Covering dimension of Cuntz semigroups

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Motivation and definition

Theorem (cf Kirchberg-Winter)

Let $n \in \mathbb{N}$. A compact metric space X satisfies $\dim(X) \leq n$ if and only if, for every open cover $X = U_1 \cup \ldots \cup U_r$, there exist open subsets $V_{j,k}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:

(i) $V_{j,k} \Subset U_j$ for each j and k.

(ii) $X = \bigcup_{j,k} V_{j,k}$.

(iii) $V_{j,k} \cap V_{j',k} = \emptyset$ for every $j \neq j'$ and k.

Question

Given a compact metric space X, let $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ be the set of lower-semicontinuous functions (i.e. $\{f \geq n\}$ open for every $n \in \overline{\mathbb{N}}$), which is a Cu-semigroup.
Can the theorem be translated to $\mathrm{Lsc}(X, \overline{\mathbb{N}})$?

Motivation and definition

Let X be compact, metric space with $\dim(X) \leq n$. For every open subset $U \subseteq X$, let χ_U denote the indicator function of U.
Given an open cover $X = U_1 \cup \ldots \cup U_r$, we get
$\chi_X \ll \chi_X \ll \chi_{U_1} + \ldots + \chi_{U_r}$.
We obtain open subsets $V_{j,k}$ such that

(i) $V_{j,k} \Subset U_j$ for each j and k. Thus, $\chi_{V_{j,k}} \ll \chi_{U_j}$.

(ii) $X = \bigcup_{j,k} V_{j,k}$, which implies $\chi_X \ll \sum_{j,k} \chi_{V_{j,k}}$.

(iii) $V_{j,k} \cap V_{j',k} = \emptyset$ for every $j \neq j'$ and, consequently, $\sum_j \chi_{V_{j,k}} \ll \chi_X$ for every k.

Motivation and definition

Whenever $\chi_X \ll \chi_X \ll \chi_{U_1} + \ldots + \chi_{U_r}$, there exist elements $\chi_{V_{j,k}}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:

(i) $\chi_{V_{j,k}} \ll \chi_{U_j}$ for each j and k.

(ii) $\chi_X \ll \sum_{j,k} \chi_{V_{j,k}}$.

(iii) $\sum_j \chi_{V_{j,k}} \ll \chi_X$ for each k.

Let $n \in \mathbb{N}$ and let X be a compact metric space with $\dim(X) \leq n$. Then, whenever $x' \ll x \ll y_1 + \ldots + y_r$, there exist elements $z_{j,k}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:

(i) $z_{j,k} \ll y_j$ for each j and k.

(ii) $x' \ll \sum_{j,k} z_{j,k}$.

(iii) $\sum_j z_{j,k} \ll x$ for each k.

Motivation and definition

Definition

Let S be a Cu-semigroup. Given $n \in \mathbb{N}$, we write $\dim(S) \leq n$ if, whenever $x' \ll x \ll y_1 + \ldots + y_r$, there exist elements $z_{j,k}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:

(i) $z_{j,k} \ll y_j$ for each j and k.

(ii) $x' \ll \sum_{j,k} z_{j,k}$.

(iii) $\sum_j z_{j,k} \ll x$ for each k.

We set $\dim(S) = \infty$ if there is no n such that $\dim(S) \leq n$. Otherwise, we let $\dim(S)$ for the smallest n such that $\dim(S) \leq n$.

Remark

Given a compact, metric space X, we have

$$\dim(\mathrm{Lsc}(X, \overline{\mathbb{N}})) \leq \dim(X).$$

Examples

$\dim(\mathrm{Lsc}(X, \overline{\mathbb{N}})) = \dim(X)$, X compact metric space

We already 'know' that $\dim(\mathrm{Lsc}(X, \overline{\mathbb{N}})) \leq \dim(X)$.
Let $n = \dim(\mathrm{Lsc}(X, \overline{\mathbb{N}}))$.
To see $\dim(X) \leq n$, let $U_1, \ldots, U_r$ be a finite open cover of X. We have $\chi_X \ll \chi_X \ll \chi_{U_1} + \ldots + \chi_{U_r}$.
We get elements $z_{j,k}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:

(i) $z_{j,k} \ll \chi_{U_j}$ for each j, k. Thus, $z_{j,k} = \chi_{V_{j,k}}$ with $V_{j,k} \Subset U_j$.

