on to the tensor product of compact convex sets, let us briefly turn our attention to a structural result by Effros, Handelman and Shen. Apart from 0, the 'simplest' ordered abelian group one can think of is $\mathbb{Z}$ or, more general, finite powers of $\mathbb{Z}$. The category OAG has inductive limits which are based on the algebraic inductive limit. The next step would be to consider the class of inductive limits of finite powers of $\mathbb{Z}$. Theorem 4.6 states that this class coincides with the class of all dimension groups. There is one crucial point in the proof of the main theorem where said theorem is employed.

A partially ordered abelian group G is said to be unperforated if for all $g \in G$, the assumption that some nonzero multiple of g is positive implies that g is positive.

4.4 Definition A partially ordered abelian group G is called simplicial if there exists $n \in \mathbb{N}$ such that G is isomorphic to $\mathbb{Z}^n$ with the usual order and addition.
We say that G is a dimension group if it is directed, unperforated and has the Riesz interpolation property.

4.5 Example If K is a Choquet simplex, then both $(\mathrm{Aff}(K), \leq)$ and $(\mathrm{Aff}(K), <^=)$ are dimension groups.

4.6 Theorem ([EHS80, 2.2]) A partially ordered abelian group is a dimension group if and only if it is the inductive limit of simplicial groups.

4.3 A tensor product for compact convex sets

4.7 Lemma Let K_1 and K_2 be compact convex sets. For any map $\varphi\colon \mathrm{Aff}(K_1) \times \mathrm{Aff}(K_2) \to \mathbb{R}$, the following are equivalent:
1) φ is a OAG-bimorphism $(\mathrm{Aff}(K_1), \leq) \times (\mathrm{Aff}(K_2), \leq) \to (\mathbb{R}, \leq)$.
2) φ is a OAG-bimorphism $(\mathrm{Aff}(K_1), <^=) \times (\mathrm{Aff}(K_2), <^=) \to (\mathbb{R}, \leq)$.
In particular, we see that

$$S((\mathrm{Aff}(K_1), \leq, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), \leq, 1)) = S((\mathrm{Aff}(K_1), <^=, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), <^=, 1)).$$

Proof. The implication 1) $\Rightarrow$ 2) is trivial, so let us show that the implication 2) $\Rightarrow$ 1) holds. Let $f \in \mathrm{Aff}(K_1)_+$, $g \in \mathrm{Aff}(K_2)_+$. For every $n \in N^*$, we have that

$$\begin{aligned}
\varphi(f, g) &= \varphi(f + \tfrac{1}{n}, g + \tfrac{1}{n}) - \varphi(f + \tfrac{1}{n}, \tfrac{1}{n}) - \varphi(\tfrac{1}{n}, g + \tfrac{1}{n}) + \varphi(\tfrac{1}{n}, \tfrac{1}{n}) \\
&= \varphi(f + \tfrac{1}{n}, g + \tfrac{1}{n}) - \tfrac{1}{n}\varphi(f + \tfrac{1}{n}, 1) - \tfrac{1}{n}\varphi(1, g + \tfrac{1}{n}) + \varphi(\tfrac{1}{n}, \tfrac{1}{n}) \\
&\geq 0 - \tfrac{1}{n}\varphi(f + 2, 1) - \tfrac{1}{n}\varphi(1, g + 2) + 0 = -\tfrac{1}{n}(\varphi(f + 2, 1) + \varphi(1, g + 2)).
\end{aligned}$$

Passing to the supremum yields $\varphi(f, g) \geq 0$, as desired. □

4.8 Definition For two compact convex sets K_1 and K_2, we define their tensor product as

$$\begin{aligned}
K_1 \otimes K_2 &:= S((\mathrm{Aff}(K_1), \leq, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), \leq, 1)) \\
&= S((\mathrm{Aff}(K_1), <^=, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), <^=, 1)).
\end{aligned}$$

The equality is given by Lemma 4.7.

Let K_1 and K_2 be two compact convex sets. If we apply Proposition 4.3 to $(\mathrm{Aff}(K_1), 1)$ and $(\mathrm{Aff}(K_2), 1)$, we obtain a map

$$K_1 \times K_2 \xrightarrow{\cong} S(\mathrm{Aff}(K_1), 1) \times S(\mathrm{Aff}(K_2), 1) \to S((\mathrm{Aff}(K_1), 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), 1)) = K_1 \otimes K_2$$

that is jointly continuous, affine in each variable and injective. It maps $(x, y) \in K_1 \times K_2$ to the unique state $x \otimes y$ on $(\mathrm{Aff}(K_1), 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), 1)$ that maps an elementary tensor $f \odot g$ to $f(x) \cdot f(y)$.

Remark The tensor product defined in 4.8 is known as the biprojective tensor product of K_1 and K_2. There are two more notions of tensor products of compact convex sets, the so called biinjective tensor product and the so called projective tensor product. In general, these tensor products disagree. However, they coincide if either K_1 or K_2 is a Choquet simplex. The projective tensor product of K_1 and K_2 has the universal property that continuous and affine maps from the projective tensor product into a different compact convex set K correspond to maps $K_1 \times K_2 \to K$ that are continuous and affine in each variable. For more information about these tensor products, we refer to [NP69] (in that paper, the biprojective tensor product is denoted by $K_1 \square K_2$).

4.9 Theorem Let K_1 and K_2 be compact convex sets. If one of them is a Choquet simplex, then

$$\partial_e(K_1 \otimes K_2) = \{x \otimes y \mid x \in \partial_e K_1,\ y \in \partial_e K_2\}.$$

In particular $\partial_e(K_1 \otimes K_2)$ is homeomorphic to $\partial_e K_1 \times \partial_e K_2$.

Proof. The first statement is proved in [NP69, 2.1]. It follows from the first statement that the map $\partial_e K_1 \times \partial_e K_2 \to \partial_e(K_1 \otimes K_2)$, which maps (x, y) to $x \otimes y$, is well defined and surjective. We know already that tensoring is continuous and injective. Since we are only dealing with compact Hausdorff spaces, this map is a homeomorphism. □

Let us prove that the tensor product of two Choquet simplices is again a Choquet simplex. On the level of groups with order unit, one would hope that the tensor product of two interpolation groups is an interpolation group. Unfortunately, this turns out to be wrong. In [Weh96, 1.5], Wehrung gives an example of two (torsion-free, directed) interpolation groups G and H such that $G \otimes^{\mathrm{OAG}} H$ does not have interpolation. However, if we assume that one of the ordered groups is unperforated, then the tensor product has interpolation, as shown in Proposition 4.11, which is taken from [Weh96, 1.7].

4.10 Proposition ([Weh96, 1.2]) Let G and H be partially ordered abelian groups, and consider the directed subgroups $G^{\mathrm{dir}} := G_+ - G_+$ and $H^{\mathrm{dir}} := H_+ - H_+$. If either G or H is unperforated, then $(G \otimes^{\mathrm{OAG}} H)_+ \cong (G^{\mathrm{dir}} \otimes^{\mathrm{OAG}} H^{\mathrm{dir}})_+$.

4.11 Proposition Suppose that G and H are two interpolation groups, one of which is unperforated. Then $G \otimes^{\mathrm{OAG}} H$ has interpolation.

Proof. Assume that G is unperforated. Recall that $G \otimes^{\mathrm{OAG}} H$ has interpolation if and only if $(G \otimes^{\mathrm{OAG}} H)_+$ has the Riesz decomposition property. Using Proposition 4.10, we may assume that both G and H are directed. In this case, G is a dimension group. By Theorem 4.6, we may express G as an inductive limit of simplicial groups G_i. It is stated in [GH86, 2.2] that the functor $\_ \otimes^{\mathrm{OAG}} H$ preserves inductive limits. It follows that $G \otimes^{\mathrm{OAG}} H \cong \varinjlim(G_i \otimes^{\mathrm{OAG}} H)$. For each i, the ordered group $G_i \otimes^{\mathrm{OAG}} H$ has interpolation (since $\mathbb{Z}^n \otimes^{\mathrm{OAG}} H \cong H^n$ and H has interpolation). It is not hard to see that $G \otimes^{\mathrm{OAG}} H$ must have interpolation as well. □

4.12 Corollary If K_1 and K_2 are Choquet simplices, then $K_1 \otimes K_2$ is a Choquet simplex.

Proof. Since both $\mathrm{Aff}(K_1)$ and $\mathrm{Aff}(K_2)$ are unperforated interpolation groups, it follows from Proposition 4.11 that $\mathrm{Aff}(K_1) \otimes^{\mathrm{OAG}} \mathrm{Aff}(K_2)$ is an interpolation group. Consequently, $K_1 \otimes K_2 = S((\mathrm{Aff}(K_1), 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), 1))$ is a Choquet simplex. □

4.13 Corollary If K_1 and K_2 are Bauer simplices, then $K_1 \otimes K_2$ is a Bauer simplex. In fact $K_1 \otimes K_2 \cong M_1^+(\partial_e K_1 \times \partial_e K_2)$. (We may write this as $M_1^+(X) \otimes M_1^+(Y) \cong M_1^+(X \times Y)$.)

Proof. We know from Corollary 4.12 that $K_1 \otimes K_2$ is a Choquet simplex. Since K_1 and K_2 are Bauer simplices, the product $\partial_e K_1 \times \partial_e K_2$, which according to Theorem 4.9 is homeomorphic to $\partial_e(K_1 \otimes K_2)$, is compact. Therefore $K_1 \otimes K_2$ is a Bauer simplex. We use Proposition 3.17 at the first step to deduce $K_1 \otimes K_2 \cong M_1^+(\partial_e(K_1 \otimes K_2)) \cong M_1^+(\partial_e K_1 \times \partial_e K_2)$. □

4.4 The main theorem

Let G and H be partially ordered abelian groups. The map $G_+ \times H_+ \to (G \otimes^{\mathrm{OAG}} H)_+$, $(g, h) \mapsto g \odot h$ is a PoM-bimorphism that induces a natural PoM-morphism

$$G_+ \otimes^{\mathrm{PoM}} H_+ \to (G \otimes^{\mathrm{OAG}} H)_+,$$

which by the definition of the positive cone in $G \otimes^{\mathrm{OAG}} H$ is always surjective. In general, this map is not a PoM-isomorphism (or equivalently injective). The problem is that $G_+ \otimes^{\mathrm{PoM}} H_+$ may fail to be cancellative. A counterexample can be found in [APT18], just above section B.4.

