Almost finiteness, $\mathcal{Z}$-stability and the dynamical Cuntz Semigroup for non-commutative Coefficients

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Based on joint work with

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Conjecture (Toms-Winter)

For a simple separable unital nuclear C*-algebra A tfae:

1. A has finite nuclear dimension;
2. $A \cong A \otimes \mathcal{Z}$, ie A is $\mathcal{Z}$-stable;
3. A has strict comparison, ie $d_\tau([a]) < d_\tau([b])$ $\forall \tau$ trace implies $[a] \preceq [b]$ in Cuntz semigroup of A.

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Now known (work of many): (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) and (2) $\Leftarrow$ (3) in many cases (uniform property Γ).

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A major source of examples of simple nuclear C^*-algebras are **crossed products**. Given actions

$$\alpha : G \curvearrowright X \qquad \text{or} \qquad \alpha : G \curvearrowright A$$

we can associate their crossed product algebras

$$C(X) \rtimes_\alpha G \qquad \text{or} \qquad A \rtimes_\alpha G.$$

There are good criteria (minimality) for their simplicity; they are always nuclear if A is and if G is amenable; the UCT is automatic for $C(X) \rtimes_\alpha G$.

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Question/Problem

Find criteria for $\alpha : G \curvearrowright A$ to ensure classifiability of $A \rtimes_\alpha G$.

Most desirable to have conditions/criteria mirroring the conditions in the Toms-Winter Conjecture/Theorem.

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1. Dynamical versions of $\dim_{nuc} A < \infty$**:** Rokhlin dimension $\dim_{Rok}(\alpha)$ defined for actions $\alpha : G \curvearrowright A$ for various classes of amenable groups, eg residually finite ones, may be regarded as analogue to $\dim_{nuc}(A)$.

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$\dim_{Rok}(\alpha) < \infty$ leads to crossed products of finite nuclear dimension.

(But involving also certain asymptotic dimension estimates for the group and $\dim_{nuc}(A)$.)

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2. Dynamical comparison for open sets for actions on $\alpha : G \curvearrowright X$ on spaces (say, cpt metric)

Definition (Kerr (using ideas by Winter))

Given $\alpha : G \curvearrowright X$ we define for open sets $U, V \subseteq X$

$$U \preceq_G V$$

if $\forall$ compact $K \subseteq U$, $\exists U_1, ..., U_k \subseteq X$ open, $\exists g_1, ..., g_k \in G$ s.t.
$K \subseteq U_1 \cup \ldots \cup U_k$
and
$g_1 U_1 \ldots g_k U_k$ are disjoint subsets in V.

3. Dynamical version of $\mathcal{Z}$-stability for actions on spaces: almost finite actions

Definition (Kerr (Winter), slightly modified)

Open tower: (S, B) (=(shape, base)), where $S \subset\subset G$ and $B \subseteq X$ open s.t. the sets gB, where $g \in S$ are pairwise disjoint.

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For G amenable, $\alpha : G \curvearrowright X$ is **almost finite** if

$\forall F \subset\subset G$, $\forall \varepsilon > 0$, $\forall V \subseteq X$ open nonempty, $\exists$ open castle in X s.t.,

1. diam(each level) $< \varepsilon$,
2. each shape is (F, ε)-Følner,
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Strict comparison for open sets using this relation reads: $\mu(U) < \mu(V)$ for all G-invariant Borel probability measures implies $U \preceq_G V$.

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3. should therefore perhaps be replaced by strict comparison in the dynamical Cuntz semigroup, and we will do so subsequently.

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The latter result has recently also been established for all elementary amenable groups (Kerr-Naryshkin).

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Start off with usual Cuntz comparison: for $a, b \in A_+$ (or $M_\infty(A)_+$), write $a \preceq_{Cu} b$ or just $a \preceq b$ if $\inf_{x \in A} \|a - x^* bx\| = 0$, leading to

$$W(A) := (M_\infty(A)_+, \oplus, \preceq_{Cu}); \quad Cu(A) := ((A \otimes K)_+, \oplus, \preceq_{Cu}).$$

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if for any $\varepsilon > 0$ there is $\delta > 0$, $n \in \mathbb{N}$, elements $g_1, \ldots, g_n \in G$ and positive elements $x_1, \ldots, x_n \in M_\infty(A)$ such that

$$(a - \varepsilon)_+ \preceq \bigoplus_i \alpha_{g_i}(x_i) \text{ and } \bigoplus_i x_i \preceq (b - \delta)_+$$

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Here is the concrete description of G-subequivalence in this setting:

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For $a, b \in A_+$, we say that a is *dynamically subequivalent* to b, in symbols $a \precsim_G b$ if there is a sequence of positive elements $x_1, \ldots, x_n \in M_\infty(A)$ with

$$a = x_1 \precsim_0 x_2 \precsim_0 \cdots \precsim_0 x_{n-1} \precsim_0 x_n = b.$$

Finally, we say that a is *dynamically equivalent* to b, in symbols $a \sim_G b$, provided $a \precsim_G b$ and $b \precsim_G a$.

