Tracially complete C*-algebras

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The trace-kernel quotient

Throughout: A is a C^*-algebra with $T(A)$ (tracial states) compact and non-empty.

ω is a free filter (usually either ∞ or an ultrafilter).

$$A^\omega := \ell_\infty(A)/\{(a_n) : \lim_{n\to\omega} \sup_{\tau\in T(A)} \tau(a_n^* a_n) = 0\}$$

Applications: - Tams-Winter conjecture (Matui-Sato,..)

simple, nuclear, strict comp + unif. prop. Γ $\Rightarrow$ $\mathcal{Z}$-stable

- Schafhauser's pf of QD thm
- New approach to classification

The tracial completion

For $X \subseteq T(A)$, define $\|a\|_{2,X} := \sup_{\tau \in X} \tau(a^* a)^{\frac{1}{2}}$

a seminorm on A (a norm if X is faithful)

Define the tracial completion of A w.r.t. X by adjoining limit points of all $\|\cdot\|$-bounded, $\|\cdot\|_{2,X}$-Cauchy seqs.

Fact: $A^\omega \cong \left(\overline{A}^{T(A)}\right)^\omega$ (to be defined).

Tracially complete C^*-algebras

Motivation: what is the right abstract structure for $\overline{A}^{\tau(A)}$?

Def'n: A tracially complete C^*-alg. is a pair (M, X) where M is a unital C^*-alg., $X \subseteq T(M)$ is closed, convex, s.t.:

(i) $\|\cdot\|_{2,X}$ is a norm

(ii) the unit ball of M is $\|\cdot\|_{2,X}$-complete.

Eg. $\cdot$ $(\overline{A}^{\tau(A)}, T(A))$ is a tc. C*-alg.

$\cdot$ $(N, \{\tau\})$ where N is a vN alg., τ a faithful trace.

Given (M, X), define $M^\omega := \ell_\infty(M)/\{(a_n) : \lim_\omega \|a_n\|_{2,X} = 0\}$

X^ω = closed convex hull of limit traces.

(M^ω, X^ω) is tc. complete.

Factorial tracially complete C*-algebras

When τ is a trace, we let π_τ be its GNS representation.

Defn: A tracially complete C*-algebra (M, X) (X closed convex) is factorial if:

(i) X is a face in $T(M)$ or equivalently

(ii) $\pi_\tau(M)''$ is a factor, $\forall \tau \in \partial_e X$.

E.g. $(\widetilde{A}^{\tau(A)}, T(A))$ is factorial

E.g. (Ozawa) Let Y be a compact H. space, N a II_1-factor.

$M := C_\sigma(Y, N) := \{ f : Y \to N \mid \|\cdot\|$-bounded and strongly cts. (i.e. $\|\cdot\|_2$-cts.$\}$

$X \cong \mathrm{Prob}(Y)$ $\quad$ (M, X) is a factorial tr.c. C*-alg.

E.g. $(N, \{\tau\})$ is not factorial, if N is a non-factor.

Q: Is (M^ω, X^ω) factorial?

E.g. "Toy example": Fix $\Theta : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$

Define $M := \{ (a_n) \in \ell_\infty(\mathbb{R}) : a_n \to \Theta(a_1 \oplus a_2)$ strongly$\}$

X = obvious traces.

$\partial_e X = \{ \tau_i : i \in \mathbb{N} \}$ $\quad$ $\tau_i((a_n)) = \tau_{\mathbb{R}}(a_i)$

$T_i \rightarrow$ convex comb. of T_1, T_2.

CPoU / local-to-global transfer

From now on, (M, X) is always a factorial tracially complete C*-algebra.

Consider properties of the form:

P: $\forall \varepsilon > 0 \ \exists x_1, \dots x_r \in M$ s.t. $\|p_i(a_1, \dots, a_k, x_1, \dots, x_r)\|_{2,X} < M_i \cdot \varepsilon$ $\quad i = 1, \dots, l$

P is defined in terms of NC *-polys p_i and $M_i \in \mathbb{R}$. It is a property of (M, X), $a_1, \dots, a_k$.

E.g. If a, b are projs., $\forall \varepsilon \ \exists x \ \|a - x^*x\|_{2,X} < \varepsilon$, $\|b - xx^*\|_{2,X} < \varepsilon$.

- Classification of maps from f.d. C*-algs.
- $(2,X)$-versions of vr0, sr1 ...

Def'n/Prop: (Farah, Hart, Hirshberg, Schafhauser, T, Vaccaro)
(M, X) has CPoU if for any property as above, any $a_1, \dots, a_k \in M$, if it holds in all fibres, it holds in M.

i.e., in $\pi_\tau(M)''$, using $\|\cdot\|_{2,\tau}$ instead of $\|\cdot\|_{2,X}$.

$\forall \tau \in \partial_e X$

Cor: M^ω has sr 1, vr 0, classification of maps from f.d. domains, every unitary is an exponential, assuming CPoU.

Also, $T(M^\omega) \cong X^\omega$, so (M^ω, X^ω) is factorized.

Property Γ

Defn: (M, X) has **property Γ** if there is a proj. $p \in M^\omega \cap M'$ s.t.

$$\tau(pa) = \tfrac{1}{2}\tau(a) \qquad \forall \tau \in X^\omega$$

Thm: (CCEGSW; CETWW in case of $(\bar{A}^{T(A)}, T(A))$, A nuclear) $\Gamma \Rightarrow$ CPoU.

(in this case, it's called uniform prop Γ for A.)

Eg. $L\mathbb{F}_2$ no Γ

$C_0([0,1], L\mathbb{F}_2)$ no CPoU.

Amenability

Defn: (M, X) is **amenable** if $(2, \infty)$-CPAP
equiv. $\pi_2(M)$ amen $\forall \tau \in X$.

E.g. $(\overline{A}^{\tau(A)}, \tau(A))$ amenable if A nuclear.

amenable + CPoU + no type I quotients + sep. $\Rightarrow$ AF & classifiable
(Invariant is X)

Questions

- Is there any factorial (M, X) s.t. $X \neq T(M)$?
  - Special cases: · $(\bar{A}^X, X)$
    · $C_0(Y, \mathcal{R})$
    · Assuming CPoU?
  - Cuntz semigroup related question: does M have strict comparison w.r.t. X? (i.e., $d_\tau(a) < d_\tau(b)$ $\forall \tau \in X \overset{?}{\Rightarrow} a \lesssim b$?
- Amenable $\Rightarrow \Gamma$, for factorial tracially complete C^*-algebras with no type I quotients?
  - Special cases: · $(\bar{A}^X, X)$
    · $(\bar{A}^X, X)$, assuming Cu-regularity
    · X Baire (the W^*-bundle triviality question)
- $sr\,1$, $rr\,0$ for M, assuming CPoU? $C_0(X, \mathcal{R})$
- Is every unitary an exponential, assuming CPoU?
- Is (M^ω, X^ω) factorial, when (M, X) is?
- If $\exists p \in M^\omega \cap M'$, $\tau(p) = \frac{1}{2}$ $\forall \tau \in X^\omega$, does (M, X) have Γ?

$$\left\{ \tau(pa) = \tfrac{1}{2}\tau(a) \quad \begin{array}{l} \forall \tau \in X^\omega, \\ a \in M \end{array} \right\}$$