For compact convex sets K_1 and K_2, consider the following special cases:

$$\mathrm{Aff}(K_1)_+ \otimes^{\mathrm{PoM}} \mathrm{Aff}(K_2)_+ \to ((\mathrm{Aff}(K_1), \leq) \otimes^{\mathrm{OAG}} (\mathrm{Aff}(K_2), \leq))_+,$$
$$\mathrm{Aff}(K_1)^0_{++} \otimes^{\mathrm{PoM}} \mathrm{Aff}(K_2)^0_{++} \to ((\mathrm{Aff}(K_1), <^=) \otimes^{\mathrm{OAG}} (\mathrm{Aff}(K_2), <^=))_+.$$

Also, by definition of $K_1 \otimes K_2$, we have the following two GOU-morphisms (the underlying maps are the same):

$$\iota\colon (\mathrm{Aff}(K_1), \leq, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), \leq, 1) \to (\mathrm{Aff}(K_1 \otimes K_2), \leq, 1),$$
$$\iota\colon (\mathrm{Aff}(K_1), <^{=}, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), <^{=}, 1) \to (\mathrm{Aff}(K_1 \otimes K_2), \leq, 1).$$

For $f \in \mathrm{Aff}(K_1)$ and $g \in \mathrm{Aff}(K_2)$, we set $f \otimes g := \iota(f \odot g)$. Notice that $(f \otimes g)(x \otimes y) = f(x) \cdot g(y)$ holds for all $x \in K_1$ and $y \in K_2$.
In order to prove the main theorem, we study the maps

$$\pi\colon \mathrm{Aff}(K_1)^0_{++} \otimes^{\mathrm{PoM}} \mathrm{Aff}(K_2)^0_{++} \to ((\mathrm{Aff}(K_1), <^{=}) \otimes^{\mathrm{OAG}} (\mathrm{Aff}(K_2), <^{=}))_+,$$
$$\iota\colon (\mathrm{Aff}(K_1), <^{=}, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), <^{=}, 1) \to (\mathrm{Aff}(K_1 \otimes K_2), \leq, 1).$$

In the case of arbitrary groups with order unit, the natural map ι is rarely surjective. We will show that that the image of ι is a dense subspace of $\mathrm{Aff}(K_1 \otimes K_2)$, which is good enough. One would hope that ι, at least as a map

$$\iota\colon (\mathrm{Aff}(K_1), \leq, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), \leq, 1) \to (\mathrm{Aff}(K_1 \otimes K_2), \leq, 1),$$

is an embedding. This is in general not the case, not even for nice compact convex sets. For example, by setting $K_1 = K_2 = M_1^+([0,1])$, ι may be identified with the map

$$\mathcal{C}([0,1]; \mathbb{R}) \otimes^{\mathrm{OAG}} \mathcal{C}([0,1]; \mathbb{R}) \to \mathcal{C}([0,1]^2; \mathbb{R}),$$

which is not an embedding (see [Fre72, 4.7]). Fortunately, we can prove that ι is not far away from being an embedding. Finally, we would like π to be an isomorphism. While it is not known to me whether or not this is the case for arbitrary compact convex sets, we will prove that this is true under the assumption that K_1 or K_2 is a Choquet simplex.
For the remainder of this chapter, we denote the image of ι by $\mathcal{A}$. Also, we denote the image of $\iota \circ \pi$ by $\mathcal{B}$.

4.14 Lemma Let K_1 and K_2 be compact convex sets, and consider the map

$$\omega\colon \mathrm{Aff}(K_1) \times \mathrm{Aff}(K_2) \to \mathrm{Aff}(K_1 \otimes K_2),\ (f, g) \mapsto f \otimes g.$$

For all $f, f' \in \mathrm{Aff}(K_1)$, $g, g' \in \mathrm{Aff}(K_2)$ and $\lambda, \mu \in \mathbb{R}$, the following holds:
1) $1 \otimes 1 = 1$,
2) $(f + f') \otimes g = f \otimes g + f' \otimes g$, and $f \otimes (g + g') = f \otimes g + f \otimes g'$,
3) $f \otimes g \geq 0$ if $f, g \geq 0$, and $f \otimes g > 0$ if $f, g > 0$,
4) $(\lambda f) \otimes (\mu g) = \lambda\mu(f \otimes g)$,

5) ω is continuous in each variable.

Furthermore, if either K_1 or K_2 is a Choquet simplex, it is true that:

6) $\|f \otimes g\| \leq \|f\| \cdot \|g\|$,

7) ω is jointly continuous.

Proof. Points 1), 2) and 3) follow directly from the definition of GOU-bimorphism. Point 4) is clear for $\lambda, \mu \in \mathbb{Q}$. It will follow from 5) that the formula is true for arbitrary $\lambda, \mu \in \mathbb{R}$. In order to prove 5), let $h \in \mathrm{Aff}(K_2)$. For now, let us assume that h is positive. Let $e \in \mathrm{Aff}(K_1)$, $\varepsilon > 0$. Choose $\delta \in \mathbb{Q}$, $\delta > 0$ such that $\delta \cdot \|1 \otimes h\| \leq \varepsilon$. If e' lies in $\mathrm{Aff}(K_1)$ and satisfies $\|e - e'\| \leq \delta$, we have that $-\delta \leq e - e' \leq \delta$. Since h is positive, we obtain

$$(e - e') \otimes h \leq \delta \otimes h = \delta(1 \otimes h) \leq \delta \cdot \|1 \otimes h\| \leq \varepsilon,$$

and similarly $-\varepsilon \leq (e - e') \otimes h$, whence $\|e \otimes h - e' \otimes h\| \leq \varepsilon$. Note that we used 4) at the second step. This is not circular reaoning however, since δ is rational. We have shown that the map $\omega(\_, h)$ is continuous if h is positive. The general case follows by writing h as a difference of positive elements. Thus, ω is continuous in the first variable. By the symmetry of this argument, ω is continuous in each variable.

Now suppose that either K_1 or K_2 is a Choquet simplex. In this case, we have that

$$\partial_e(K_1 \otimes K_2) = \{x \otimes y \mid x \in \partial_e K_1, y \in \partial_e K_2\},$$

according to Theorem 4.9. It follows that

$$\begin{aligned}\|f \otimes g\| &= \max_{z \in K_1 \otimes K_2} |(f \otimes g)(z)| = \max_{z \in \partial_e(K_1 \otimes K_2)} |(f \otimes g)(z)| \\ &= \max_{x \in \partial_e K_1, y \in \partial_e K_2} |(f \otimes g)(x \otimes y)| = \max_{x \in \partial_e K_1, y \in \partial_e K_2} |f(x) \cdot g(y)| \\ &\leq (\max_{x \in \partial_e K_1} |f(x)|) \cdot (\max_{y \in \partial_e K_2} |g(y)|) = (\max_{x \in K_1} |f(x)|) \cdot (\max_{y \in K_2} |g(y)|) = \|f\| \cdot \|g\|,\end{aligned}$$

which proves 6). Then 7) will easily follow from this. □

The following theorem is due to Jellett (see [Jel68, p. 224]). An alternative proof can be found in [Goo86, 7.4].

4.15 Theorem Let K be a compact convex set, and let $V \subseteq \mathrm{Aff}(K)$ be a linear subspace. If V contains 1 and separates the points of K, then V is dense in $\mathrm{Aff}(K)$.

It follows from Lemma 4.14 that $\mathcal{A}$ is a subspace of $\mathrm{Aff}(K_1 \otimes K_2)$ containing 1. It is trivial that $\mathcal{A}$ separates the points of $K_1 \otimes K_2$. Thus, we obtain the following corollary.

4.16 Corollary $\mathcal{A}$ is a dense subspace of $\mathrm{Aff}(K_1 \otimes K_2)$.

4.17 Corollary If K_1 and K_2 are two metrizable compact convex sets, then $K_1 \otimes K_2$ is metrizable.

Proof. Since K_1 and K_2 are metrizable, $\mathrm{Aff}(K_1)$ and $\mathrm{Aff}(K_2)$ are separable according to Proposition 2.24. Suppose that $B_1 \subseteq \mathrm{Aff}(K_1)$ and $B_2 \subseteq \mathrm{Aff}(K_2)$ are countable dense subsets. We claim that the countable set

$$B := \left\{ \sum_{i=0}^{n} f_i^{(1)} \otimes f_i^{(2)} \mid n \in \mathbb{N} \text{ and } f_i^{(j)} \in B_j \text{ for all } i \text{ and } j \right\}$$

is dense in $\mathrm{Aff}(K_1 \otimes K_2)$. The map $\mathrm{Aff}(K_1) \times \mathrm{Aff}(K_2) \to \mathrm{Aff}(K_1 \otimes K_2),\ (f, g) \mapsto f \otimes g$ is continuous in each variable, according to Lemma 4.14. It follows from this that we can approximate each elementary tensor in $\mathrm{Aff}(K_1 \otimes K_2)$ by elements in B. Thus, we can approximate each element in $\mathcal{A}$ by elements in B. Since $\mathcal{A}$ lies dense in $\mathrm{Aff}(K_1 \otimes K_2)$, B must also be a dense subset of $\mathrm{Aff}(K_1 \otimes K_2)$. We have shown that $\mathrm{Aff}(K_1 \otimes K_2)$ is separable. Using Proposition 2.24 a second time, we deduce that $K_1 \otimes K_2$ is metrizable. □

4.18 Proposition Let (G, u) be a nonzero, unperforated partially ordered abelian group with order unit, and consider the natural map $\iota\colon (G, u) \to (\mathrm{Aff}(S(G, u)), 1)$. For $g \in G$, we have $\iota(g) > 0$ if and only if g is an order unit in G.

Proof. [Goo86, 4.13]. □

4.19 Lemma Let G be a partially ordered vector space, and let H be a partially ordered abelian group. There exists a unique OAG-bimorphism

$$m\colon \mathbb{R} \times (G \otimes^{\mathrm{OAG}} H) \to G \otimes^{\mathrm{OAG}} H,$$

that satisfies $m(t, g \odot h) = (t \cdot g) \odot h$ for all $t \in \mathbb{R}$, $g \in G$ and $h \in H$. When equipped with this scalar multiplication, $G \otimes^{\mathrm{OAG}} H$ carries the structure of a partially ordered vector space. In particular, $G \otimes^{\mathrm{OAG}} H$ is unperforated.

Proof. Let $t \in \mathbb{R}$. The map

$$G \times H \to G \otimes^{\mathrm{OAG}} H,\ (g, h) \mapsto (t \cdot g) \odot h$$

is a AG-bimorphism. If t is positive, then this map is a OAG-bimorphism. Let us denote the corresponding AG-morphism $G \otimes^{\mathrm{OAG}} H \to G \otimes^{\mathrm{OAG}} H$ by m_t. Again, if t is positive, then m_t is a OAG-morphism. Then define $m\colon \mathbb{R} \times (G \otimes^{\mathrm{OAG}} H) \to G \otimes^{\mathrm{OAG}} H$ by the formula $m(t, x) := m_t(x)$. Using the universal property of the tensor product, one checks that the equations $m_1 = \mathrm{id}$, $m_{t+t'} = m_t + m_{t'}$ and $m_{t \cdot t'} = m_t \circ m_{t'}$ are satisfied for all $t, t' \in \mathbb{R}$. It follows from this that m has all the properties of a scalar multiplication. Moreover, m is an OAG-bimorphism since m_t is a OAG-morphism whenever t is positive. The claimed uniqueness of m is clear. Thus, $G \otimes^{\mathrm{OAG}} H$ can be considered as a partially ordered vector space. It is trivial that any partially ordered vector space is unperforated. $\square$

Now that we understand ι well enough, let us turn our attention to the map

$$\pi\colon \mathrm{Aff}(K_1)^0_{++} \otimes^{\mathrm{PoM}} \mathrm{Aff}(K_2)^0_{++} \to ((\mathrm{Aff}(K_1), <^=) \otimes^{\mathrm{OAG}} (\mathrm{Aff}(K_2), <^=))_+.$$

More general, we try to find a condition for which the natural map $G_+ \otimes^{\mathrm{PoM}} H_+ \to (G \otimes^{\mathrm{OAG}} H)_+$ is a PoM-isomorphism.

For the upcoming Lemma, we need the following: given an abelian monoid M, let $\mathrm{Gr}(M)$ denote the Grothendieck completion. The universal map $\delta\colon M \to \mathrm{Gr}(M)$ is an embedding if and only if M is cancellative (i.e. $x + z = y + z$ implies that $x = y$, for all $x, y, z \in M$). If M is both cancellative and conical, it follows that $\delta(M)$ is a conical submonoid of $\mathrm{Gr}(M)$. In this case, we can consider the partially ordered abelian group $\mathrm{OGr}(M) := (\mathrm{Gr}(M), \leq_{\delta(M)})$. Observe that $\mathrm{OGr}(M)$ is directed. The map $\delta\colon M \to \mathrm{OGr}(M)_+$ is a AM-isomorphism. If M is a PoM, then δ is a PoM-isomorphism if and only if M is algebraically ordered. Both the category AM and the category PoM have inductive limits which are based on the algebraic inductive limit. Let $\mathrm{OAG_{dir}}$ denote the full subcategory of OAG, consisting of directed ordered abelian groups. It is not too hard to see that $\mathrm{OAG_{dir}}$ is closed under the inductive limit in OAG.