(ii) $\chi_X \ll \sum_{j,k} \chi_{V_{j,k}}$.This implies $X = \cup_{j,k} V_{j,k}$.

(iii) $\sum_j \chi_{V_{j,k}} \ll \chi_X$ for each k, which shows $V_{j,k} \cap V_{j',k} = \emptyset$ for every $j \neq j'$ and k.

Thus, $\dim(X) \leq \dim(\mathrm{Lsc}(X, \overline{\mathbb{N}}))$.

Examples

Recall that a positively ordered monoid is said to satisfy the *Riesz decomposition property* if, whenever $x \leq y_1 + \ldots + y_r$, there exist elements $z_j \leq y_j$ such that $x = z_1 + \ldots + z_r$.

Remark

$\dim(S) = 0$ if, whenever $x' \ll x \ll y_1 + \ldots + y_r$, there exist $z_j \ll y_j$ such that $x' \ll z_1 + \cdots + z_r \ll x$.

In particular, a zero-dimensional Cu-semigroup always satisfies (O6).

$\overline{\mathbb{N}}$ and $E_k = \{0, 1, \ldots, k, \infty\}$

The compact elements are dense and satisfy Riesz decomposition. This implies dimension 0.

Examples

$S = \{0, 2, 3, \infty\}$

Consider $3 \ll 3 \ll 2 + 2$.
Since one cannot find elements $z_0, z_1 \ll 2$ such that $3 = z_0 + z_1$, we have $\dim(S) \neq 0$.
In fact, $\dim(S) = 1$.

$\mathrm{Cu}(\mathcal{W}) \cong [0, \infty]$

$[0, \infty)$ is dense and satisfies Riesz decomposition.
This implies $\dim([0, \infty]) = 0$.

$S = \mathrm{Lsc}(\{x, y\}, \overline{\mathbb{N}})_{++} \cup \{0\}$

$1 + (n+1)\chi_{\{x\}} \ll 1 + (n+1)\chi_{\{x\}} \ll 1 + \ldots_{n+2} + 1$ implies $\dim(S) > n$ for every $n \in \mathbb{N}$. Thus, $\dim(S) = \infty$.

Permanence properties

Lemma

Let S, T be Cu-semigroups. Then,

$$\dim(S \oplus T) = \max\{\dim(S), \dim(T)\}.$$

Lemma

Let S be a Cu-semigroup and let I be an ideal of S. Then,

$$\dim(I), \dim(S/I) \leq \dim(S).$$

Proposition

Let $S = \varinjlim_{\lambda \in \Lambda} S_\lambda$ be an inductive limit of Cu-semigroups. Then $\dim(S) \leq \liminf_\lambda \dim(S_\lambda)$.

Permanence properties

Question

Is there a bound of $\dim(S)$ on $\dim(I)$ and $\dim(S/I)$? For example, do we have $\dim(S) \leq \dim(I) + \dim(S/I) + 1$?
Is it true that $\dim(\mathrm{Cu}(\tilde{A})) \leq \dim(\mathrm{Cu}(A)) + 1$?

Question

Given two Cu-semigroups S, T, what is the dimension of $S \otimes T$?
Do we have $\dim(S \otimes T) \leq (\dim(S) + 1)(\dim(T) + 1) - 1$?
What can we say about $\dim(\mathrm{Cu}(A \otimes B))$?

Given two Cu-semigroups S, T, we say that S is a *retract* of T if there exist a Cu-morphism $\iota\colon S \to T$ and a generalized Cu-morphism $\sigma\colon T \to S$ such that $\sigma \circ \iota = \mathrm{id}_S$.

Lemma

Let S, T be Cu-semigroups with S a retract of T. Then, $\dim(S) \leq \dim(T)$.

Permanence properties

A non-zero element x in a Cu-semigroup S is *soft* if for every $x' \ll x$ there exists a nonzero $t \in S$ such that $x' + t \ll x$. We denote the set of soft elements by S_{soft}.
Let S be a countably based, simple, weakly cancellative Cu-semigroup. Then, S_{soft} is a retract of S.