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$W_G(A)$ is the semigroups given by the equivalence classes of $\preceq_G$ thus defined.

To explain this differently, note that $\preceq_{Cu}$ is the smallest (ie strongest) additive pre-order $\preceq$ on A_+ st

1. $\overline{aAa} \subseteq \overline{bAb}$ (write $a \preceq_{her} b$) $\Rightarrow a \preceq b$
2. $x^*ax \preceq a, \forall x \in A, a \in A_+$
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4. $g \cdot a := \alpha_g(a) \preceq a$ (hence $g \cdot a \sim a$), $\forall g \in G$

Recall strict comparison in A or $W(A)$

Note: $a \preceq b \Rightarrow d_\tau(a) \leq d_\tau(b)$ $\forall$ trace τ ($d_\tau(a) := \lim_n \tau(a^{1/n})$).
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$\alpha : G \curvearrowright A$ is said to have **dynamical strict comparison** (DSC) if whenever $d_\tau(a) < d_\tau(b)$, $\forall$ G-invariant trace τ, we have $a \preceq_G b$ (in $W_G(A)$).

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We'll next define non-commutative castles and our generalisation of almost finiteness which we call almost elementariness.

We will use $\preceq_G$ to measure smallness of the complement of castles.

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Definition (Winter, Z)

$dim_{nuc}(A) \leq d$ iff $\forall F \subset\subset A$, $\forall \varepsilon > 0$, $\exists$ finite-dimensional C*-algebra C and cpc maps $\psi : A \to C = C_0 \oplus \ldots \oplus C_d$ and order zero maps $\varphi_i : C_i \to A$ such that $\|(\varphi_0 + \ldots + \varphi_d) \circ \psi(a) - a\| < \varepsilon$, $\forall a \in F$.

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Theorem (Hirshberg-Orovitz)

A simple unital nuclear C-algebra A is $\mathcal{Z}$-stable if* $\forall F \subset\subset A$, $\forall \varepsilon > 0$, $\forall n$, $\forall b \in A_+ \setminus \{0\}$, $\exists$ *cpc* $\perp$ $\varphi : M_n \to A$ *s.t.* $\|[F, \varphi((M_n)_{\leq 1}]\| < \varepsilon$ *and* $1 - \varphi(1) \preceq_{Cu} b$.

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Indeed the space is being approximated by finer and finer castles exhausting the space tracially further and further whilst the towers of the castle correspond to larger and larger chunks of the group left almost invariant.

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Definition (BPWZ)

A castle for $\alpha : G \curvearrowright A$ is a c.p.c. $\perp$ map $\lambda : \bigoplus_{i=1}^{k} M_{n_i} \otimes C(S_i) \to A$, where $S_i \subset\subset G$, st $\forall i, \forall g, h \in S_i$, we have with $\lambda_i = \lambda|M_{n_i} \otimes C(S_i)$

$$\alpha_{g^{-1}} \circ \lambda_i(\cdot \otimes \delta_g) = \alpha_{h^{-1}} \circ \lambda_i(\cdot \otimes \delta_h) : M_{n_i} \to A,$$

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Theorem (BPWZ)

Let $\alpha : G \curvearrowright A$ be a minimal action, G amenable then

$$\mathrm{AE}, \ \ \mathrm{AE}_{abs}, \ \ \mathrm{AE}_{rel}, \ \ \mathrm{AE}_{abs}^{mea} + (\mathrm{DSC}), \ \ \mathrm{AE}_{rel}^{mea} + (\mathrm{DSC})$$

are all equivalent.

Our definition indeed generalises Kerr's almost finiteness:

Theorem (BPWZ)

For $A = C(X)$ and G amenable, $\alpha : G \curvearrowright A$ is almost elementary iff $\alpha : G \curvearrowright X$ is almost finite.

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Idea: a castle $\lambda : C \to A$ is always of the form $\lambda : \oplus_{i=1}^{k} C(S_i) \to C(X)$. The open supports of $\lambda_i(\delta_g)_{i=1,\dots,k; g \in S_i}$ form a castle of open sets in X.

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This is a weak/order zero version of Lin's tracial AF condition and closely related to Fu and Lin's tracial nuclear dimension.

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Any simple, nuclear, monotracial $\mathcal{Z}$-stable C-algebra satisfying the UCT is almost elementary.*

We suspect that $\mathcal{Z}$-stability and almost elementariness might be equivalent simple nuclear unital infinite-dimensional C*-algebra, thus adding another condition to the Toms-Winter conditions.

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Theorem (BPWZ)

Let $\alpha : G \curvearrowright A$ be almost elementary then the restriction of the canonical map $W_G(A) \to W(A \rtimes G)$ to the soft part is an order embedding.

Thank you very much !