4.20 Lemma Let H be a partially ordered abelian group, and let N be a positively ordered monoid.

1) The functor $\_ \otimes^{\mathrm{OAG}} H\colon \mathrm{OAG} \to \mathrm{OAG}$ preserves inductive limits.
2) The functor $\_ \otimes^{\mathrm{PoM}} N\colon \mathrm{PoM} \to \mathrm{PoM}$ preserves inductive limits.
3) The functor $(\_)_+\colon \mathrm{OAG_{dir}} \to \mathrm{PoM}$ preserves inductive limits.

Proof. 1): This is stated in [GH86, 2.2].

2): The functor $\_ \otimes^{\mathrm{PoM}} N$ is left adjoint to the functor $\mathrm{Hom_{PoM}}(N, \_)\colon \mathrm{PoM} \to \mathrm{PoM}$. It follows from general category theory that $\_ \otimes^{\mathrm{PoM}} N$ is cocontinuous. Since inductive limits are just a special case of a colimit, $\_ \otimes^{\mathrm{PoM}} N$ must preserve inductive limits.

3): Let $(G_i)_i$ be an inductive system in $\mathrm{OAG_{dir}}$, and let $M := \varinjlim (G_i)_+$ be the inductive limit of the positive cones in PoM. Suppose that $x, y, z \in M$ satisfy $x + z = y + z$. It follows from the construction of the inductive PoM-limit that there exists an index k and representatives $x', y', z' \in (G_k)_+$ for x, y, z satisfying $x' + z' = y' + z'$. Since $(G_k)_+$ is cancellative, we obtain that $x' = y'$, whence $x = y$. Thus, M is cancellative. Similarly, one can show that M is conical and algebraically ordered. It follows that M and $\mathrm{OGr}(M)_+$ are naturally isomorphic as PoMs. The following bijections are natural in partially ordered abelian groups A:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{OAG}}(\mathrm{OGr}(\varinjlim(G_i)_+), A) &\cong \mathrm{Hom}_{\mathrm{AM}}(\varinjlim(G_i)_+, A_+) \cong \varprojlim \mathrm{Hom}_{\mathrm{AM}}((G_i)_+, A_+) \\ &\cong \varprojlim \mathrm{Hom}_{\mathrm{OAG}}(G_i, A) \cong \mathrm{Hom}_{\mathrm{OAG}}(\varinjlim G_i, A).\end{aligned}$$

It follows that $\mathrm{OGr}(\varinjlim(G_i)_+) \cong \varinjlim G_i$. Note that the third step relied on the fact that each G_i is directed. If we combine all the isomorphisms, we obtain

$$\varinjlim(G_i)_+ \cong \mathrm{OGr}(\varinjlim(G_i)_+)_+ \cong (\varinjlim G_i)_+$$

as PoMs, which is the desired result. □

4.21 Proposition Let G and H be partially ordered abelian groups, and suppose that one of them is unperforated and has interpolation. Then the natural map

$$G_+ \otimes^{\mathrm{PoM}} H_+ \to (G \otimes^{\mathrm{OAG}} H)_+$$

is a PoM-isomorphism.

Proof. Since the tensor product is commutative, we may assume without loss of generality that G is unperforated and has interpolation. Then, using Proposition 4.10, we may assume without loss of generality that both G and H are directed. In this case, G is a dimension group. Write G as the inductive limit of a system $(G_i)_i$ of simplicial groups. For any $n \in \mathbb{N}$, we have that $\mathbb{Z}^n \otimes^{\mathrm{OAG}} H \cong H^n$. As H is directed, it follows that $G_i \otimes^{\mathrm{OAG}} H$ is directed for each i. Applying Lemma 4.20, we find that

- $\varinjlim(G_i \otimes^{\mathrm{OAG}} H) \cong G \otimes^{\mathrm{OAG}} H$,
- $\varinjlim(G_i \otimes^{\mathrm{OAG}} H)_+ \cong (G \otimes^{\mathrm{OAG}} H)_+$,
- $\varinjlim(G_i)_+ \cong G_+$,
- $\varinjlim((G_i)_+ \otimes^{\mathrm{PoM}} H_+) \cong G_+ \otimes^{\mathrm{PoM}} H_+$.

For every i, the map $(G_i)_+ \otimes^{\mathrm{PoM}} H_+ \to (G_i \otimes^{\mathrm{OAG}} H)_+$ is an isomorphism, because G_i is a simplicial group. Using that this isomorphism is natural at the second step, we deduce

$$G_+ \otimes^{\mathrm{PoM}} H_+ \cong \varinjlim((G_i)_+ \otimes^{\mathrm{PoM}} H_+) \cong \varinjlim(G_i \otimes^{\mathrm{OAG}} H)_+ \cong (G \otimes^{\mathrm{OAG}} H)_+.$$

The composition of these PoM-isomorphisms is given by the natural map. □

Let us summarize our findings in the following corollary.

4.22 Corollary Let K_1 and K_2 be compact convex sets, and consider

$$\pi\colon \mathrm{Aff}(K_1)^0_{++} \otimes^{\mathrm{PoM}} \mathrm{Aff}(K_2)^0_{++} \to ((\mathrm{Aff}(K_1), <^=) \otimes^{\mathrm{OAG}} (\mathrm{Aff}(K_2), <^=))_+,$$
$$\iota\colon (\mathrm{Aff}(K_1), <^=, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), <^=, 1) \to (\mathrm{Aff}(K_1 \otimes K_2), \leq, 1).$$

The image of ι is a dense subspace of $\mathrm{Aff}(K_1 \otimes K_2)$. Set $G \coloneqq (\mathrm{Aff}(K_1), <^=) \otimes^{\mathrm{OAG}} (\mathrm{Aff}(K_2), <^=)$. If $x, y \in G$ satisfy $\iota(x) < \iota(y)$, then $y - x$ lies in G_+. If K_1 or K_2 is a Choquet simplex, then π is a PoM-isomorphism.
Analogous statements hold for

$$\mathrm{Aff}(K_1)_+ \otimes^{\mathrm{PoM}} \mathrm{Aff}(K_2)_+ \to ((\mathrm{Aff}(K_1), \leq) \otimes^{\mathrm{OAG}} (\mathrm{Aff}(K_2), \leq))_+,$$
$$(\mathrm{Aff}(K_1), \leq, 1) \otimes^{\mathrm{GOU}} (\mathrm{Aff}(K_2), \leq, 1) \to (\mathrm{Aff}(K_1 \otimes K_2), \leq, 1).$$

4.23 Lemma Let S be a Cu-semigroup and suppose that $D \subseteq S$ is a dense submonoid, that is for every $s_1, s_2 \in S$ with $s_1 \ll s_2$, there exists $d \in D$ such that $s_1 \leq d \ll s_2$. The restriction of the way below relation in S to D is an auxiliary relation, and $(D, \ll)$ is a W-semigroup. If T is a Cu-semigroup, then every W-morphism $(D, \ll) \to (T, \ll)$ extends uniquely to a Cu-morphism $S \to T$. In particular $\gamma(D, \ll) \cong S$.

Proof. It is clear that that the restriction of $\ll$ onto D is an auxiliary relation, and that $(D, \ll)$ satisfies (W3). The axioms (W1), (W2) and (W4) will be satisfied since D is dense in S.
Suppose that T is a Cu-semigroup and that $\varphi\colon (D, \ll) \to (T, \ll)$ is a W-morphism. Given $s \in S$, choose an increasing sequence $(d_n)_{n\in\mathbb{N}} \subseteq D$ with supremum s. Then $(\varphi(d_n))_{n\in\mathbb{N}}$ is an increasing sequence in T, so it has a supremum which we denote by $\tilde{\varphi}(s)$. We will show that this value does not depend on the choice of the increasing sequence. Let $(e_n)_{n\in\mathbb{N}} \subseteq D$ be another increasing sequence with supremum s. For $k \in \mathbb{N}$, let $t \in T$ such that $t \ll \varphi(d_k)$. By the continuity of φ, we may choose an element $a \in D$ such that $a \ll d_k$ while also $t \leq \varphi(a)$. As $d_k \leq \sup_n d_n = s = \sup_n e_n$, there exists an index i such that $a \leq e_i$. It follows that $t \leq \varphi(a) \leq \varphi(e_i) \leq \sup_n \varphi(e_n)$. Since t was arbitrary, we can deduce that $\varphi(d_k) \leq \sup_n \varphi(e_n)$. Passing to the supremum, we get $\sup_n \varphi(d_n) \leq \sup_n \varphi(e_n)$. This argument is symmetric, hence $\sup_n \varphi(d_n) = \sup_n \varphi(e_n)$.
By the argument above, the map $\tilde{\varphi}\colon S \to T$ is well defined. It is clear that $\tilde{\varphi}$ is an extension of φ. A straightforward, but tedious, computation shows that $\tilde{\varphi}$ is a Cu-morphism. We omit this. Uniqueness follows from the fact that D is dense in S. □

Remark In the situation of Lemma 4.23, the restriction of the way-below relation in S to D agrees with the way-below relation in D, because D is dense in S. One application of Lemma 4.23 is the following: If K is a metrizable compact convex set, then $(\mathrm{Aff}(K)^0_{++}, \ll)$ is a W-semigroup satisfying $\gamma(\mathrm{Aff}(K)^0_{++}, \ll) \cong \mathrm{LAff}(K)^0_{++}$.

We are now in the position to prove the main theorem. Most of the work is done in the following Lemma. Recall that we use $\mathcal{A}$ to denote the image of ι and $\mathcal{B}$ to denote the image of $\iota \circ \pi$.

4.24 Lemma Let K_1 and K_2 be compact convex sets. The following holds:
1) $\mathcal{B} = \mathcal{A} \cap \mathrm{Aff}(K_1 \otimes K_2)^0_{++}$,
2) $\mathcal{B}$ is a dense submonoid of $\mathrm{LAff}(K_1 \otimes K_2)^0_{++}$.
Now suppose that both K_1 and K_2 are metrizable, and that either K_1 or K_2 is a Choquet simplex. Then:
3) $(\mathcal{B}, \ll)$ is a W-semigroup, and we have that

$$\gamma((\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes^{\mathrm{PreW}} (\mathrm{Aff}(K_2)^0_{++}, \ll)) \cong \gamma(\mathcal{B}, \ll) \cong \mathrm{LAff}(K_1 \otimes K_2)^0_{++}.$$