Proposition

Let S be a countably based, simple, weakly cancellative Cu-semigroup. Then,

$$\dim(S_{\mathrm{soft}}) \leq \dim(S) \leq \dim(S_{\mathrm{soft}}) + 1.$$

$Z = \mathrm{Cu}(\mathcal{Z}) \cong [0, \infty] \sqcup \overline{\mathbb{N}}$

Since $Z_{\mathrm{soft}} \cong [0, \infty]$, we have $0 \leq \dim(Z) \leq 1$. Considering $1 \ll 1 \ll 0.6 + 0.6$, one sees that $\dim(Z) \neq 0$.

Permanence properties

Recall that an *R-multiplication* is a scalar multiplication on the semigroup with natural compatibility conditions.

Theorem

Let S be a Cu-semigroup. Then,

(i) If S has $\{0, \infty\}$-multiplication, $\dim(S) = 0$.

(ii) If S has $[0, \infty]$-multiplication, $\dim(S) = 0$.

(iii) If S has Z-multiplication, $\dim(S) \leq 1$.

Corollary

Let A be a C^*-algebra. Then,

(i) If A is purely infinite, $\dim(\mathrm{Cu}(A)) = 0$.

(ii) If A is $\mathcal{W}$-stable, $\dim(\mathrm{Cu}(A)) = 0$.

(iii) If A is $\mathcal{Z}$-stable, $\dim(\mathrm{Cu}(A)) \leq 1$.

Commutative and subhomogeneous C^*-algebras

Theorem

Let A be a C^*-algebra. Then, $\dim(\mathrm{Cu}(A)) \leq \dim_{\mathrm{nuc}}(A)$.

Lemma

Let X be a compact metric space. Then, $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a retract of $\mathrm{Cu}(C(X))$.

Proposition

Let X be a compact metric space. Then,

$$\dim(X) = \dim(\mathrm{Cu}(C(X))).$$

Theorem

Let A be a subhomogeneous C^*-algebra. Then,

$$\dim(\mathrm{Cu}(A)) = \dim_{\mathrm{nuc}}(A).$$

Zero dimensional Cu-semigroups

Lemma

Let S be a Cu-semigroup. Then, $\dim(S) = 0$ if and only if whenever $x' \ll x \ll y_1 + y_2$ there exist elements $z_1 \ll y_1$ and $z_2 \ll y_2$ such that $x' \ll z_1 + z_2 \ll x$.

Proposition

Let A be a real rank zero C^*-algebra. Then, $\dim(\mathrm{Cu}(A)) = 0$.

Example

The C^*-algebra $A = \mathcal{B}(\ell^2(\mathbb{N}))$ satisfies $\dim(\mathrm{Cu}(A)) = 0$ but $\dim_{\mathrm{nuc}}(A) = \infty$.

Zero dimensional Cu-semigroups

Proposition

Let S be a zero-dimensional Cu-semigroup satisfying (O5) and weak cancellation and let c be a compact element in S. Then, the ideal generated by c is algebraic.

Corollary

Let A be a unital, stable rank one C^*-algebra. Then, $\dim(\mathrm{Cu}(A)) = 0$ if and only if A is of real rank zero.

Example

Let $\widetilde{\mathcal{W}}$ be the unitization of the Jacelon-Razak algebra $\mathcal{W}$. Then,

$$0 \neq \dim(\mathrm{Cu}(\widetilde{\mathcal{W}})) \leq \dim_{\mathrm{nuc}}(\widetilde{\mathcal{W}}) = 1.$$

Zero dimensional Cu-semigroups

Proposition

Let A be a separable, simple, $\mathcal{Z}$-stable C^*-algebra. Then we have $\dim(\mathrm{Cu}(A)) \leq 1$. Moreover, $\dim(\mathrm{Cu}(A)) = 0$ if and only if A has real rank zero or if A is stably projectionless.

Lemma

Let S be a simple, weakly cancellative Cu-semigroup satisfying (O5). If $\dim(S) = 0$, then S is either algebraic or soft.

Zero dimensional Cu-semigroups

An element x in a Cu-semigroup is *complementable* if $y = x + r$ whenever $x \ll y$.

Theorem

Let S be a countably based, simple, soft, weakly cancellative Cu-semigroup satisfying (O5) and (O6). Then, $\dim(S) = 0$ if and only if the set of complementable elements is sup-dense.

Theorem

Let S be a countably based, simple, weakly cancellative Cu-semigroup satisfying (O5) and (O6). Then, $\dim(S) = 0$ if and only if S is a retract of a countably based, simple, weakly cancellative, algebraic Cu-semigroup satisfying (O5) and (O6).