Proof. Set $G := (\mathrm{Aff}(K_1), <^=) \otimes^{\mathrm{OAG}} (\mathrm{Aff}(K_2), <^=)$, and $P := \mathrm{Aff}(K_1)^0_{++} \otimes^{\mathrm{PoM}} \mathrm{Aff}(K_2)^0_{++}$.
1) It follows from Lemma 4.14 that $\mathcal{B}$ is contained in $\mathcal{A} \cap \mathrm{Aff}(K_1 \otimes K_2)^0_{++}$. For the converse inclusion, let $f \in \mathcal{A} \cap \mathrm{Aff}(K_1 \otimes K_2)^0_{++}$, and assume that $f > 0$. Choose $f' \in G$ such that $\iota(f') = f$. Then f' lies in G_+, so the surjectivity of π allows us to find an element $f'' \in P$ such that $\pi(f'') = f'$. But then $(\iota \circ \pi)(f'') = f$, so f lies in $\mathcal{B}$.
2) Clearly $\mathcal{B}$ is a submonoid of $\mathrm{LAff}(K_1 \otimes K_2)^0_{++}$. Suppose that $f, h \in \mathrm{LAff}(K_1 \otimes K_2)^0_{++}$ satisfy $f \ll h$, and assume that $h \neq 0$. Choose $e \in \mathrm{Aff}(K_1 \otimes K_2)$ and $\varepsilon > 0$ such that $f < e - \varepsilon < e < e + \varepsilon < h$. Since $\mathcal{A}$ is dense in $\mathrm{Aff}(K_1 \otimes K_2)$, we can choose an element $e' \in \mathcal{A}$ such that $\|e' - e\| < \varepsilon$. This entails that $-\varepsilon < e' - e < \varepsilon$, implying that $f < e' < h$. Now e' lies in $\mathcal{A} \cap \mathrm{Aff}(K_1 \otimes K_2)^0_{++}$, which is equal to $\mathcal{B}$ by part 1). Additionally, $f \leq e' \ll h$, as desired.
3) Since both K_1 and K_2 are assumed to be metrizable, $K_1 \otimes K_2$ is metrizable according to Corollary 4.17. Then $\mathrm{LAff}(K_1 \otimes K_2)^0_{++}$ is a Cu-semigroup, and it follows from 2) and Lemma 4.23 that $(\mathcal{B}, \ll)$ is a W-semigroup satisfying $\gamma(\mathcal{B}, \ll) \cong \mathrm{LAff}(K_1 \otimes K_2)^0_{++}$. If we assume that either K_1 or K_2 is a Choquet simplex, then $\pi \colon P \to G_+$ is a PoM-isomorphism. Using Lemma 4.14, one can check that the map

$$\psi \colon (\mathrm{Aff}(K_1)^0_{++}, \ll) \times (\mathrm{Aff}(K_2)^0_{++}, \ll) \to (\mathcal{B}, \ll), \ \psi(f, g) := f \otimes g$$

is a W-bimorphism. By the universal property of the tensor product, there exists a unique W-morphism $\tilde{\psi} \colon (\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes^{\mathrm{PreW}} (\mathrm{Aff}(K_2)^0_{++}, \ll) \to (\mathcal{B}, \ll)$ such that the diagram

$$\begin{array}{ccc} (\mathrm{Aff}(K_1)^0_{++}, \ll) \times (\mathrm{Aff}(K_2)^0_{++}, \ll) & \longrightarrow & (\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes^{\mathrm{PreW}} (\mathrm{Aff}(K_2)^0_{++}, \ll) \\ & \overset{\psi}{\searrow} & \downarrow \tilde{\psi} \\ & & (\mathcal{B}, \ll) \end{array}$$

commutes. However, the same diagram commutes if we replace $\tilde{\psi}$ by $\iota \circ \pi$. Using the universal property of the tensor product on the PoM-level, we conclude that $\tilde{\psi} = \iota \circ \pi$. Thus, $\iota \circ \pi$ is a W-morphism.

Set $T := (\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes^{\mathrm{PreW}} (\mathrm{Aff}(K_2)^0_{++}, \ll)$. Let S be a Cu-semigroup, and suppose that $\varphi\colon T \to (S, \ll)$ is a W-morphism. We claim that there exists a unique W-morphism $\tilde{\varphi}\colon (\mathcal{B}, \ll) \to (S, \ll)$ such that $\varphi = \tilde{\varphi} \circ \iota \circ \pi$. Define $\tilde{\varphi}$ as follows: Given $c \in \mathcal{B}$, choose $c' \in P$ such that $(\iota \circ \pi)(c') = c$. Then set $\tilde{\varphi}(c) := \varphi(c')$. Let us show that this value does not depend on the choice of c'. In general, Lemma 4.14 implies that $f \otimes g = 0$, for $f \in \mathrm{Aff}(K_1)^0_{++}$ and $g \in \mathrm{Aff}(K_2)^0_{++}$, can only occur if $f = 0$ or $g = 0$. It follows that the only choice for c' in the case $c = 0$ is $c' = 0$. Now assume that $c > 0$. Let $c'' \in P$ be another element such that $(\iota \circ \pi)(c'') = c$. Let $s \in S$ such that $s \ll \varphi(c')$. By the continuity condition of φ, there exists an element $e' \in T$ such that $e' \prec c'$ (where $\prec$ denotes the auxiliary relation in T) while also $s \leq \varphi(e')$. As $\iota \circ \pi$ preserves the auxiliary relation, we see that $(\iota \circ \pi)(e') \ll (\iota \circ \pi)(c') = (\iota \circ \pi)(c'') \neq 0$, hence $(\iota \circ \beta)(e') < (\iota \circ \beta)(c'')$. We use Corollary 4.22 to deduce $\beta(e') \leq \beta(c'')$, and then $e' \leq c''$. It follows that $s \leq \varphi(e') \leq \varphi(c'')$. Passing to the supremum yields $\varphi(c') \leq \varphi(c'')$. Then, by the symmetry of the argument, we conclude that $\varphi(c') = \varphi(c'')$. This shows that $\tilde{\varphi}$ is well defined. Clearly, $\tilde{\varphi}$ is the only map such that $\varphi = \tilde{\varphi} \circ \iota \circ \beta$, so it remains to show that $\tilde{\varphi}$ is a PreW-morphism. It is clear that $\tilde{\varphi}$ is a monoid homomorphism. Also, $\tilde{\varphi}$ satisfies the continuity condition since φ does. Let $c, d \in (\mathcal{B}, \ll)$ such that $c \ll d$. We will show that $\tilde{\varphi}(c) \ll \tilde{\varphi}(d)$. This is clear for $d = 0$, so assume that $d > 0$. Choose $c', d' \in P$ such that $(\iota \circ \pi)(c') = c$ and $(\iota \circ \pi)(d') = d$. We may find an element $e' \in P$ such that $e' \prec d'$ while also $(\iota \circ \beta)(c') = c < (\iota \circ \beta)(e')$. It follows again that $c' \leq e' \prec d'$, thus $\tilde{\varphi}(c) = \varphi(c') \ll \varphi(d') = \tilde{\varphi}(d)$. Finally, let $c, d \in (\mathcal{B}, \ll)$ such that $c \leq d$. Let us prove that $\tilde{\varphi}(c) \leq \tilde{\varphi}(d)$. As $(\mathcal{B}, \ll)$ is a W-semigroup, we find a $\ll$-increasing sequence $(c_n)_n$ in $\mathcal{B}$ with supremum c. By the previous part, the sequence $(\tilde{\varphi}(c_n))_n \subseteq S$ is increasing, and the inequality $\sup_n \tilde{\varphi}(c_n) \leq \tilde{\varphi}(c)$ holds. Let $s \in S$ such that $s \ll \tilde{\varphi}(c)$. By the continuity condition of $\tilde{\varphi}$, there exists an index k such that $s \leq \tilde{\varphi}(c_k) \leq \sup_n \tilde{\varphi}(c_n)$. Since s was arbitrary, we get $\tilde{\varphi}(c) \leq \sup_n \tilde{\varphi}(c_n)$. It follows that $\tilde{\varphi}(c) = \sup_n \tilde{\varphi}(c_n) \leq \tilde{\varphi}(d)$. This proves the claim.

Overall, we have shown that there are bijections

$$
\begin{aligned}
&\mathrm{Hom}_{\mathrm{Cu}}(\gamma((\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes^{\mathrm{PreW}} (\mathrm{Aff}(K_2)^0_{++}, \ll)), S) \\
\cong &\mathrm{Hom}_{\mathrm{PreW}}((\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes^{\mathrm{PreW}} (\mathrm{Aff}(K_2)^0_{++}, \ll), (S, \ll)) \\
\cong &\mathrm{Hom}_{\mathrm{PreW}}((\mathcal{B}, \ll), (S, \ll)) \\
\cong &\mathrm{Hom}_{\mathrm{Cu}}(\gamma(\mathcal{B}, \ll), S)
\end{aligned}
$$

which are natural in Cu-semigroups S. It follows that

$$
\gamma((\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes^{\mathrm{PreW}} (\mathrm{Aff}(K_2)^0_{++}, \ll)) \cong \gamma(\mathcal{B}, \ll),
$$

as desired. □

4.25 Theorem Let K_1 and K_2 be two metrizable compact convex sets. If either K_1 or K_2 is a Choquet simplex, there is a Cu-isomorphism

$$\mathrm{LAff}(K_1)^0_{++} \otimes \mathrm{LAff}(K_2)^0_{++} \cong \mathrm{LAff}(K_1 \otimes K_2)^0_{++}.$$

Proof. We use Lemma 4.23 at the first step, Theorem 1.25 at the second step and Lemma 4.24 at the third step to obtain

$$\begin{aligned}\mathrm{LAff}(K_1)^0_{++} \otimes \mathrm{LAff}(K_2)^0_{++} &\cong \gamma(\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes \gamma(\mathrm{Aff}(K_2)^0_{++}, \ll)\\ &\cong \gamma((\mathrm{Aff}(K_1)^0_{++}, \ll) \otimes^{\mathrm{PreW}} (\mathrm{Aff}(K_1)^0_{++}, \ll))\\ &\cong \mathrm{LAff}(K_1 \otimes K_2)^0_{++},\end{aligned}$$

which is the desired result. □

4.26 Corollary Let X and Y be metrizable compact Hausdorff spaces. There is a natural Cu-morphism

$$\mathrm{Lsc}(X)^0_{++} \otimes \mathrm{Lsc}(Y)^0_{++} \cong \mathrm{Lsc}(X \times Y)^0_{++}.$$

Proof. As X is a metrizable compact Hausdorff space, $M_1^+(X)$ is a metrizable Bauer simplex and satisfies $\mathrm{Lsc}(X)^0_{++} \cong \mathrm{LAff}(M_1^+(X))^0_{++}$. An analogous statement holds for Y. According to Corollary 4.13, we have that $M_1^+(X) \otimes M_1^+(Y) \cong M_1^+(X \times Y)$. We apply the main theorem at the second step to deduce

$$\begin{aligned}\mathrm{Lsc}(X)^0_{++} \otimes \mathrm{Lsc}(Y)^0_{++} &\cong \mathrm{LAff}(M_1^+(X))^0_{++} \otimes \mathrm{LAff}(M_1^+(Y))^0_{++}\\ &\cong \mathrm{LAff}(M_1^+(X) \otimes M_1^+(Y))^0_{++}\\ &\cong \mathrm{LAff}(M_1^+(X \times Y))^0_{++}\\ &\cong \mathrm{Lsc}(X \times Y)^0_{++},\end{aligned}$$

as desired. □

5 The Cu-semigroup $L(F(S))$

In this chapter, our goal is to determine which Cu-semigroups are of the form $\mathrm{LAff}(K)^0_{++}$, for some metrizable compact convex set K. This problem requires us to come up with a meaningful way to assign to each Cu-semigroup (with a few additional properties) a compact convex set. In the case of groups with order unit, the compact convex set consists of all normalized positive homomorphisms into $\mathbb{R}$. Similarly, in the case of Cu-semigroups, the compact convex set consists of all functionals that are normalized at a certain element in the Cu-semigroup. Our main reference for this chapter is [Rob13].

5.1 Functionals

Recall that a functional on a Cu-semigroup S is a generalized Cu-morphism $\lambda\colon S \to [0,\infty]$, i.e. a map that preserves the zero element, addition, order and suprema of increasing sequences. There are two trivial functionals on S: the functional λ_0 that maps every element to zero and the functional λ_∞ that maps every nonzero element to ∞. We denote the set of functionals on S by $F(S)$. When equipped with pointwise order and addition, $F(S)$ becomes a positively ordered monoid. Any upward directed subset of $F(S)$ has a supremum, which is just given by the pointwise supremum. Thus, $F(S)$ is a dcpo, and the dcpo-structure is compatible with addition. The formula $(t \cdot \lambda)(s) := t \cdot \lambda(s)$ (for $t \in (0,\infty)$, $\lambda \in F(S)$ and $s \in S$) defines a scalar multiplication $(0,\infty) \times F(S) \to F(S)$. We extend this scalar multiplication to $[0,\infty] \times F(S)$ by setting

$$(0 \cdot \lambda)(s) := \begin{cases} 0 & \text{if } \lambda(s') < \infty \text{ for all } s' \ll s \\ \infty & \text{otherwise} \end{cases}, \quad (\infty \cdot \lambda)(s) := \begin{cases} 0 & \text{if } \lambda(s) = 0 \\ \infty & \text{otherwise} \end{cases},$$

for all $\lambda \in F(S)$ and $s \in S$. Finally, we can endow $F(S)$ with a certain topology that turns $F(S)$ into a compact Hausdorff space. This is done in the next Theorem.

5.1 Theorem ([ERS11, 4.8]) Let S be a Cu-semigroup. There exists a topology on $F(S)$ in which a net $(\lambda_i)_i \subseteq F(S)$ converges to a functional $\lambda \in F(S)$ if and only if for all $s', s \in S$ satisfying $s' \ll s$, the inequality

$$\limsup_i \lambda_i(s') \leq \lambda(s) \leq \liminf_i \lambda_i(s)$$

holds. When equipped with this topology, $F(S)$ becomes a compact Hausdorff space. If S is countably based, then $F(S)$ is second countable, hence metrizable.

5.2 Lemma Let S be a Cu-semigroup. Addition $F(S) \times F(S) \to F(S)$ and scalar multiplication $(0,\infty) \times F(S) \to F(S)$ are jointly continuous.

The proof of Lemma 5.2 is elementary and will be omitted. In [ERS11], it is erroneously stated that the scalar multiplication $[0,\infty] \times F(S) \to F(S)$ is jointly continuous (where $[0,\infty]$ is equipped with the obvious topology that makes it homeomorphic to $[0,1]$). For $S = [0,\infty]$ and $t \in [0,\infty]$, let λ_t be the unique functional on S that maps 1 to t. It is easy to see that the sequence $(\lambda_{\frac{1}{n}})$ converges to λ_0, yet $\infty \cdot \lambda_{\frac{1}{n}} = \lambda_\infty$ does not converge to $\infty \cdot \lambda_0 = \lambda_0$. This counterexample was taken from [APT18, 5.2.1]. Similarly, the sequence (λ_n) converges to λ_∞, but $0 \cdot \lambda_n = \lambda_0$ does not converge to $0 \cdot \lambda_\infty = \lambda_\infty$. Thus, scalar multilpication with 0 or with ∞ is discontinuous.

We would like to study maps $\varphi\colon F(S) \to [0,\infty]$ that preserve the structure of $F(S)$. Possible requirements for φ could include:

1) φ is a monoid homomorphism.
2) φ is homogeneous for elements in $(0,\infty)$.
3) φ is lower semicontinuous.
4) φ is order-preserving.
5) φ preserves suprema of upward directed subsets of $F(S)$.

In [Rob13], Robert considers maps $\varphi\colon F(S) \to [0,\infty]$ that satisfy the conditions 1), 2) and 3). He denotes the set of all these maps by $\mathrm{Lsc}(F(S))$ (this is not to be confused with the definition of $\mathrm{Lsc}(X)$ how it appears in this thesis). We will consider a different set: let $\mathrm{LPoM}(F(S))$ denote the set of all lower semicontinuous PoM-morphisms $\varphi\colon F(S) \to [0,\infty]$, i.e. all φ that satisfy the conditions 1), 3) and 4). We claim that such a φ automatically satisfies conditions 2) and 5). Let $(\lambda_i)_i \subseteq F(S)$ be an increasing net of functionals, and let us denote the supremum by λ. For all $s', s \in S$ with $s' \ll s$, we have that

$$\limsup_i \lambda_i(s') = \sup_i \lambda_i(s') = \lambda(s') \leq \lambda(s) = \sup_i \lambda_i(s) = \liminf_i \lambda_i(s),$$

because the net is increasing. This shows that $(\lambda_i)_i$ converges to λ. Since φ is order-preserving, the net $(\varphi(\lambda_i))_i \subseteq [0,\infty]$ is increasing, and $\varphi(\lambda)$ is an upper bound. We use this at the second and third step and the lower semicontinuity of φ at the first step to deduce

$$\varphi(\lambda) \leq \liminf_i \varphi(\lambda_i) = \sup_i \varphi(\lambda_i) \leq \varphi(\lambda),$$

hence $\varphi(\lambda) = \sup_i \varphi(\lambda_i)$. This proves that φ satifies condition 5). Let $\mu \in F(S)$, $t \in (0,\infty)$.

It follows from 1) that $\varphi(t \cdot \mu) = t \cdot \varphi(\mu)$ holds if t is rational. The general case follows using condition 5). Thus, φ also satisfies condition 2).

This shows that our $\mathrm{LPoM}(F(S))$ is contained in Robert's $\mathrm{Lsc}(F(S))$. It is proved in [Rob13, 2.2.3] that $F(S)$ is algebraically ordered if S satisfies $(\mathcal{O}5')$ (the weak form of almost algebraic order). In this case, every monoid homomorphism from $F(S)$ into $[0,\infty]$ (or any other positively ordered monoid) is automatically order-preserving. It follows that Robert's $\mathrm{Lsc}(F(S))$ agrees with our $\mathrm{LPoM}(F(S))$ if S satisfies $(\mathcal{O}5')$.

We equip $\mathrm{LPoM}(F(S))$ with pointwise order and addition, giving it the structure of a positively ordered monoid. Again, the pointwise supremum of any upward directed subset of $\mathrm{LPoM}(F(S))$ lies in $\mathrm{LPoM}(F(S))$. Therefore $\mathrm{LPoM}(F(S))$ is a dcpo, and the dcpo-structure is compatible with the addition. Also, we equip $\mathrm{LPoM}(F(S))$ with the obvious scalar multiplication $(0,\infty) \times \mathrm{LPoM}(F(S)) \to \mathrm{LPoM}(F(S))$. This scalar multiplication satisfies all the compatibility conditions you would hope for.

Any element $s \in S$ defines a map $\hat{s}\colon F(S) \to [0,\infty]$ by setting $\hat{s}(\lambda) := \lambda(s)$. It is easy to check that $\hat{s}$ lies in $\mathrm{LPoM}(F(S))$ and that the canonical map $S \to \mathrm{LPoM}(F(S))$, $s \mapsto \hat{s}$ is a PoM-morphism that preserves suprema of increasing sequences. If s is compact, then $\hat{s}$ is continuous.

5.2 The realification of a Cu-semigroup

It is natural to ask whether we can recover S from $\mathrm{LPoM}(F(S))$ or more general from $F(S)$. The answer to this question is no, because different Cu-semigroups may have the same functionals. Consider for example the Cu-semigroups $\overline{\mathbb{N}}$ and $[0,\infty]$. Any functional on $\overline{\mathbb{N}}$ extends uniquely to a functional on $[0,\infty]$. The underlying reason for this is that any functional $\lambda\colon [0,\infty] \to [0,\infty]$ satisfies $\lambda(t) = t \cdot \lambda(1)$ for any $t \in (0,\infty)$; hence λ is uniquely determined by the value $\lambda(1)$. Thus, there are canonical bijections $F([0,\infty]) \cong F(\overline{\mathbb{N}}) \cong [0,\infty]$, and these bijections preserve all relevant structures.

More general, if S is a Cu-semigroup and if there is a meaningful way to multiply elements in S with elements in $(0,\infty)$, the equation $\lambda(t \cdot s) = t \cdot \lambda(s)$ holds for all $t \in (0,\infty)$, $s \in S$ and $\lambda \in F(S)$. This is captured in the upcoming definition. We can assign to each Cu-semigroup S another Cu-semigroup $R(S)$ for which there is a meaningful way of multiplying with elements in $(0,\infty)$ and which has a certain universal property. It will follow from this universal property that $F(R(S)) \cong F(S)$. We refer to $R(S)$ as the realification of S. In the example above, $[0,\infty]$ is the realification of $\overline{\mathbb{N}}$. The realification of S turns out to be a useful tool for studying the natural map $S \to \mathrm{LPoM}(F(S))$.

The following definition is a special case of [APT18, 7.1.3].

5.3 Definition and Proposition Let S be a Cu-semigroup. A $[0,\infty]$-multiplication on S is a Cu-bimorphism $m\colon [0,\infty]\times S\to S$ such that the equations

$$m(t_1, m(t_2, s)) = m(t_1\cdot t_2, s) \text{ and } m(1, s) = s$$

hold for all $t_1, t_2 \in [0,\infty]$ and $s\in S$. A $[0,\infty]$-multiplication is unique: if m and m' are two $[0,\infty]$-multiplications on S, then $m = m'$. Moreover, for any $n\in\mathbb{N}$ and $s\in S$, we have that $m(n,s) = n\cdot s$. It is therefore justified to write $t\cdot s$ instead of $m(t,s)$ (for $t\in[0,\infty]$ and $s\in S$).
A Cu-semigroup S is said to have $[0,\infty]$-multiplication if there exists a $[0,\infty]$-multiplication on S. Since the $[0,\infty]$-multiplication on S is unique, we will regard it as a property of S rather than part of the data.
Let S and T be two Cu-semigroups with $[0,\infty]$-multiplication. Any generalized Cu-morphism $\varphi\colon S\to T$ preserves the $[0,\infty]$-multiplications, i.e. $\varphi(t\cdot s) = t\cdot\varphi(s)$ holds for any $t\in[0,\infty]$ and $s\in S$.

Proof. For $i=1,2$, suppose that m_i is a real multiplication on a Cu-semigroup S_i. Additionally, let $\varphi\colon S_1\to S_2$ be a generalized Cu-morphism. Let $s\in S_1$. For any $n\in\mathbb{N}$, we have that

$$m_1(n,s) = m_1(\sum_{i=1}^{n} 1, s) = \sum_{i=1}^{n} m_1(1,s) = \sum_{i=1}^{n} s = n\cdot s$$

and similarly $m_2(n,\varphi(s)) = n\cdot\varphi(s)$, hence

$$\varphi(m_1(n,s))) = \varphi(n\cdot s) = n\cdot\varphi(s) = m_2(n,\varphi(s)).$$

Now let $n\in\mathbb{Z}$ and $k\in\mathbb{N}^*$. The computation

$$\begin{aligned}\varphi(m_1(k^{-1}n,s)) &= m_2(k^{-1}, m_2(k,\varphi(m_1(k^{-1}n,s)))) = m_2(k^{-1},\varphi(m_1(k, m_1(k^{-1}n,s))))\\ &= m_2(k^{-1},\varphi(m_1(n,s))) = m_2(k^{-1}, m_2(n,\varphi(s))) = m_2(k^{-1}n,\varphi(s))\end{aligned}$$

shows that the formula $\varphi(m_1(q,s)) = m_2(q,\varphi(s))$ holds for every rational number q. Finally, let $t\in\mathbb{R}$. Choose an increasing sequence $(q_n)_{n\in\mathbb{N}}$ of rational numbers with supremum t. We have that

$$\varphi(m_1(t,s)) = \varphi(\sup_n m_1(q_n,s)) = \sup_n \varphi(m_1(q_n,s)) = \sup_n m_2(q_n,\varphi(s)) = m_2(t,\varphi(s)),$$

proving that φ preserves the $[0,\infty]$-multiplications.
In the special case of $S_1 = S_2 = S$ and $\varphi = \mathrm{id}$, this shows that the real multiplication on S is unique. □

5.4 Example The Cu-semigroup $[0,\infty]$ has $[0,\infty]$-multiplication, where multiplication with 0 and ∞ is defined by $0\cdot s := 0$, for all $s\in[0,\infty]$, and $\infty\cdot s = \infty$, for all nonzero s.
If K is a metrizable compact convex set, then $\mathrm{LAff}(K)^0_{++}$ has $[0,\infty]$-multiplication, by setting $(t\cdot f)(x) := t\cdot f(x)$, for all $t\in[0,\infty]$, $f\in\mathrm{LAff}(K)^0_{++}$ and $x\in K$.

5.5 Theorem For every Cu-semigroup S, there exists a Cu-semigroup $R(S)$ with $[0,\infty]$-multiplication and a generalized Cu-morphism $\gamma\colon S \to R(S)$ such that the following universal property holds:

For any Cu-semigroup T that has $[0,\infty]$-multiplication and for any generalized Cu-morphism $\varphi\colon S \to T$, there exists a unique generalized Cu-morphism $\tilde{\varphi}\colon R(S) \to T$ satisfying $\varphi = \tilde{\varphi} \circ \gamma$. If φ is a Cu-morphism, then $\tilde{\varphi}$ is a Cu-morphism as well.

As usual, $R(S)$ is uniquely determined up to Cu-isomorphism. In fact, $(R(S), \gamma)$ can be realized as $[0,\infty] \otimes S$ and $\gamma\colon S \to [0,\infty] \otimes S, \gamma(s) = 1 \otimes s$. Thus, S has $[0,\infty]$-multiplication if and only if $S \cong [0,\infty] \otimes S$. We refer to $(R(S), \gamma)$ as the realification of S.

Proof. Set $R(S) \coloneqq [0,\infty] \otimes S$, and define $\gamma\colon S \to R(S)$ by the formula $\gamma(s) \coloneqq 1 \otimes s$. It is clear that γ is a generalized Cu-morphism. To show that $R(S)$ has $[0,\infty]$-multiplication, let $t \in [0,\infty]$. The map $[0,\infty] \times S \to R(S)$, $(\alpha, s) \mapsto (t \cdot \alpha) \otimes s$ is a generalized Cu-bimorphism. Let us denote the corresponding generalized Cu-morphism $R(S) \to R(S)$ by m_t. Then we define $m\colon [0,\infty] \times R(S) \to R(S)$ by setting $m(t,x) \coloneqq m_t(x)$. If $\alpha \otimes s \in R(S)$ is an elementary tensor, we have that $m(t, \alpha \otimes s) = (t \cdot \alpha) \otimes s$. It is fairly straightforward, but tedious, to show that m is indeed a $[0,\infty]$-multiplication on $R(S)$. We will omit the argument.

Let T be a Cu-semigroup with $[0,\infty]$-multiplication, and let $\varphi\colon S \to T$ be a generalized Cu-morphism. The map $[0,\infty] \times S \to T$, which is defined by $(\alpha, s) \mapsto \alpha \cdot \varphi(s)$, is a generalized Cu-bimorphism. The corresponding generalized Cu-morphism $\tilde{\varphi}\colon R(S) \to T$ clearly satisfies $\varphi = \tilde{\varphi} \circ \gamma$, so only uniqueness remains to be shown. Let $\psi\colon R(S) \to T$ be another generalized Cu-morphism satisfying $\varphi = \psi \circ \gamma$. Using that any generalized Cu-morphism preserves $[0,\infty]$-multiplication at the second step, we deduce

$$\psi(\alpha \otimes s) = \psi(\alpha \cdot (1 \otimes s)) = \alpha \cdot \psi(1 \otimes s) = \alpha \cdot \varphi(s) = \tilde{\varphi}(\alpha \otimes s),$$

for any elementary tensor $\alpha \otimes s \in R(S)$, implying that $\psi = \tilde{\varphi}$. □

Since the Cu-semigroup $[0,\infty]$ has $[0,\infty]$-multiplication, it follows from the universal property of the realification of a Cu-semigroup S that there exists a natural bijection $F(S) \cong F(R(S))$. This bijection is compatible with all the relevant structures (this is easy, albeit not entirely trivial). The induced map $\mathrm{LPoM}(F(S)) \cong \mathrm{LPoM}(F(R(S)))$ is also a bijection that is compatible with all the structures. The diagram

$$\begin{array}{ccc} S & \longrightarrow & \mathrm{LPoM}(F(S)) \\ \big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ R(S) & \longrightarrow & \mathrm{LPoM}(F(R(S))) \end{array}$$

is commutative. From now on, we will not differentiate between $F(S)$ and $F(R(S))$, between

$\mathrm{LPoM}(F(S))$ and $\mathrm{LPoM}(F(R(S)))$ and between the maps $S \to \mathrm{LPoM}(F(S))$ and $R(S) \to \mathrm{LPoM}(F(S))$.

5.3 The range of the natural map

Given two maps $f, g\colon F(S) \to [0, \infty]$, we write $f \lhd g$ if there exists some $\varepsilon > 0$ such that $f \leq (1-\varepsilon)g$ and if f is continuous at all points where g is finite. In [Rob13], Robert uses $L(F(S))$ to denote the set of all $f \in \mathrm{Lsc}(F(S))$ for which there exists a $\lhd$-increasing sequence in $\mathrm{Lsc}(F(S))$ with supremum f (where $\mathrm{Lsc}(F(S))$ is defined as in [Rob13]). In this thesis, we use $L(F(S))$ to denote the set of all $f \in \mathrm{LPoM}(F(S))$ for which there exists a $\lhd$-increasing sequence in $\mathrm{LPoM}(F(S))$ with supremum f. Again, in general our $L(F(S))$ is contained in Robert's $L(F(S))$, and both agree if S satisfies ($\mathcal{O}$5').

If $f \lhd g$, then $f \ll g$, where $\ll$ denotes the sequential way-below relation in $\mathrm{LPoM}(F(S))$. We refer to [ERS11, 5.1]. An adapted proof shows that this is even true for the non-sequential way-below relation. One can show that $L(F(S))$ is a submonoid of $\mathrm{LPoM}(F(S))$ that is closed under scalar multiplication and suprema of increasing sequences.

The bijection $\mathrm{LPoM}(F(S)) \cong \mathrm{LPoM}(F(R(S)))$ is compatible with the $\lhd$-relations, so it restricts to a bijection $L(F(S)) \cong L(F(R(S)))$. Thus, we will not differentiate between $L(F(S))$ and $L(F(R(S)))$.

Lemma 5.6 and Proposition 5.7 are adapted from [Rob13, 3.1.6].

5.6 Lemma Let S be a Cu-semigroup with $[0,\infty]$-multiplication. If $a, c \in S$ satisfy $a \ll c$, then there exists $b \in S$ such that $a \leq b \ll c$ while also $\hat{a} \leq \hat{b} \lhd \hat{c}$.

Proof. Choose some element $d \in S$ and $\varepsilon > 0$ such that $a \ll d \ll (1-\varepsilon)c \leq c$. Then, by induction, we may choose elements $s_r \in S$, where r runs through the dyadic rationals in $[0,1]$, such that $s_0 = a$, $s_1 = d$ and $s_r \ll s_{r'}$ whenever $r < r'$. For $n \in \mathbb{N}^*$, set

$$b_n := \frac{1}{2^n} \cdot \sum_{k=0}^{2^n-1} s_{\frac{k}{2^n}} \quad \text{and} \quad \tilde{b}_n := \frac{1}{2^n} \cdot \sum_{k=1}^{2^n} s_{\frac{k}{2^n}}.$$

Although not entirely obvious, it is easy to check that the following conditions are satisfied:

- The sequence $(b_n)_n \subseteq S$ is $\ll$-increasing.
- The sequence $(\tilde{b}_n)_n \subseteq S$ is $\ll$-decreasing.
- For every n, the inequality $a \leq b_n \leq \tilde{b}_n \leq d$ holds.

Let $b := \sup_n b_n \in S$. Then $a \leq b \leq d \ll (1-\varepsilon)c \leq c$, whence $a \leq b \ll c$ and $\hat{a} \leq \hat{b} \leq (1-\varepsilon)\hat{c}$. It remains to show that $\hat{b}$ is continuous at each point where $\hat{c}$ is finite. Let $\lambda \in F(S)$ such that $\hat{c}(\lambda) < \infty$, and let $(\lambda_i)_i \subseteq F(S)$ be a net that converges to λ. For any $m \in \mathbb{N}^*$, we have that

$b_m \ll b$, hence

$$\limsup_i \widehat{b_m}(\lambda_i) = \limsup_i \lambda_i(b_m) \leq \lambda(b) = \hat{b}(\lambda)$$

by the description of the convergence in $F(S)$. As $d \ll c$, we obtain $\limsup_i \hat{d}(\lambda_i) \leq \hat{c}(\lambda)$ with the same reasoning. Moreover, we have that $b \leq \tilde{b}_m \leq b_m + \frac{d}{2^m}$ (we used that the sequence $(\tilde{b}_n)_n$ is decreasing at the first step), hence $\hat{b} \leq \widehat{b_m} + \frac{1}{2^m}\hat{d}$. We obtain that

$$\limsup_i \hat{b}(\lambda_i) \leq (\limsup_i \widehat{b_m}(\lambda_i)) + \frac{1}{2^m}(\limsup_i \hat{d}(\lambda_i)) \leq \hat{b}(\lambda) + \frac{\hat{c}(\lambda)}{2^m}.$$

Since $\hat{c}(\lambda)$ is finite and m is arbitrary, we conclude that $\limsup_i \hat{b}(\lambda_i) \leq \hat{b}(\lambda)$. This shows that $\hat{b}$ is upper semicontinuous (and hence continuous) in λ. Therefore $\hat{a} \leq \hat{b} \lhd \hat{c}$, as desired. □

5.7 Proposition Let s be an element in some Cu-semigroup S. There exists a sequence $(s_n)_n \subseteq R(S)$ such that the sequence $(\widehat{s_n})_n \subseteq \mathrm{LPoM}(F(S))$ is $\lhd$-increasing with supremum $\hat{s}$. The sequence $(s_n)_n$ may be chosen to be $\ll$-increasing. In particular, $\hat{s}$ lies in $L(F(S))$.

Proof. We may assume without loss of generality that S has $[0,\infty]$-multiplication. Choose a $\ll$-increasing sequence $(t_n)_n \subseteq S$ with supremum s. For each n, apply Lemma 5.6 to obtain an element $s_n \in S$ such that $t_n \leq s_n \ll t_{n+1}$ while also $\widehat{t_n} \leq \widehat{s_n} \lhd \widehat{t_{n+1}}$. The sequence $(s_n)_n$ is $\ll$-increasing with supremum s, and the sequence $(\widehat{s_n})_n$ is $\lhd$-increasing with supremum $\hat{s}$. □

5.8 Theorem ([Rob13, 3.2.1]) Let S be a Cu-semigroup satisfying ($\mathcal{O}$5'). Then $L(F(S))$ is a Cu-semigroup, and the natural map $R(S) \to L(F(S))$ is a Cu-isomorphism.

Proof. In [Rob13], the Cu-semigroup S is always assumed to satisfy ($\mathcal{O}$5'). Remember that in this case, there is no difference between Robert's $\mathrm{Lsc}(F(S))$ and our $\mathrm{LPoM}(F(S))$ and between Robert's $L(F(S))$ and our $L(F(S))$. Robert defines $S_{\mathbb{R}}$ to be the subset of $\mathrm{LPoM}(F(S))$ that consists of suprema of increasing sequences coming from the set $\{\frac{\hat{s}}{n} \mid n \in \mathbb{N}^*, s \in S\}$. He shows that $S_{\mathbb{R}}$ is a Cu-semigroup and that $S_{\mathbb{R}} = L(F(S))$. It is shown in [APT18, 7.5.9] that the natural map $R(S) \to S_{\mathbb{R}}$ is a Cu-isomorphism. □

5.4 The image of the functor $\mathrm{LAff}(\_)^0_{++}$

For a fixed element $u \in S$, let $F_u(S)$ denote the set of all functionals on S that are normalized at u, that is all functionals $\lambda \in F(S)$ such that $\lambda(u) = 1$. Then $F_u(S)$ is a convex subset of $F(S)$ in the sense that $t \cdot \lambda + (1-t) \cdot \mu$ lies in $F_u(S)$ for all $\lambda, \mu \in F_u(S)$ and $t \in [0,1]$. Notice that $F_u(S)$ is a closed (and therefore compact) subset of $F(S)$ if $\hat{u}$ is continuous. The following lemma demonstrates that we can recover $F(S)$ from $F_u(S)$ if S is simple.

5.9 Lemma Let S be a simple Cu-semigroup, and let $u \in S \setminus \{0\}$ such that $\hat{u}$ is continuous. Then $F(S) = ((0,\infty) \cdot F_u(S)) \cup \{\lambda_0, \lambda_\infty\}$.

Proof. Let $\lambda \in F(S)$. Suppose first that $\lambda(u) = 0$. Let $s \in S$. Every $s' \ll s$ is way-below ∞. Since u is nonzero and S is simple, there exists $n \in \mathbb{N}$ such that $s' \leq nu$, implying that $\lambda(s') = 0$. Passing to the supremum yields $\lambda(s) = 0$, whence $\lambda = \lambda_0$. If $\lambda(u) \in (0,\infty)$, then $\frac{\lambda}{\lambda(u)}$ lies in $F_u(S)$, so λ lies in $(0,\infty) \cdot F_u(S)$. Finally, suppose that $\lambda(u) = \infty$. Let us assume that $\lambda \neq \lambda_\infty$. This implies that $0 \cdot \lambda = \lambda_0$, because S is simple. The sequence $(\frac{\lambda}{n})_n$ converges to λ_0. Since $\hat{u}$ is continuous, we get that

$$0 = \hat{u}(\lambda_0) = \lim_n \hat{u}(\frac{\lambda}{n}) = \lim_n \frac{\lambda(u)}{n} = \infty,$$

which is a contradiction. Therefore $\lambda = \lambda_\infty$. □

The following lemma is adapted from [Rob13, 3.2.3].

5.10 Lemma Let S be a simple and nonzero Cu-semigroup. Set $C := F(S) \setminus \{\lambda_\infty\}$, and let V denote the set of all maps $C \to [0,\infty)$ that are additive, continuous and homogeneous for elements in $[0,\infty)$. A net $(\lambda_i)_i \subseteq C$ converges to a functional $\lambda \in C$ if and only if $(\varphi(\lambda_i))_i$ converges to $\varphi(\lambda)$ for all $\varphi \in V$.

Proof. In order to prove the nontrivial implication, let $a, b \in S$ satisfy $a \ll b$. We have to show that the inequalities

$$\limsup_i \lambda_i(a) \leq \lambda(b) \leq \liminf_i \lambda_i(b)$$

hold. Choose $c \in S$ such that $a \ll c \ll b$, and let $\varepsilon > 0$. It is shown in [Rob13, 2.2.5] that $(1-\varepsilon)\hat{c} \ll \hat{b}$. Using Proposition 5.7, we may choose $d, e \in R(S)$ such that $(1-\varepsilon)\hat{c} \ll \hat{d} \lhd \hat{e} \ll \hat{b}$. We claim that $\hat{e}$ is finite on C. To see this, choose $x \in S$ such that $x \ll b$ and such that $\hat{e} \ll \hat{x}$ (this is possible since $\hat{b} = \sup_{y \ll b} \hat{y}$). If λ lies in C, we may choose an element $s \in S \setminus \{0\}$ such that $\lambda(s) < \infty$. Since S is simple and since $x \ll \infty$, there exists $n \in \mathbb{N}$ such that $x \leq ns$, implying that $\hat{x}(\lambda) < \infty$. Therefore $\hat{x}$ is finite on C, and the same can be said for $\hat{e}$.
We conclude that the restriction of $\hat{d}$ to C lies in V. Using our assumption at the second step yields

$$(1-\varepsilon) \limsup_i \hat{a}(\lambda_i) \leq \limsup_i \hat{d}(\lambda_i) = \hat{d}(\lambda) \leq \hat{b}(\lambda).$$

As ε may be arbitrary small, we obtain $\limsup_i \lambda_i(a) \leq \lambda(b)$. Similarly, we have that

$$(1-\varepsilon)\hat{c}(\lambda) \leq \hat{d}(\lambda) = \liminf_i \hat{d}(\lambda_i) \leq \liminf_i \hat{b}(\lambda_i),$$

and therefore $\lambda(c) \leq \liminf_i \lambda_i(b)$ by passing to the supremum of all $\varepsilon > 0$. Since $c \ll b$ is arbitrary, and since λ preserves suprema of increasing sequences, we deduce that $\lambda(b) \leq \liminf_i \lambda_i(b)$. Overall, we have shown that $(\lambda_i)_i$ converges to λ, as desired. □

5.11 Proposition Let S be a simple Cu-semigroup, and let $u \in S$ such that $\hat{u}$ is continuous. Then $F_u(S)$ is a compact convex subset of a locally convex, Hausdorff, real topological vector space. If S is countably based, then $F_u(S)$ is metrizable. If S satisfies $(\mathcal{O}5')$ and $(\mathcal{O}6)$, then $F_u(S)$ is a Choquet simplex.

Proof. The statements are clear if $u = 0$, so let us assume that $u \neq 0$. Suppose that λ, μ, η in $C := F(S) \setminus \{\lambda_\infty\}$ satisfy $\lambda + \eta = \mu + \eta$. Then λ and μ agree on the ideal $\overline{\{s \in S \mid \eta(s) < \infty\}}$. Since $\eta \neq \lambda_\infty$ and since S is simple, this ideal is equal to S, so $\lambda = \mu$. Therefore C is a cancellative monoid, so C can be understood as a submonoid of its Grothendieck completion, which we denote by E. We endow E with the obvious scalar multiplication, giving it the structure of a real vector space. Let V be the set of all maps $C \to [0, \infty)$ that are additive, continuous and homogeneous for elements in $[0, \infty)$. For each $\varphi \in V$, we define a seminorm on E by setting $\|\lambda - \mu\|_\varphi := |\varphi(\lambda) - \varphi(\mu)|$. We equip E with the topology induced by these seminorms, giving E the structure of a locally convex real topological vector space. By Lemma 5.10, the topology of C as a subset of $F(S)$ agrees with the topology of C as a subset of E. Since C is hausdorff, E is a Hausdorff space as well. Therefore, we can regard $F_u(S)$ as a subset of the locally convex, Hausdorff, real topological vector space E. Also, $F_u(S)$ is contained in a hyperplane which misses the origin, so it follows from Lemma 5.9 that the cone with base $F_u(S)$ is given by C.
If S is countably based, then $F(S)$ is metrizable according to Theorem 5.1, so $F_u(S)$ is metrizable as well. Under the assumption that S satisfies $(\mathcal{O}5')$ and $(\mathcal{O}6)$, it is shown in [Rob13, 4.1.2] that $F(S)$ is a lattice. Then C must be a lattice as well, so $F_u(S)$ is a Choquet simplex. □

5.12 Proposition Suppose that S is a countably based, simple Cu-semigroup satisfying $(\mathcal{O}5')$. Let $u \in S \setminus \{0\}$ such that $\hat{u}$ is continuous. The restriction map

$$\rho \colon L(F(S)) \to \mathrm{LAff}(F_u(S))^0_{++}$$

is a Cu-isomorphism.

Proof. It is clear that the restriction of an element in $L(F(S))$ to $F_u(S)$ is lower semicontinuous and affine. However, it is not immediately clear that such a restriction is strictly positive (or zero). According to Theorem 5.8, every element in $L(F(S))$ has the form $\hat{a}$ for some $a \in R(S)$. Let us suppose that $a \neq 0$. We claim that $\rho(\hat{a}) > 0$. To see this, we may choose $s \in S \setminus \{0\}$ and $n \in \mathbb{N}^*$ such that $\frac{\hat{s}}{n} \leq \hat{a}$, implying that $\frac{1}{n}\rho(\hat{s}) \leq \rho(\hat{a})$. Thus, it suffices to show that $\rho(\hat{s}) > 0$. Let $\lambda \in F_u(S)$ and choose $u' \ll u$ such that $\lambda(u') > 0$. Since S is simple and s is nonzero, there exists $m \in \mathbb{N}$ such that $u' \leq m \cdot s$. It follows that

$$0 < \lambda(u') \leq m \cdot \lambda(s) = m \cdot \rho(\hat{s})(\lambda),$$

whence $\rho(\hat{s}) > 0$, as desired.

Let us show that any element $f \in \mathrm{LAff}(F_u(S))^0_{++}$ extends uniquely to some element $\tilde{f}$ in $L(F(S))$. We have seen in the proof of Proposition 5.11 that the cone with base $F_u(S)$ is given by $F(S) \setminus \{\lambda_\infty\}$ (under the assumption that $u \neq 0$). Therefore f extends uniquely to a lower semicontinuous PoM-morphism $\tilde{f}\colon F(S) \setminus \{\lambda_\infty\} \to [0, \infty]$. We extend $\tilde{f}$ to $F(S)$ by setting $\tilde{f}(\lambda_\infty) := 0$ if $f = 0$ and $\tilde{f}(\lambda_\infty) := \infty$ if $f > 0$. It is clear that $\tilde{f}$ lies in $\mathrm{LPoM}(F(S))$, so it remains to show that $\tilde{f}$ actually lies in $L(F(S))$. Choose a $\ll$-increasing sequence $(f_n)_n$ in $\mathrm{Aff}(F_u(S))^0_{++}$ with supremum f. Let $n \in \mathbb{N}$. Note that $f_n \ll f_{n+1}$ entails that there exists some $\varepsilon > 0$ such that $f_n \leq (1-\varepsilon) f_{n+1}$, whence $\tilde{f_n} \leq (1-\varepsilon)\widetilde{f_{n+1}}$. Since f_n is continuous, the extension onto $F(S) \setminus \{\lambda_\infty\}$ is continuous as well. It follows that the sequence $(\tilde{f_n})_n$ is $\lhd$-increasing with supremum $\tilde{f}$. In particular, $\tilde{f}$ lies in $L(F(S))$.

We have seen that ρ is a PoM-isomorphism. In general, any PoM-isomorphism of Cu-semigroups is automatically a Cu-isomorphism. Thus, ρ is a Cu-isomorphism. □

We know already that we can recover any compact convex set K from $\mathrm{Aff}(K)$ by applying the state functor. We will see in the upcoming Proposition that we can also recover K from $\mathrm{LAff}(K)^0_{++}$ by taking normalized functionals.

5.13 Proposition Let K be a metrizable compact convex set and consider $1 \in \mathrm{LAff}(K)^0_{++}$. Then $\hat{1}$ is continuous. The natural map

$$\kappa\colon K \to F_1(\mathrm{LAff}(K)^0_{++}),\ x \mapsto \mathrm{ev}_x$$

is an affine homeomorphism.

Proof. Suppose that $(\lambda_i)_i \subseteq F(\mathrm{LAff}(K)^0_{++})$ is a net that converges to some functional λ in $F(\mathrm{LAff}(K)^0_{++})$. All $t \in (0,1)$ satisfy $t \ll 1$ (in $\mathrm{LAff}(K)^0_{++}$), so the inequality

$$t \cdot \limsup_i \lambda_i(1) = \limsup_i \lambda_i(t) \leq \lambda(1)$$

holds. Passing to the supremum yields $\limsup_i \hat{1}(\lambda_i) \leq \hat{1}(\lambda)$. We have shown that $\hat{1}$ is upper semicontinuous, hence continuous.

It is clear that κ is affine. The injectivity follows from the Hahn-Banach theorem. In order to show that κ is continuous, let $(x_i)_i \subseteq K$ be a net that converges to some point $x \in K$. Let $f, h \in \mathrm{LAff}(K)^0_{++}$ satisfy $f \ll h$. Choose $g \in \mathrm{Aff}(K)$ such that $f \leq g \ll h$. If we use the continuity of g at the second and the lower semicontinuity of h at the fourth step, we obtain

$$\limsup_i f(x_i) \leq \limsup_i g(x_i) = g(x) \leq h(x) \leq \liminf_i h(x_i),$$

so $\kappa(x_i)$ converges to $\kappa(x)$, as desired.

For the surjectivity, let $\lambda \in F_1(\mathrm{LAff}(K)^0_{++})$. Notice that the restriction of λ to $\mathrm{Aff}(K)^0_{++}$ is real-valued (because 1 is an order unit in $\mathrm{Aff}(K)^0_{++}$). The function $\varphi := \mathrm{Grot}(\lambda|_{\mathrm{Aff}(K)^0_{++}})$ lies in $S(\mathrm{Aff}(K), <^=, 1) = S(\mathrm{Aff}(K), \leq, 1)$. By Proposition 2.11, there exists an element $x \in K$ such that $\varphi = \mathrm{ev}_x$. But then $\kappa(x) = \lambda$ (since they agree on the dense subset $\mathrm{Aff}(K)^0_{++}$), so κ is indeed surjective. □

5.14 Corollary A metrizable compact convex set K is a Choquet simplex if and only if $\mathrm{LAff}(K)^0_{++}$ satisfies ($\mathcal{O}6$).

Proof. If K is a Choquet simplex, then $\mathrm{LAff}(K)$ is an inf-semilattice according to Proposition 3.12. It follows easily from this that $\mathrm{LAff}(K)^0_{++}$ satisfies ($\mathcal{O}6$). The argument is omitted. Conversely, assume that $\mathrm{LAff}(K)^0_{++}$ satisfies ($\mathcal{O}6$). It then follows from Propositions 5.13 and 5.11 that $K \cong F_1(\mathrm{LAff}(K)^0_{++})$ is a Choquet simplex. □

Recall that a Cu-semigroup S is called almost unperforated if for all $x, y \in S$, the condition $(k+1)x \leq ky$ for some $k \in \mathbb{N}$ implies that $x \leq y$. We say that S is almost divisible if for all $k \in \mathbb{N}$ and $x, y \in S$ satisfying $x \ll y$, there exists $z \in S$ such that $x \leq (k+1)z$ and $kz \leq y$.

5.15 Theorem ([APT18, 7.5.4.]) A Cu-semigroup S has $[0,\infty]$-multiplication if and only if S is almost unperforated, almost divisible, and every element of S is soft.

5.16 Theorem For a Cu-semigroup S, the following are equivalent:

1) There exists a metrizable compact convex set K such that $S \cong \mathrm{LAff}(K)^0_{++}$.
2) S is simple, countably based and satisfies ($\mathcal{O}5'$). Moreover, S is almost unperforated, almost divisible, and every element of S is soft.

Proof. It is clear that 1) implies 2). Now assume that 2) holds. If $S = 0$, we can choose $K = \emptyset$. Now assume that $S \neq 0$. Then we can choose $a, b \in S$ such that $0 \neq a \ll b \ll \infty$. By Theorem 5.15, S has $[0,\infty]$-multiplication. Lemma 5.6 allows us to find an element $u \in S$ such that $a \leq u \ll b$ while also $\hat{a} \leq \hat{u} \lhd \hat{b}$. Clearly, u is nonzero. Since we have chosen b to be way below ∞, and since S is simple, $\hat{b}$ is finite on $F(S) \setminus \{\lambda_\infty\}$. Therefore $\hat{u}$ is continuous on $F(S) \setminus \{\lambda_\infty\}$. As $\hat{u}$ is automatically continuous in λ_∞, we have that $\hat{u}$ is continuous. Then $K := F_u(S)$ is a metrizable compact convex set according to Proposition 5.11. We use that S has $[0,\infty]$-multiplication at the first step, Theorem 5.8 at the second step and Proposition 5.12 at the third step to deduce

$$S \cong R(S) \cong L(F(S)) \cong \mathrm{LAff}(K)^0_{++},$$

which is the desired result. □

Remark It follows from Theorem 5.16 and the main theorem that the full subcategory of Cu, consisting of simple, countably based, almost unperforated, almost divisible Cu-semigroups satisfying $(\mathcal{O}5')$ and $(\mathcal{O}6)$ and which consist only of soft elements is closed under the tensor product in Cu.

6 An application of the main theorem

One of the main motivations for studying $\mathrm{LAff}(K)^0_{++}$ is that this semigroup appears naturally as the soft part of Cuntz semigroups coming from sufficiently nice C^*-algebras. In this chapter, we use the main theorem to relate $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B)$ to $\mathrm{Cu}(A \otimes_{\min} B)$, for C^*-algebras A and B. This is merely meant to be an outlook on what could be done with the main theorem, so we will not provide any proofs. In fact, the argumentation is speculative at times. The reader should keep that in mind.

If A is any unital C^*-algebra, then both the set of all normalized traces $\mathrm{T}_1(A)$ and the set of all normalized quasitraces $\mathrm{QT}_1(A)$ can be regarded as Choquet simplices (see [Sak71, 3.1.18] and [BH82, II.4.4]). If A is separable, then $\mathrm{T}_1(A)$ and $\mathrm{QT}_1(A)$ are metrizable. Now suppose that A and B are unital, simple, separable, stably finite, $\mathcal{Z}$-stable C^*-algebras. By Proposition 1.32, their Cuntz semigroups can be computed as

$$\mathrm{Cu}(A) \cong V(A) \sqcup_{\gamma_A} \mathrm{LAff}(\mathrm{QT}_1(A))^0_{++}, \text{ and}$$
$$\mathrm{Cu}(B) \cong V(B) \sqcup_{\gamma_B} \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++},$$

where γ_A and γ_B are the obvious composition maps. Since all those properties pass to the minimal tensor product, we also have that

$$\mathrm{Cu}(A \otimes_{\min} B) \cong V(A \otimes_{\min} B) \sqcup_{\gamma} \mathrm{LAff}(\mathrm{QT}_1(A \otimes_{\min} B))^0_{++},$$

with the obvious composition map γ. It follows from the main theorem and the fact that the tensor product of two Choquet simplices is again a Choquet simplex that

$$\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++} \cong \mathrm{LAff}(\mathrm{QT}_1(A) \otimes \mathrm{QT}_1(B))^0_{++}$$

is a simple semigroup in the category $\mathcal{D}$ (see below Definition 1.27 on page 14). Let us assume that $V(A) \otimes^{\mathrm{PoM}} V(B)$ is a simple semigroup in the category $\mathcal{C}$, and that the canonical PoM-morphism

$$\gamma_A \otimes \gamma_B \colon V(A) \otimes^{\mathrm{PoM}} V(B) \to \mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++}$$

is a composition map. We can then compose these semigroups to obtain a Cu-semigroup

$$(V(A) \otimes^{\mathrm{PoM}} V(B)) \sqcup_{\gamma_A \otimes \gamma_B} (\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++}),$$

and it is hopefully not too far-fetched to assume that

$$\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \cong (V(A) \otimes^{\mathrm{PoM}} V(B)) \sqcup_{\gamma_A \otimes \gamma_B} (\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++}).$$

For arbitrary C^*-algebras A and B, there is a natural Cu-morphism $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\min} B)$ that factors through $\mathrm{Cu}(A \otimes_{\max} B)$. This Cu-morphism is known to be a Cu-isomorphism if one of the C^*-algebras is an AF-algebra (cf. [APT18, 6.4.13]). In general, however, it is not an isomorphism. Any Cu-morphism maps compact elements to compact elements and soft elements to soft elements. In our case, we obtain a Cu-morphism

$$\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++} \to \mathrm{LAff}(\mathrm{QT}_1(A \otimes_{\min} B))^0_{++}.$$

In the case that $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\min} B)$ is a Cu-isomorphism, it follows that

$$\begin{aligned}\mathrm{LAff}(\mathrm{QT}_1(A) \otimes \mathrm{QT}_1(B))^0_{++} &\cong \mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++} \\ &\cong \mathrm{LAff}(\mathrm{QT}_1(A \otimes_{\min} B))^0_{++}.\end{aligned}$$

We can then apply Proposition 5.13 to deduce $\mathrm{QT}_1(A \otimes_{\min} B) \cong \mathrm{QT}_1(A) \otimes \mathrm{QT}_1(B)$. To the author's knowledge, there is no complete description of $\mathrm{QT}_1(A \otimes_{\min} B)$ for general C^*-algebras A and B. On the other hand, there is this result for traces, which is due to Guichardet [Gui69, Proposition 22]: for all unital C^*-algebras A and B, we have that

$$\partial_e \mathrm{T}_1(A \otimes_{\min} B) = \{\tau_A \otimes \tau_B \mid \tau_A \in \partial_e \mathrm{T}_1(A), \tau_B \in \partial_e \mathrm{T}_1(B)\}.$$

In view of Theorem 4.9, this suggests that $\mathrm{T}_1(A \otimes_{\min} B) \cong \mathrm{T}_1(A) \otimes \mathrm{T}_1(B)$ holds for all unital C^*-algebras A and B. Haagerup proved that every quasitrace on a unital, exact C^*-algebra is a trace [Haa14]. Again, this suggests that $\mathrm{QT}_1(A \otimes_{\min} B) \cong \mathrm{QT}_1(A) \otimes \mathrm{QT}_1(B)$ holds for all unital, exact C^*-algebras.

We return to our original assumptions on A and B. Let us assume that there is a natural isomorphism $\mathrm{QT}_1(A \otimes_{\min} B) \cong \mathrm{QT}_1(A) \otimes \mathrm{QT}_1(B)$. We obtain a natural Cu-isomorphism

$$\begin{aligned}\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++} &\cong \mathrm{LAff}(\mathrm{QT}_1(A) \otimes \mathrm{QT}_1(B))^0_{++} \\ &\xrightarrow{\cong} \mathrm{LAff}(\mathrm{QT}_1(A \otimes_{\min} B))^0_{++}.\end{aligned}$$

There is a canonical PoM-morphism $V(A) \otimes^{\mathrm{PoM}} V(B) \to V(A \otimes_{\min} B)$ that factors through $V(A \otimes_{\max} B)$. If we combine these two maps, we obtain a Cu-morphism ω such that the diagram

$$\begin{array}{ccc}\mathrm{Cu}(A \otimes_{\min} B) & \xrightarrow{\cong} & V(A \otimes_{\min} B) \sqcup_\gamma \mathrm{LAff}(\mathrm{QT}_1(A \otimes_{\min} B))^0_{++} \\ \uparrow & & \uparrow \omega \\ \mathrm{Cu}(A) \otimes \mathrm{Cu}(B) & \xrightarrow{\cong} & (V(A) \otimes^{\mathrm{PoM}} V(B)) \sqcup_{\gamma_A \otimes \gamma_B} (\mathrm{LAff}(\mathrm{QT}_1(A))^0_{++} \otimes \mathrm{LAff}(\mathrm{QT}_1(B))^0_{++})\end{array}$$

commutes. In the case that $V(A) \otimes^{\mathrm{PoM}} V(B) \to V(A \otimes_{\min} B)$ is an isomorphism, then ω should be an isomorphism as well. Consequently, the natural map $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\min} B)$ is also an isomorphism. Thus, we should be able to derive a criterion for which the natural Cu-morphism $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\min} B)$ is a Cu-isomorphism.

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